A CONVOLUTION STRUCTURE FOR ALMOST INVARIANT OPERATORS*

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1. Let G be a compact Abelian group and let $\{(t, \gamma_n)\}$, $n = 0, \pm 1, \pm 2, \ldots$, be the set of characters on G. On the lines of Deleeuw [2], we say that T is an operator on G, provided

$$TR_t = (t, \gamma_n) R_t T, \; t \in G,$$

where R_t is the translation operator on G defined by

$$(R_t f)(s) = f(s - t); \; s, t \in G.$$

Let B be a translation invariant dense linear subspace of $L_1(G)$. We suppose that B is a Banach space under a norm $\| \cdot \|_B$ satisfying the conditions

$$\| f \|_{L_1}(G) \neq \| f \|_B,$$

$$\| R_t f \|_B = \| f \|_B, \; f \in B \text{ and } t \in G;$$

$$\lim_{t \to 0} \| R_t f - f \|_B = 0,$$

and B is closed under multiplication by $\{(t, \gamma_n)\}$, i.e., if $f \in B$, then the function $M_n f$ given by

$$(M_n f)(t) = (t, \gamma_n) f(t), \; t \in G,$$

belongs to B.

Let $\mathfrak{L}$ be the Banach algebra of bounded linear operators on B with respect to the norm $\| \cdot \|_{\mathfrak{L}}$. An operator T in $\mathfrak{L}$ is called almost invariant, if

$$\lim_{t \to 0} \| TR_t - R_t T \|_{\mathfrak{L}} = 0.$$

We denote the set of almost invariant operators in $\mathfrak{L}$ by $\mathfrak{L}_*$.

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In fact, $\mathfrak{L}_*$ is a closed subalgebra of $\mathfrak{L}$. For, let $T_1, T_2 \in \mathfrak{L}_*$ and λ, μ be any constants, then we have

$$(\lambda T_1 + \mu T_2) R_t - R_t(\lambda T_1 + \mu T_2) = \lambda(T_1 R_t - R_t T_1) + \mu(T_2 R_t - R_t T_2) \Rightarrow$$

$$0 \leqq |\lambda| \, \| T_1 R_t - R_t T_1 \|_{\mathfrak{L}} + |\mu| \, \| T_2 R_t - R_t T_2 \|_{\mathfrak{L}} \to 0, \text{ as } t \to 0.$$

Hence $\mathfrak{L}_*$ is linear.

Also, for any $T_n \in \mathfrak{L}_*$, we have

$$\| TR_t - R_t T \|_{\mathfrak{L}} \leq \| TR_t - T_n R_t \|_{\mathfrak{L}} + \| T_n R_t - R_t T_n \|_{\mathfrak{L}} + \| R_t T_n - R_t T \|_{\mathfrak{L}},$$

which tends to zero for $T_n \to T$.

This implies that T is continuous in $\mathfrak{L}_*$.

2. Let C be the class of functions such that

$$C = \left\{ Q : Q(t) = \sum_{-\infty}^{\infty} \alpha_k \cdot (t, \gamma_k) \right\},$$

where $\alpha_k > 0$, $\alpha_k \to 0$ as $k \to \pm \infty$ and $\Delta^2 \alpha_k \geqq 0$.

We observe that the series $\Sigma \alpha_k (t, \gamma_k)$ is uniformly convergent. Hence every function in C is continuous.

We define the Fourier series associated with an almost invariant operator T by

$$S(T) = \sum_{-\infty}^{\infty} a_n(T) \cdot (t, \gamma_n) ,$$

where

$$a_n(T) = \int_G (t, \gamma_n) R_{-t} T R_t \, dt .$$

In a recent paper Deleeuw [2] has proved that the Fourier series of an almost invariant operator on a circle group is (C, I) summable to T in the operator norm.

The object of the present paper is to study the convolution structure for an almost invariant operator. We shall prove the undermentioned:

Theorem. The following statements about an operator T are equivalent:

i) $T \in \mathfrak{L}_*$.

ii) $S(T)$ *is summable* (C, I) *to* T *in the operator norm.*

iii) $T = P * Q$; $P \in \mathfrak{L}_*$ *and* $Q \in C$.

3. We shall use the following lemma in the proof of our theorem:

Lemma. If a series Σu_n is summable (C, I) to f in a Banach space B, Φ is a positive increasing function satisfying the conditions

$$\int_0^\infty \frac{1}{\Phi(t)}\, dt < \infty,$$

$$\mu_n = \Phi^{-1}(\| \sigma_n - f \|^{-1}),$$

σ_n being the (C, I) mean of Σu_n; then there exists a sequence $\{\lambda_n\}$, $0 < \lambda_n \leq \mu_n$, $\Delta^2 \lambda_n \leq 0$, $\lambda_n \uparrow \infty$; such that the series $\Sigma \lambda_n u_n$ is summable (C, I) in B.

For the proof see [I].

