WHY MIKUSINSKI'S CALCULUS?

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Introduction

Operational calculus will be referred to as the Heaviside calculus for solving linear differential equations or system of equations. The original form of this calculus is known not to be a satisfactory mathematical model. The usual form of this operational calculus is based on Laplace-transform, however, attemptions have been made to an algebraic foundation in which Laplace-transform provides only a computational or realization device.

A well-known algebraic foundation of the operational calculus is due to *J. Mikusinski* [1]. The purpose of this paper is to fit the Mikusinski's calculus into the usual frame of functional analysis.

Two different representations of translation invariant operators in the discrete case will be described to motivate our investigations.

1. The discrete case

Let S_+ be the linear space of finite sequences of real numbers. If $\{a_n\} \in S_+$, then

$$\{a_n\} = a_0, a_1, a_2, \ldots, a_N, 0, 0, \ldots$$

i.e. for every $\{a_n\} \in S_+$ there is a N, such that $a_k = 0$ for $k > N$. Let U be the *translation operator*

$$U\{a_n\} = 0, a_0, a_1, \ldots, a_{N-1}, a_N, 0, \ldots$$

i.e. the zeroth element of $U\{a_n\}$ is 0, and the k-th element is a_{k-1} if $k = 1, 2, \ldots$

The linear operator T is called *translation-invariant* if $UT = TU$. In the following the traslation-invariant operators in S_+ will be characterized.

If T is translation-invariant, e_0 is the unit sequence 1, 0, 0, . . . and

$$Te_0 = a_0, a_1, a_2, \dots, a_N, 0, 0, \dots$$

then

$$Te_m = TU^m e_0 = U^m T e_0$$

where e_m is the sequence with one in the m-th place and zeros in other places. Consequently, on the basis of e_k; $k = 1, 2, \dots$ the matrix $\mathbf{T}$

$$\mathbf{T} = \begin{pmatrix} a_0 & 0 & 0 & \dots \\ a_1 & a_0 & 0 & \dots \\ \vdots & & & \\ a_N & a_{N-1} & a_{N-2} & \dots \\ 0 & a_N & a_{N-1} & \dots \\ 0 & 0 & a_N & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

i.e. if t_{ik} is the ik-th element of T, then

$$t_{ik} = \begin{cases} a_{i-k} & \text{if } k \le i \le N + k \\ 0 & \text{otherwise.} \end{cases}$$

Particularly, for the translation operator, $Ue_0 = e_1$ and hence

$$U = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

The translation-invariant operators form a commutative algebra and if

$$y = y_0, y_1, \dots, y_M, 0, 0, \dots$$

then the n-th element of Ty is $\sum_{i=0}^{n} a_{n-i}\, y_i$. Ty is called the *(discrete) convolution* of a and y and denoted by $a * y$. Since

$$\sum_{i=0}^{n} a_{n-i}\, y_i = \sum_{i=0}^{n} y_{n-i}\, a_i,$$

the convolution is commutative, i.e. $a * y = y * a$.

It is worth to mention that the operator T and his infinite matrix T is determined by the *finite* sequence

$$a = a_0, a_1, \ldots, a_N, 0, 0, \ldots$$

Because of the representation $Ty = y * a$, the operator T is called *the convolution operator a*. The connection between convolution and translation operator is

$$Ty = y * a = \left(\sum_{k=0}^{\infty} a_k \ U^k \right) y$$

where the sum eventually consists of finite members depending on a.

It follows from this connection, that if φ is a multiplicative linear functional of the commutative algebra of translation-invariant operators (or, what is the same, convolution operators), then

$$\varphi(T) = \varphi(a) = \sum_{k=0}^{\infty} a_k \varphi(U)^k.$$

Hence, if $\varphi(U) = z$, then $\varphi(T)$ is a certain value of the polynomial $\sum_{k=0}^{N} a_k z^k$ and thus, because of the totality of multiplicative linear functionals, the algebra of convolution operators is transformed into the algebra of (complex) polynomials called the Z-transform of a.

