ON GLOBAL METHODS TO SOLVE DIFFERENCE-DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENTS

By

A. Bleyer

Department of Mathemathics of the Faculty of Electrical Engineering
Technical University, Budapest

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1. Introduction

In the recent time interest is focussed on investigations into difference-differential equations, because of the wide scale of applications. Nevertheless the numerical methods for solving this kind of equations have not developed so far. Here it is attempted to find a general theorem for the equation $y'(t) = F[t, y(t), y(t-\omega)]$ $(\omega > 0)$, and the convergence of a spline method. Finally some problems are discussed concerning the numerical stability of the presented method.

2. (ω, h)-step method

Let us consider the difference-differential equation

$$u'(t) = F[t, u(t-\omega)] \tag{1}$$

with initial function $u(t) = g(t)$ for $0 \leq t \leq \omega$. Assume nice properties on $F(t, u, v)$ by which the uniqueness and existence of (1) have been made sure. For instance $F(t, u, v)$ is assumed to be analytic and $g'(\omega - 0) = F[\omega, g(\omega), g(0)]$. Let $\omega = hn$ (where n is a fixed integer) and $t_k = kh$, $k = 0, 1, \ldots$ u_k stands for the computed value and $u(t_k)$ means the exact value of the solution of (1) at $t = t_k$.

An (ω, h)-step method means first to determine the discrete values u_k for $k = (n+1), (n+2) \ldots, 2n$, and a well defined interpolant of $u(t)$ based on point t_k and on computed values u_k. Then a solution is based on $[\omega, 2\omega]$, denoted by $s_1(t)$. Second, the method is repeated on $[2\omega, 3\omega]$ having initial function $s_1(t)$ on $[0, 2\omega]$. Repeating the same (ω, h)-step for any large t yields the interpolant. Since $t_k - \omega = (k-n)\,h = t_{k-n}$ we have $u(t_k - \omega) = u_{k-n} = g(t_{k-n})$ for $n < k < 2n$, and if $nl \leq k \leq n(l+1)$ then $u(t_k - \omega) = u_{k-n} = s_l(t_{k-n})$, where $s_l(t)$ is the interpolant of $u(t)$ in the interval $l\omega \leq t \leq (l+1)\omega$.

Let

$$u_{k+1} - u_k = h\Phi(t_k, u_k, u_{k-n}, u_{k+1}, h) \tag{2}$$

for $k = (n + 1), (n + 2) \ldots$. The function $\Phi(t, u, v, w, h)$ is called the *implicit increment function* of the method. If $e_k = u_k - u(t_k) = 0\,(h^p)$, p being the greatest number for which the relation holds, then p is the *discretization order* of the method. If $u \in C^2$, $\vartheta(u, t)$ is the used interpolant on $(l\omega, (l + 1)\,\omega)$ with $\vartheta(u, t_k) = u(t_k)$ then $\| \vartheta^{(i)}(u, t) - u^{(i)}(t) \|_C = 0\,(h^{q-i})$, $(i = 0, 1)$, q is said to be the *order* of interpolation.

It is also assumed to have a "good" interpolation method where, if $| u(t_k) - v(t_k) | < Kh^r$ for $nl \leq k \leq n(l + 1)$ then $\| \vartheta^{(i)}(u, t) - \vartheta^{(i)}(v, t) \|_C = 0(h^{r-i})$, $(i = 0,1)$, with $r > 1$ holds.

Let $\| s_l^{(i)}(t) - u^{(i)}(t) \|_C = 0(h^{G_i})$ $(i = 0,1)$. $[G_0, G_1]$ is said to be the *global order* of (ω, h) — step method. A *global method* on $[a, b]$ is called *convergent* if $G_0 > 1$ and $G_1 > 0$. We will prove the following theorem on convergence.