4. *Proof of the theorem.* First if will be shown that

$$i) \Rightarrow ii).$$

The n-th (C, I) mean $\sigma_n(T)$ of the Fourier series of T is given by

$$\begin{aligned}\sigma_n(T) &= \sum_{n=-N}^{+N} \left(1 - \frac{|n|}{N+1}\right) (\varkappa, \gamma_n)\, a_n(T) = \\ &= \int_G \left[\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) (\varkappa, \gamma_n)(t, \gamma_n)\right] \cdot R_{-t} T R_t \, dt = \\ &= \int_G K_n(\varkappa, t)\, R_{-t} T R_t \, dt,\end{aligned} \tag{4.1}$$

where $K_n(\varkappa, t)$ is the n-th Fejér kernel.

Therefore, we get

$$\sigma_n(T) - T = \int_G K_n(\varkappa, t)\, \{R_{-t} T R_t - T\}\, dt$$

i.e., $$\| \varkappa_n(T) - T \|_{\mathfrak{L}} \leq \| K_n(\varkappa, t) \|_{L_1(G)} \| R_{-t} T R_t - T \|_{\mathfrak{L}} = 0\ (1),$$

since $K_n(\varkappa, t)$ is an approximate identity.

Next, we shall show that $(ii) \Rightarrow (iii)$.

Let us consider, e.g. the series

$$\sum \lambda_n(\varkappa, \gamma_n)\, a_n(T) = \sum (\varkappa, \gamma_n)\, \beta_n(P), \tag{4.2}$$

By the lemma in section 3, the series (4.2) is summable (C, I) in $\mathfrak{L}_*$ to, e.g. the value P.

Thus, we have

$$\begin{aligned}\beta_n(P) &= \int_G (t, \gamma_n)\, R_{-t}\, PR\, dt = \\ &= \int_G [R_{-t}\, PR_t - \sigma_k(P)]\, (t, \gamma_n)\, dt + \\ &+ \int_G \sigma_k(P)\, (t, \gamma_n)\, dt = I_1 + I_2\, .\end{aligned} \tag{4.3}$$

We now observe that $I_1 \to 0$ in the operator norm.

Also, for $k \geqq n$, we have

$$I_2 = \int_G \left[\sum_{m=-k}^{k} \left(1 - \frac{|m|}{k+2}\right) \cdot \lambda_m(t, \gamma_m)\, a_m(T)\right] \cdot (t, \gamma_n)\, dt = \frac{(k-m+1)\,\lambda_n\, a_n(T)}{k+1}$$

by the orthonormality of the set $\{t, \gamma_n\}$: $n = 0, \pm 1, \ldots$

Hence,

$$\beta_n(P) = \lambda_n \cdot a_n(T)$$

i.e.,

$$a_n(T) = \lambda_n^{-1} \cdot \beta_n(P).$$

Choosing $\alpha_n = \lambda_n^{-1}$ in item 2:

$$\begin{aligned}\Delta^2 \alpha_n &= \{\lambda_{n+1}(\lambda_{n+2} + \lambda_n) - 2\,\lambda_n\,\lambda_{n+2}\}/(\lambda_n\,\lambda_{n+1}\,\lambda_{n+2}) \geqslant \\ &\geqslant \frac{1}{2} \cdot \frac{(\lambda_n - \lambda_{n+2})^2}{\lambda_n\,\lambda_{n+1}\,\lambda_{n+2}} \, . \\ &\geqslant 0.\end{aligned}$$

Thus, we infer that every term in the series $S(T)$ is a convolution of the terms in the series

$$\sum_{-\infty}^{\infty} \beta_n(P) \cdot (t, \gamma_n)$$

and

$$\sum_{-\infty}^{\infty} \alpha_n(t, \gamma_n).$$

Hence:

$$T = P * Q.$$

Finally, we have to prove that $(iii) \Rightarrow (i)$. The convolution of an operator P in $\mathfrak{L}_*$ and a function $Q \in C$ are defined by the vector valued integral

$$P * Q = \int_G R_t P R_{-t}\, Q\, dt.$$

In fact, the above integral is defined as the limit of the sum

$$\sum_{i=1}^{n} R_{-t_i} P R_{t_i} Q(t_i) \, \Delta t_i \, .$$

Since $\mathfrak{L}_*$ is a norm closed linear subspace of $\mathfrak{L}$ and $R_t PR_{-t} \in \mathfrak{L}_*$, therefore $P * Q = T$ is an almost invariant operator.

Thus: $T \in \mathfrak{L}_*$.

This completes the proof of the theorem.

Summary

A theorem is presented about the construction of almost invariant operators, introduced by Deleeuw [2], for translation-invariant operators and multiplication operators.

References

1. Bryant, J.: On convolution and Fourier series, Duke Mathematical Journal, Vol. 34 (1967), 117–122.
2. Deleeuw, K.; A harmonic analysis for operators. Illinois Mathematical Journal, Vol. 19 (1975), 593–606.

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