2. The maximal extension of translation-invariant operators

Let C_+ be the class of continuous functions of the real line which are zero for $t < 0 \cdot C_+$ is a linear space over the field of complex numbers with tha usual linear operations.

The sequence f_n is *convergent* in C_+ if there is an $f \in C_+$ such that $f_n \to f$ uniformly on every finite closed $[0, t_0]$.

Remark. The reason why C_+ has been chosen for the domain of translation-invariant operators is to diminish the problems connected with the classical, so-called hard analysis. Starting from a more general class of functions and many different convergences leads to the same structure.

The operators

$$U_\tau: \quad [U_\tau f](t) = f(t - \tau); \qquad \tau \geq 0$$

are called *translation operators.*

The linear operator T is *translation-invariant* if

$$f \in D(T) \Rightarrow U_\tau f \in D(T) \text{ and } U_\tau T = T U_\tau; \quad \tau \geq 0$$

where $D(T)$ is the domain of T.

Remark. It is essential that T is *not* everywhere defined on C_+ in contrast to the discrete case. Overcoming the difficulties yields T defined on but a small part of C_+ and that different operators are defined, in general, on different parts of C_+ is the main subject of our investigations.

We shall deal with *closed* operators. The operator T is closed if

$$f_n \in D(T);\ n = 1, 2, \ldots, f_n \to f \text{ and } Tf_n \to g$$

it follows, that $f \in D(T)$ and $Tf = g$.

A continuous operator T with closed domain is a closed one. Particularly, a continuous operator defined everywhere is a closed one. On the other hand, a closed operator is not necessarily continuous. The simple example for this is the differential operator $T = \dfrac{d}{dt}$ in C_+.

The *convolution* of the functions $f, g \in C_+$ is defined as

$$f * g = \int_{-\infty}^{+\infty} f(t - \tau)\, g(\tau)\, d\tau\ ;$$

in fact:

$$f * g = \int_0^t f(t - \tau)\, g(\tau)\, d\tau$$

since $g(\tau) = 0$ if $\tau < 0$ and $f(t - \tau) = [U_\tau f](t) = 0$ if $\tau > t$.

The following axioms of multiplication are satisfied by the convolution:

I. $f * g = g * f \quad f, g \in C_+$
II. $f * (g * h) = (f * g) * h \quad f, g, h \in C_+$
III. $(f + g) * h = f * h + g * h \quad f, g, h \in C_+$
IV. $f * g = \emptyset$ if and only if $f = \emptyset$, or $g = \emptyset$

where $\emptyset$ is a function of a zero value everywhere.

There is a closed connection between translation and convolution.

Let $f, g \in C_+$ and for a fixed t:

$$\lambda_k = \frac{k}{n} t, \quad k = 0, 1, 2, \ldots, n;$$

then the sums

$$h_n(t) = \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\, f(\lambda_k)\, U_{\lambda_k}\, g(t)$$

are the rectangle approximation of the integral $f * g$ for a fixed t and

$$h_n \to f * g.$$

The operator $T: Tg = f * g$, where $f \in C_+$, is called a ***convolution operator***.

The operator T is called *multiplier* if $g \in D(T) \Rightarrow f * g \in D(T)$ for every $g \in C_+$ and

$$T(f * g) = f * Tg.$$

It follows from the axioms I and II satisfied by the convolution, that every convolution operator is a multiplier. Moreover, T is a multiplier if $Tf * \; = f * T$ for every convolution operator f.

Theorem 1. Every closed translation-invariant operator is a multiplier.

Proof. If T is translation-invariant and $g \in D(T)$, then

$$Th_n = \sum_{k=1}^{n} (\lambda_k - \lambda_{k-1})\, f(\lambda_k)\, U_{\lambda_k}\, Tg$$

hence, from the connection between translation and convolution:

$$Th_n \to f * Tg.$$

On the other hand, $f * g \in D(T)$ and $T(f*g) = f*Tg$ since T is closed.