Theorem 1. *Let $\Phi(t, u, v, w, h)$ be the implicit increment function of the method applied for (1). Assume $F(t, u, v)$ to be an analytic function and a good interpolation with order $q \geq 2$. The function $\Phi(t, u, v, w, h)$ satisfies a Lipschitz condition for $0 \leq h < h_0$, $t \geq \omega$, $-\infty < u, v, w < \infty$, that is, there exists $L > 0$ such that*

$$| \Phi(t, u, v, w, h) - \Phi(t, \bar{u}, \bar{v}, \bar{w}, h) | < L\{ | u - \bar{u} | + | v - \bar{v} | + | w - \bar{w} | \} \tag{3}$$

Moreover Φ_h^1 is continuous at $h = 0$. Then the condition

$$\Phi(t, u, v, w, 0) = F(t, u, v) \tag{4}$$

is necessary and sufficient for the convergence. The condition (4) is said to be the *consistence relation.*

Proof: The idea of the proof comes from the result by **P.** Henrici [1]. At first we prove that (4) is sufficient. Assume

$$\Phi(t, u(t), u(t - \omega), u(t), 0) = H\big(t, u(t), u(t - \omega)\big)$$

Whenever $u(t)$ is a solution of (1) then $u(t - \omega) = g(t - \omega)$; hence $H(t, u(t), g(t - \omega)) = \tilde{h}(t, u(t))$. According to the theorem of Cauchy-Peano there exists a solution of the initial value problem $h(t, z) = z'$, $z(\omega) = z_0$. Let $z(t)$ the solution of this problem.

$$z_{k+1} = z_k + h\Phi(t_k, z_k, g(t_k - \omega), z_{k+1}, h) \tag{5}$$

$k = (n + 1), (n + 2) \ldots, t_k$ is defined as above, then $e_k = z_k - z(t_k)$ can be proven to tend to zero as $h \to 0$; more exactly $e_k = 0(h^2)$ will be proven.

Denote

$$z(t_{k+1}) = z(t_k) + h\,\frac{z(_{k+1}) - z(t_k)}{h} = z(t_k) + h\Delta\left(t_k, z(t_k), h\right) \tag{6}$$

then

$$e_{k+1} = e_k + h\left[\Phi(t_k, z_k, g(t_k - \omega), z_{k+1}, h) - \Delta(t_k, z(t_k), h)\right] \tag{7}$$

By the mean-value theorem there is θ such that $0 \leq \theta \leq 1$ and $\Delta(t_k, z(t_k), h) = = \tilde{h}(t_k + \theta h, z(t_k + \theta h))$. Writing

$$\begin{aligned}
&\Phi(t_k, z_k, g(t_k - \omega), z_{k+1}, h) - \Delta(t_k, z(t_k), h) = \\
&= \Phi(t_k, z_k, g(t_k - \omega), z_{k+1}, h) - \Phi(t_k, z(t_k), g(t_k - \omega), z(t_{k+1}), h) + \\
&+ \Phi(t_k, z(t_k), g(t_k - \omega), z(t_{k+1}), h) - \Phi(t_k, z(t_k), g(t_k - \omega), z(t_{k+1}), 0) \\
&+ \Phi(t_k, z(t_k), g(t_k - \omega), z(t_{k+1}), 0) - \tilde{h}\left((t_k + \theta h), z(t_k + \theta h)\right).
\end{aligned}$$

By virtue of (3) and (4) and uniform continuity of $\Phi(x, u, v, w, h)$ on the compact set $\{0 \leq h \leq h_0, t \in [\omega, 2\omega], u = z(t), v = g(t - \omega), w = z(t + h)\}$ we obtain

$$\begin{aligned}
\varphi(h) = \max_{t\in[\omega,2\omega]} &\left[\Phi\left(t, z(t), g(t - \omega), z(t + h), h\right) - \right. \\
&\left. - \Phi\left(t, z(t), g(t - \omega), z(t + h), 0\right)\right] = 0(h)
\end{aligned} \tag{8}$$

continuous at $h = 0$ and $\varphi(h) \to 0$ as $h \to 0$.

