A GENERALIZATION OF THE FIRST-ORDER LOGIC

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This paper (and its continuation) deals with certain probabilistic generalizations of the classical first-order logic and with the connections between the generalizations. To lock for the historical genesis of our subject would mean go back to the beginning of the idea of probability, since it would be very difficult to draw the historical borderline between the concept of the probability and that of the inference by probability. The latter relationship remained implicit by the rise of modern symbolic logic and exact theories of probability. Those studies can be considered as new ones which were inspired by the ideas of either the recursion theory or of the model theory and the theory of probability together and which examine the connections among these ideas by the implements of the latest mathematics. This interdisciplinar topic is very far reaching. Roughly speaking, that part of the mathematical logic, whose "probabilistic generalization" is to be outlined in this paper, belongs to the classical model theory. Such generalizations are referred to as "probabilistic logics" (among others). In Section 3 two such concrete generalizations (with identical concepts of the model) will be referred to.

The term "probabilistic logic" will be used with several different meanings on this paper. (This procedure is natural and inevitable, since there is no unified terminology in this field.) One of its meanings is the heuristic one, other meanings correspond to certain well defined logics. Regardless of this ambiguity, it was endeavoured to use this term in a natural and at the same time precise way.

The main view point of the present paper is that of mathematical logic, nevertheless aspects of (the foundation of) the theory of probability seem to deserve to be discussed, involving, however, only those parts of the theory of probability, which are familiar for a reader of average mathematical erudition. The same level is required for understanding the algebraic or measure-theoretic aspects (this framework would not be sufficient for the discussion of algorithm-theoretic or topological, etc. aspects.).

In the second part of this study, which will be published separately, we shall discuss the connection of the topic above with the foundation of the theory of probability — among others the logical interpretations of the event algebras, the possibilities of substitution and generalization of the event algebras by the devices of the mathematical logic.

The present paper is a survey. To my best knowledge no survey has been published on this subject, however, it is justified by the timeliness of the modern mathematical ideas as well as by the numerous results accumulated. Neither my results nor the sources of other theorems and definitions will be referred to separately. This procedure is justified by the *character* of this paper. An outline will be given rather than a precise treatment, taking the risk of being careless at formal issues and at the same time didactic in approaching certain concepts. In this relatively short paper the usual "theorem — definition" structure would not cope with the aim of being understandable for others than specialists. For more details see References as well as the Doctoral Thesis by the author, the latter containing precise references on the author's own results.

Section 1 deals with the probabilistic generalization of the concepts of the first order logic of the classical syntactic system, Section 2 with the classical semantic system. After introducing a concept of the model, Section 3 examine two types of the "probability logic".

1. An approach from the viewpoint of the classical syntactics

The first and perhaps the most classic approach to our subject is the following: How can the axiomatic method, one of the foundations of the mathematical logic be generalized into probabilistic direction? More generally: How can the concept of the theory — as a category of the mathematical logic — be generalized?

The heuristic answer will be given according to the terminology of the syntactical system of the classical logic. The essence of the classical axiomatic method is the following.

We give certain formulae (axioms) and we deduce theorems from them by applying the rules of inference of the logic; one part of the formulae in the given language can be considered as the theorems of the given system of axioms, another part as the undecidable formulae. The first and natural idea of the probabilistic generalization is to build on the classical first order logic; i.e. to use its results instead of building "in itself" a "probability logic."

Now a real number between 0 and 1 will be assigned to each formula. This real number will be (the measure of) the probability. It is a natural requirement to assign the same number to logically equivalent formulae (relative to the given system of axioms); to assign number 1 to the theorems. Further, additivity is required for logically inconsistent formulae: φ and ψ i.e. if $\neg(\varphi \wedge \psi)$ is a theorem. (Finally, if $\psi \vdash \varphi$ holds, then the probability of φ has to be greater than or equal to the probability of ψ, — as a consequence of earlier requirements.)[1] Thus, from a certain point of view, the corresponding concepts of the classical rules of inference will be the properties required from the probability above.[2]

To every classical system of axioms a logical theory can be assigned: the set of formulae deducible from the axioms. Assigning 1 or 0 to a formula when it is a theorem or its negation in the system of axioms, respectively, or even not assigning any probability to a formula otherwise, a correspondence is created between 0 and 1 in the one hand and a set of formulae on the other hand; a correspondence satisfying the requirements for probability.