If T is a multiplier, then for every $f, h \in D(T)$

$$h * Tf = (Th) * f \qquad (*)$$

i.e. a multiplier can be transferred from one side of the convolution to the other; indeed, if T is a multiplier, then

$$T(h * f) = h * Tf$$

and

$$T(h * f) = T(f * h) = f * Th = (Th) * f.$$

Remark. It is a natural question whether every operator satisfying $(*)$ is a multiplier or not. If T is everywhere defined, then the answer is yes; but in general, an operator satisfying $(*)$ is only a restriction of a multiplier.

The way how $(*)$ as a multiplier can be transferred is very similar to the definition of derivation and many other operators in distribution theory. Indeed, by the property $(*)$, a closed connection between translation-invariant operators and distributions can be demonstrated.

Theorem 2. For a given pair $f_0, g_0 \in C_+, (f_0 \neq \emptyset)$, be defined

$$(**) \qquad D = \{f: \text{ there exists } g \text{ such that } f*g_0 = g*f_0\}$$

then there exists a *unique* operator T satisfying $(*)$ such that $D = D(T)$ and $Tf = g$.

Proof. It is obvious that the operator T satisfies $(*)$. If for a certain $f \in D$, two different functions g and g' exist such that

$$f*g_0 = g*f_0$$

and

$$f*g_0 = g'*f_0$$

then

$$g*f_0 = g'*f_0$$

what means that $(g - g')*f_0 = \emptyset$. Since $f_0 \neq \emptyset$, it follows from axiom IV of the convolution that $g - g' = \emptyset$ and hence the uniqueness of T is proved.

The operator defined as in the previous theorem is called *maximal operator* determined by (f_0, g_0).

Theorem 3. Every maximal operator T is a closed translation-invariant operator.

Proof. If $f_n \to f$ and $g_n \to g$, then $f_0*g_n \to f_0*g$ and $g_0*f_n \to g_0*f$. Consequently, if for every pair (f_n, g_n) $n = 1, 2, \ldots$

$$f_0*g_n = g_0*f_n$$

then $f_0*g = g_0*f$. It follows, that $Tf = g$ for the maximal operator T determined by (f_0, g_0) and hence T is closed.

It is obvious that every translation U_τ $\tau > 0$ is a continuous operator defined everywhere and hence it is closed; moreover each U_τ $\tau > 0$ is translation-invariant. Consequently, it follows from theorem 1 that every U_τ $\tau > 0$ is a multiplier and hence if

$$g_0*f = f_0*g$$

then

$$g_0*U_\tau f = U_\tau(g_0*f) = U_\tau(f_0*g) = f_0*U_\tau g.$$

It follows that the maximal operator T determined by (f_0, g_0) is translation-invariant.

If T is a maximal operator, then it is a *maximal translation-invariant operator* in the sense that there is no translation-invariant extension for T.

As a partial result of the theorems proved in this item, it has to be pointed out that translation-invariant operators and multipliers are almost the same things; every closed translation-invariant operator is a multiplier and every multiplier can be extended to a closed translation-invariant operator.

The connection between the different types of operators investigated in this item is illustrated by the diagram in Fig. 1.

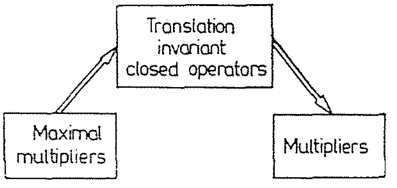

Fig. 1

3. Maximal translation-invariant operators and convolution-quotients

It follows from the Mikusinski's definition of convolution quotients and theorem 2 that the map

$$(*) \qquad T \to (Tf_0, f_0); \quad f_0 \in D(T)$$

is an $1-1$ mapping from the algebra of maximal operators to the convolution quotients. Moreover, it also follows from theorem 2 that this mapping is *onto*.

In fact, the mapping $(*)$ is an algebraic isomorphism:

Theorem 4. If $\quad T_1 \to (T_1 f_1, f_1)$ and $T_2 \to (T_2 f_2, f_2)$

then $\quad T_1 T_2 \to (T_1 f_1 * T_2 f_2, f_1 * f_2)$ and $T_1 + T_2 \to$
$\to (f_1 * T_2 f_2 + f_2 * T_1 f_1, f_1 * f_2)$.