By a similar argument

$$\psi(h) = \max_{\substack{t\in[\omega,2\omega] \\ 0 \leq \theta \leq 1}} \left[\tilde{h}\left(t, z(t)\right) - \tilde{h}\left(t + \theta h, z(t + \theta h)\right)\right] = 0(h) \tag{9}$$

continuous at $h = 0$ and tends to zero.

$$\begin{aligned}
|e_{k+1}| &< |e_k| + h \mid L(|e_k| + |e_{k+1}|) + \varphi(h) + \psi(h) \mid \leq \\
&\leq |e|_k(1 + hL) + hL|e_{k+1}| + h\left(|\varphi(h)| + |\psi(h)|\right).
\end{aligned}$$

Hence

$$|e_{k+1}| \leq |e_k|\,\frac{1 + hL}{1 - hL} + \frac{h}{1 - hL}\left(|\varphi(h)| + |\psi(h)|\right) \tag{10}$$

Therefore by (8), (9) and (10)

$$|e_k| < \frac{h}{1 - hL}\left(|\varphi(h)| + |\psi(h)|\right)\frac{\exp\left(\dfrac{1 + 2hL}{1 - hL}\right) - 1}{h\,\dfrac{1 + 2hL}{1 - hL}} = 0(h^2) \tag{11}$$

if $h < h_0$ whenever $hL < 1/2$.

Hence in the case $H(t, u(t), u(t-\omega)) = F(t, u(t), u(t-\omega))$ and $t \in [\omega, 2\omega]$ we have $u(t) \equiv z(t)$.

Now let $\vartheta_{1,k}$ be the interpolant function having values u_k at $t = t_k$, then $||\vartheta_{1,k}(t) - u(t)||_C \leq ||\vartheta_{1,k}(t) - \vartheta(u, t)||_C + ||\vartheta(u, t) - u(t)||_C \leq 0(h^2) + 0(h^q) = 0(h^{G_1})$ where $G_1 = \min\{2, q\} \geq 2$ by assumption.

Since the interpolation method is good,

$$||\vartheta_{1,k}^{(i)}(t) - u^{(i)}(t)||_C = 0(h^{G_i}) \quad i = 0, 1$$

hold, where $G_i \geq 2 - i$, the method is convergent on $[\omega, 2\omega]$. Denote by $s_1(t)$ the function defined on $[0, 2\omega]$

$$s_1(t) = \begin{cases} g(t) & \text{for } t \in [0, \omega] \\ \vartheta_{1,k}(t) & \text{for } t \in (\omega, 2\omega] \end{cases} \tag{12}$$

then $||s_1(t) - u(t)||_{C[0,2\omega]} = 0(h^{G_1})$.

To get the approximate solution of (1) by the method let us consider the following initial value problem

$$\begin{cases} u'(t) = F(t, u(t), u(t-\omega)) \\ u(t) = s_1(t) \text{ for } 0 \leq t \leq 2\omega \end{cases} \tag{13}$$

Then (2) is approximate solution of (13) in $[2\omega, 3\omega]$ as in the first part of the proof. Let $s_2(t)$ be the approximation of solution given by (2) and the interpolation procedure. Denote by $\tilde{u}(t)$ the exact solution of (13) and be $u(t)$ stand for the solution of (1). Let us estimate $|s_2(t_k) - u(t_k)|$ for $k = (2n+1), (2n+2), \ldots 4n$. From the previous part of the proof it follows that

$$|s_2^{(i)}(t_k) - u^{(i)}(t_k)| < Kh^{G_i} \tag{14}$$

Since $F(t, u, v)$ in analytic function, by virtue of (12):

$$F(t, u(t), u(t-\omega)) = F(t, u(t), s_1(t-\omega)) + 0(h^{G_1}) \tag{15}$$

on $2\omega \leq t \leq 3\omega$. Hence

$$|\tilde{u}(t_k) - u(t_k)| < Kh^{G_1}. \tag{16}$$

Integrating both sides of (1) and (13) from 2ω to t_k it follows from (15):

$$\begin{aligned} |u(t_k) - \tilde{u}(t_k)| &< |u(2\omega) - \tilde{u}(2\omega)| + \\ &+ 2\omega L\{||u(t-\omega) - s_1(t-\omega)||_{C[2\omega,3\omega]} + 0(h^{G_1})\}. \end{aligned} \tag{17}$$

Comparing (14), (16) and (17) we obtain:

$$\|s_2^{(i)}(t) - u^{(i)}(t)\|_{C[2\omega,3\omega]} = 0(h^{G_i}).$$

Repeating the presented argument the sufficiency of theorem is seen likewise on any finite interval $[a, b] \subset [0, \infty)$.