Since the concept of the theory in the sense of the mathematical logic is more general than that defined by the classical system of axioms (e.g. it can be defined as a subset of formulae of the language closed with respect to deduction; in the case of a consistent theory it is also true that a formula or its negation is either a theorem of the theory or is undecidable) leaving aside the framework provided by the classical axiomatic method; *in the generalization of the concept of theory similarly to the defining axioms cannot and need not be referred at the requirements on probabilities;* the term e.g. "*relative equivalence with respect to a theory*" or "*theorem of a theory*", etc. can be used instead.

To provide the probabilistic generalization of the concept of the theory a technical pair of concept is needed: that of the *basic Lindenbaum algebras* and the *relative Lindenbaum algebra with respect to a theory*. A consistent theory is known to define an equivalence relation (relative logical equivalence with respect to this theory) on the set of formulae of the language; and a Boolean structure can be introduced on the set of equivalence classes obtained above. For example, the theorems of the theory form the unit element of this Boolean algebra; the class consisting of the conjuction of the formulae defining the individual classes is the product of the respective elements, etc. This Boolean algebra is called the Lindenbaum algebra belonging to the language and relative to the corresponding theory. If the theory is the set of the tautologies, then the relative algebra itself is the so-called basic Lindenbaum algebra of the language. Then *the concept of the theory can be generalized as a probability on the relative Lindenbaum algebra with respect to the corresponding theory*. Thus we get back the probabilistic generalization of the theory for those special theories which are defined by axioms.

Remark: For the concept of the theory can be given a different probability generalization, based on the following fact: Every consistent theory induces a 0—1 measure on the subalgebra of its basic Lindenbaum algebra by assign-

[1] Concerning the assumption or negation of the additivity as property of the probability, see in [4]. But in the framework of the present paper, the acceptance of the additivity is natural like of the other usual axioms of the probability.

[2] Using the concept of deducibility the rules of inference of the logic are implicitly assumed. Thus the requirements for the probability are not intended to substitute them.

ing 1 to the classes representing the theorems and 0 to the classes representing the negation of the theorems, since the classes representing the theorems and their negations of the theory form a subalgebra. Generalizing this concept, a *theory* can be defined as a probability on any subalgebra of the basic Lindenbaum algebra.

This definition differs from the earlier one by the following facts: confining our attention to the classical axiomatic method, the definition it allows probabilities less than 1 for axioms; it does not assign probability to every formula; it does not rely directly on the concept of theory of the first order logic. The introduction of a subalgebra may mean closedness with respect to a classical concept of inference.

It is easy to modify our discussion above and below for this generalization of the concept of theory. The generalization outlined originally seems to be more appropriate from the points of view of principle and practice, but there is no reason to stand up for the first or second generalization, because both definitions can be proven to be essentially reducible to the examination of probability defined on the entire basic Lindenbaum algebra.

The concept of inference is one of the basic concepts of the logic. *Every consistent theory induces a concept of relative consequence*, with the following semantical approach; regarding of the concept of the corresponding consequence only those models are to be taken into account which are already the models of the corresponding theory. The concept of the non-relative consequence can be considered as one induced by the trivial theory of the logically deducible formulae, i.e. the tautology. As the probabilistic generalization of the foregoing let us consider the probability on relative Lindenbaum algebra, generalizing a theory. The *probability of inference by this theory* i.e. the probability of "ψ implies φ"; denoted by $c(\varphi, \psi)$ can be measured by the number $m(\varphi\wedge\psi)/m(\psi)$, assuming $m(\psi) \neq 0$ (where the values m assigned directly to the formulae are defined by the probability defined on the relative Lindenbaum algebra).

This is equivalent to the definition of the conditional probability well known from the theory of probability — we shall go back to its logical interpretation. For the latter the classical logical semantical approach will be more appropriate, completed by the analysis of the relationship

$$m(\varphi\wedge\psi) = c(\varphi, \psi) \cdot m(\psi) ,$$

the well-known principle of multiplication, which contains the conditional probability but avoids the division.

(*Remark:* The concept of inference by probability can be defined on an arbitrary set of formulae.)

It is known that a *consistent theory defines a concept of consequence* and *vice versa* [21; 7.9], i.e. a binary relation defined on all formulae, with appropriate properties (supplying the concept of relative consequence $\psi \overset{T}{\vdash} \varphi$) defines a theory, thus it defines a relative Lindenbaum algebra with respect to a theory just mentioned and an ordering on it.