Remark.

$$(T_1 f_1 * T_2 f_2, f_1 * f_2)$$

is known to be the *product*, and

$$(f_1 * T_2 f_2 + f_2 * T_1 f_1, f_1 * f_2)$$

the *sum* of the convolution quotients $(T_1 f_1, f_1)$ and $(T_2 f_2, f_2)$.

Proof. If $f_1 \in D(T_1)$ and $f_2 \in D(T_2)$, then $f_1 * f_2 \in D(T_1 \cdot T_2)$ and

$$T_1 T_2(f_1 * f_2) = T_1 T_2(f_2 * f_1) =$$
$$T_2(f_2 * T_1 f_1) = T_2(T_1 f_1 * f_2) = T_1 f_1 * T_2 f_2.$$

If $f_1 \in D(T_1)$ and $f_2 \in D(T_2)$, then $f_1 * f_2 \in D(T_1 + T_2)$ and

$$(T_1 + T_2)(f_1 * f_2) = T_1(f_2 * f_1) + T_2(f_1 * f_2) = f_2 * T_1 f_1 + f_1 * T_2 f_2.$$

4. The convergence of maximal operators

Let $\{T_n\}$ be a sequence of maximal operators, $f \in D(T_n)$ $n = 1, 2, \ldots$ and the sequence $\{T_n f\}$ be convergent in C_+. Then $\{T_n\}$ is called *convergent in one point*. In this case $T = \lim T_n$ is defined as the maximal operator given as in Theorem 2 by the pair (f, g) where $g = \lim T_n f$.

$T = \lim T_n$ is independent of the choice of $f \in D(T_n)$ $n = 1, 2, \ldots$. Indeed, if $f_0 \neq f$ and $f_0 \in D(T_n)$ $n = 1, 2, \ldots$ then

$$f_0 * T_n f = f * T_n f_0 \quad n = 1, 2, \ldots$$

moreover if $T_n f \to g$ and $T_n f_0 \to g_0$, then

$$f_0 * g = f * g_0$$

and hence (f_0, g_0) defines the same maximal operator T as the pair (f, g).

It is easy to show that the convergence in one point is equivalent to the Mikusinski's convergence.

Now let $T(\lambda)$ be an *operator function* i.e. a function from a subset of real numbers into the algebra of maximal operators. $T(\lambda)$ is continuous over an interval (α, β) if there is $f \in C_+$ such that $T(\lambda) f \in C_+$ for every $\lambda \in (\alpha, \beta)$ and $T(\lambda) f$ is a continuous function on $(\alpha, \beta) \times R$, where R is the real line. $T(\lambda)$ is *derivable* on (α, β) if $T(\lambda) f$ is derivable on $(\alpha, \beta) \times R$. In this case

$$T'(\lambda): \qquad T'(\lambda) f = \frac{\delta}{\delta \lambda} T(\lambda) f.$$

It can be shown, that continuity and derivative are consistent with the convergence defined in the beginning of this item and hence, continuity and derivative are equivalent to the Mikusinski's ones.

Example. For the translation operator $U(\lambda)$ $\lambda \geq 0$:

$$U(\lambda)'f = \frac{d}{d\lambda} f(t - \lambda) = - f'(t - \lambda)$$

particularly, if D is the differential operator d/dt, then the value of $U'(\lambda)$ at $\lambda = 0$ is

$$- Df = - f'.$$

Consequently, the translation operator $U(\lambda)$ satisfies the differential equation

$$(*) \qquad \frac{d}{d\lambda} U(\lambda) = - DU(\lambda); \quad U(0) = E$$

for $\lambda \geqq 0$ and the identity operator E.