Necessity. Assume that there is (t, u) such that $t \in (\omega, 2\omega]$, $H(t, u(t), u(t-\omega)) \neq F(t, u(t), u(t-\omega))$. Then the equation $z' = H(\tau, z(\tau), g(\tau-\omega)) = h(\tau, z(\tau))$ has a solution satisfying $z(t) = u$. By virtue of the proof of sufficiency $z_k \to u(t) - z_k$ defined by (5) — as $h \to 0$. This fact obviously leads to a contradiction, since $z(t) \not\equiv u(t)$ on $[\omega, 2\omega]$ is impossible and assuming $z(t) \equiv u(t)$ on $[\omega, 2\omega]$ then $z'(t) = h(t, z(t)) = H(t, u(t), g(t-\omega)) \neq F(t, u(t), g(t-\omega)) = u'(t)$ by assumption. A similar argument shows the necessity on any interval $[a, b] \subset [0, \infty)$. The proof is complete.

Remark. From the theorem no convergence on the whole line follows since the constants are increasing, but from the proof it is seen that the constants appearing at the estimations depend only on Φ, F, g and on the length of the interval where the solution has to be approximated. (For the whole real line see Problem 1.)

3. Spline approximation

More details on spline see in [2]. A cubic Hermite spline on $I = [a, b]$ is a function of $C^2(I)$ having it knots at points $t_0 = a < t_1 < t_2 \ldots < t_n = b$ such that on each subinterval $[t_i, t_{i+1}]$ that is a cubic polynomial.

Using the Hermite interpolation theory we have

Lemma 1. *There is a unique cubic polynomial* $s(t)$ on $[b_p, e_p]$ *such that it satisfies the conditions*

$$\begin{aligned} s(b_p) &= s_b \qquad & s(e_p) &= s_e \\ s'(b_p) &= s_b' \qquad & s'(e_p) &= s_e' \end{aligned} \tag{18}$$

and it is of the form;

$$\begin{aligned} s(t) = {} & s_b\left[\frac{(t-e_p)^2}{h^2} + 2\frac{(t-e_p)^2(t-b_p)}{h^3}\right] + \\ & + s_e\left[\frac{(t-b_p)^2}{h^2} - 2\frac{(t-e_p)(t-b_p)^2}{h^3}\right] + s_e'\frac{(t-b_p)(t-e_p)^2}{h^2} \\ & + s_e'\frac{(t-b_p)^2(t-e_p)^2}{h^2} \,. \end{aligned} \tag{19}$$

where $h = e_p - b_p$.

From (19) we obtain

$$s''(b_p) = -\frac{6}{h^2} s_{b_p} + \frac{6}{h^2} s_{e_p} - \frac{4}{h} s'_{b_p} - \frac{2}{h} s'_{e_p} \tag{20a}$$

$$s''(e_p) = \frac{6}{h^2} s_{b_p} - \frac{6}{h^2} s_{e_p} + \frac{2}{h} s'_{e_p} + \frac{4}{h} s'_{e_p}. \tag{20b}$$

Therefore, if $s(t)$ is a cubic Hermite spline on $[t_0, t_1] \cup [t_1, t_2]$

$$t_i = t_0 + ih, \; s(t_i) = s_i, s'(t_i) = s'_i$$

then

$$s_2 - s_0 = \frac{h}{3} (s'_0 + 4s'_1 + s'_2) . \tag{21}$$

(This formula is the well-known Simpson-formula)
Using (21) to (1) on $[\omega, 2\omega]$ with step length $h = \omega/n$ we obtain

$$\begin{aligned} u_{k+2} - u_k - \frac{h}{3} [F(t_k, u_k, g(t_k - \omega)) + 4F(t_{k+1}, u_{k+1}, g(t_{k+1} - \omega)) + \\ + F(t_{k+2}, u_{k+2}, g(t_{k+2} - \omega)) \text{ for } k = (n+1) \ldots (2n-2). \end{aligned} \tag{22}$$

By assumption for F, (22) can be solved by iteration when $h < h_0$, provided u_k, u_{k+1} are known.