For the closed finite interval $[\alpha, \beta]$ and

$$\alpha = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{n-1} < \lambda_n = \beta$$

considered the formal sum

$$\sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \, T(\lambda_k) \, .$$

If there exists $f \in C_+$ such that $T(\lambda) f \in C_+$ for every $\lambda \in [\alpha, \beta]$, then

$$\sum_{k=1}^{n} (\lambda_k - \lambda_{k-1}) \, T(\lambda_k) f \in C_+$$

is the rectangle approximation of the (parametric) integral

$$\int_{\alpha}^{\beta} T(\lambda) f d\lambda \in C_+ \, .$$

If the above integral exists, e.g. if $T(\lambda)$ is a continuous function of two variables, then

$$\int_{\alpha}^{\beta} T(\lambda) \, d\lambda : \left[\int_{\alpha}^{\beta} T(\lambda) \, d\lambda \right] f = \int_{\alpha}^{\beta} T(\lambda) f d\lambda$$

is called *the integral on* $[\alpha, \beta]$ of the operator function $T(\lambda)$.

Example. The convolution operator

f: $\qquad T_f g = g * f$

is a continuous translation-invariant operator defined everywhere and hence it is maximal. It follows from the connection between translation and convolution shown in item 2, that

$$(**) \qquad f = \int_0^\infty f(\lambda)\, U(\lambda)\, d\lambda$$

where $U(\lambda)$ are the translations again.

It is emphasized that the integral $(**)$ is eventually an integral on the finite interval $[0, t]$.

5. Laplace-transform and multiplicative functionals

The previous items demonstrated the connections between the concepts of "translation-invariance", "multiplier" and "convolution quotients". Next a natural explanation will be given of the question: why the Laplace-transform is the mathematical device of constant systems?

Let φ be a continuous multiplicative linear functional (shortly, *multiplicative functional*) in the algebra of maximal operators. i.e.

I. $\varphi(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 \varphi(T_1) + \lambda_2 \varphi(T_2)$; (linear)

II. if $T_n \to T$, then $\varphi(T_n) \to \varphi(T)$; (continuous)

III. $\varphi(T_1 \cdot T_2) = \varphi(T_1)\, \varphi(T_2)$; (multiplicative)

Then, from Eq. $(*)$ of the previous item 4, it follows that the function $\varphi(U(\lambda))$ satisfies the equation

$$\varphi'(U(\lambda)) = -s\varphi(U(\lambda))$$
$$\varphi(U(0)) = 1$$

where $s = \varphi(D)$. Hence $\varphi(U(\lambda)) = e^{-s\lambda}$ and it follows from the representation $(**)$ in 4 that

$$\varphi(f) = \varphi\left[\int_0^\infty f(\lambda)\, U(\lambda)\, d\lambda\right] = \int_0^\infty f(\lambda)\, e^{-s\lambda}\, d\lambda\,.$$

Thus we have obtained

Theorem 5. The multiplicative functional φ is defined for the convolution operator f if and only if the Laplace transform of $f \in C_+$ exists in the point $s = \varphi(D)$. In this case, $\varphi(f)$ is the Laplace transform of f in the point $s = \varphi(D)$.

Now the more general case will be considered where the maximal operator T is not a convolution operator. If $f \in D(T)$ and there exists a Laplace transform of f and Tf, then

$$\varphi(Tf) = \varphi(T)\,\varphi(f)$$

since φ is multiplicative. Hence, for $\varphi(f) \neq 0$ we have

$$\varphi(T) = \frac{\varphi(Tf)}{\varphi(f)} .$$

The value $\varphi(T)$ is independent of the choice of $f \in D(T)$. Indeed, if $f_0 \in D(T)$ is different from f, $\varphi(f_0) \neq 0$ and the Laplace-transform of f_0 and Tf_0 exists, then

$$\varphi(f * Tf_0) = \varphi(f)\,\varphi(Tf_0)$$

and

$$\varphi(f_0 * Tf) = \varphi(f_0)\,\varphi(Tf);$$

hence

$$\varphi(f)\,\varphi(Tf_0) = \varphi(f_0)\,\varphi(Tf),$$

it follows:

$$\frac{\varphi(Tf_0)}{\varphi(f_0)} = \frac{\varphi(Tf)}{\varphi(f)} .$$

Summary

Description is given of translation-invariant operators in certain linear spaces of importance in the operational calculus. The connections between these classes of operators and Laplace-Transform technique and Mikusinski's calculus are investigated.

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