Hence (22) will furnish the discrete solution $s_1(t_k) = u_k$ provided u_0 and $u_1 = s_1(t_1)$ are used as starting values. This means, u_1 has to determined somehow — $u_0 = s_1(t_0) = u(\omega) = g(\omega)$ —.

Determination of u_1 and the implicit increment function. From (20a) on $[\omega, \omega + h]$ we obtain

$$\frac{6}{h^2} s_1(\omega + h_h) - \frac{2}{h} s'_1(\omega + h) = g''(\omega) + \frac{6}{h^2} s_1(\omega) + \frac{4}{h} s'_1(\omega) \tag{23}$$

in need of: $s'_1(\omega + h) = F(\omega + h, s_1(\omega + h), g(h))$;

$$\begin{aligned} s_1(\omega + h) - g(\omega) = h \Big[\frac{h}{6} g''(\omega) + \frac{2}{3} F(\omega, g(\omega), g(0)) + \\ + \frac{1}{3} F(\omega + h, s_1(\omega + h), g(h)) \Big], \end{aligned} \tag{24}$$

or on the interval $[t_k, t_{k+1}] \subset [l\omega, (l+1)\omega]$

$$s_{k+1} - s_k = h\left[\frac{h}{6} s''_{l-1}(t_{k-n}) + \frac{2}{3} F\big(t_k, s(t_k), s_{l-1}(t_{k-n})\big) + \frac{1}{3} F\big(t_{k+1}\, s_{k+1}, s_{l-1}(t_{k+1-n})\big)\right]. \tag{25}$$

Theorem 2. The cubic Hermite spline method is a convergent global method of order [2, 1]

Proof: The implicit increment function (25)

$$\Phi(t, u(t), u(t-\omega), u(t+h), h) =$$

$$= \frac{h}{6} u''(t-\omega) + \frac{2}{3} F\big(t, u(t), u(t-\omega)\big) + \frac{1}{3} F\big(t+h, u(t+h), u(t+h-\omega)\big)$$

statisfies the condition of theorem 1. The cubic Hermite interpolation gives a good interpolant with $r = 4$ (see [3]).

Remark. To use (22) $s_1(\omega + h) = u_1$ can be obtain by iteration from (25) by the analyticity of F.

4. Supplementary notes

In [3] it is proved that the higher-order spline methods ($m \geq 4$) are unstable. Relaxing the continuity restrictions it is possible to generate stable high order spline methods for numerically solving the problem (1). (For starting ideas see [4].)

Let us consider the equation

$$u'(t) + a\, u(t) + b\, u(t-\omega) = 0, \; a > |b| \tag{26}$$

For this equation the zero is a stable solution. The question naturally arises whether a numerical solution of (26) tends to zero or not. For ordinary differential equations *Dahlquist* [5] defined the so-called A-stability. After Dahlquist we define (ω, A)-stability: A numerical method (global (ω, h)-step one) is said to be (ω, A)-stable, whenever, applied to Eq. (26) with all the roots of the characteristic polynomial $z + a + be^{-\omega z} = 0$ have negative real parts, the computed u_k values tend to zero as $t_k \to \infty$ and h is fixed.

Problem 1. To give a sufficient condition for (ω, A)-stability.

Problem 2. Is the method described in item 3 (ω, A)-stable? We should note that a similar problem has been solved [4] for ordinary initial value problems.

Summary

The present paper gives a sufficient and necessary condition on implicit increment function related to global methods for solving numerically the difference-differential equation

$$y'(t) = F[t, y(t), y(t-\omega)].$$

A spline method is also discussed with respect to the above equation.

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Dr. András BLEYER H-1521 Budapest