Generalizing the foregoing, assigning a probability with prescribed properties (see [3]) to the "ψ implies φ" in the sense of the foregoing ordering for every pair φ, ψ (except if ψ is a theorem) a real correspondence is created between the pairs of elements of the relative Boolean algebra defined above and the real interval [0, 1] — *which generalizes the concept of consequence, and indirectly*, also an ordinary probability defined on the Boolean algebra is created — generalizing the concept of theory.

We remind the reader of the introduction of the conditional probability fields, well known from the theory of probability; which at the same time defines a non-conditional probability field [4]. In a certain sense not yet defined by us, these ordinary probability fields can be considered as a probabilistically generalized theory; generated by the concept of conditional probability fields, which is a certain concept of probabilistic consequence. This analogy together with analogies above provides an intuitive argument for the naturality of such a building up of the "probability logic" in which besides relying on the results of the first-order logic, *a probabilistically generalized concept of consequence is considered as a start; i.e. inference by probability is its basic concept* (assigning a given m to the formulae means *to infer by probability m.*) It is the contrary of the procedure outlined first, based on the concept of the logical theory. Therefore the basic concept of the inference by probability is the function defined on the ordered pairs of the elements of the relative algebra, of a range of intervals [0, 1], with usually five requirements [4].

Remark: The latter construction historically preceded the former one; it is due to Carnap [3], who introduced the conditional probability discussed above on the "relative" algebra with respect to the trivial theory and called it *logical probability*: the descriptive probability function was referred to as *confirmation function* and his theory as *the theory of inductive logic.* Perhaps it is not by accident that he, as a philosopher, has chosen this construction for his theory, emphasizing the basic property of the concept of probability to be essentially conditional.

The construction outlined above is due to Gaifman, Scott, Krauss, among others [10], [24], [16].

In the ordinary logic, certain logically consistent sentences determine uniquely the deducibility of certain formulae and the undecidability of others in the given system of axioms — i.e. these sentences uniquely determine a theory. However, if concrete probabilities are given on certain formulae (i.e. on classes of equivalent formulae), where the probabilities satisfy the rules of probability, then a positive result on the set of those formulae for which the probability can be extended uniquely, is known only in special cases. For example: if a probability is defined on every quantifier-free sentence, then it can be extended — under certain restrictions — to the set of sentences with quantifiers.

To get concrete probability values for a well-manageable and very big set of the formulas, of course there is another possibility (besides the first possibility, just mentioned): to put further requirement on the concept of probability. Such a requirement for example is as follows: For a language with identity and for all different individual symbols a and b it is a requirement that the probability of formula $a = b$ be 0. (*Probability with strict identity.*) The larger the set of the requirements on probability, the narrower the class satisfying them. A sufficiently high number of requirements produces *a concrete probability for every formula.* In other words: *We can infer by a concrete probability* (we can use a concrete "inductive method").

Remark: It was also Carnap, who, following the lines above, enounced those probabilistic axioms under which the well-defined special distribution-family (called *continuum of inductive methods*) is applicable.

By the way, perhaps the highlight of Carnap's whole work is the following: While the theory of probability had long been surrounded by a bit of mystery, even today it is often not accepted as pure mathematics, only as applied mathematics, together with heuristics). Carnap's construction consistently endeavours to maximal exactness of mathematical logic. Carnap not only restored the theory of probability as a pure science (this had already been done) but he raised it to the rank of a logical theory (the term "inductive logic" partly refers to this fact) centered around the concept of "inference by probability", the so-called "inductive inference".

2. An approach from the view point of the classical semantics

Let us approach the concepts outlined above from the side of the semantical system rather than from the syntactical one of the classical logic.

Suppose to have a general first-order language. In the classical logic we are interested (among others) in the following question: Considering a fixed model, where certain formulae (axioms and hypotheses) are true, do they imply the truth of another prefixed formula?

More precisely: Is the conclusion true *in every model* where the hypotheses are true? In other words: Is the conclusion (or negation) on the hypotheses valid or not?

The concept of the probability assigned to the formulae by classical semantical way will generalize the classical concept of inference. For example: in the case of closed formulae if neither the conclusion nor its negation are implied by the hypotheses, our concept provides the "probability" of the inference, namely: the degree by which "the hypotheses are true" implies "the conclusion (or its negation) is true". In other words: *choosing a model at random from models where the hypotheses are true, what is the probability that the conclusion is true?*

Examine the correctness of this description and for a while forget about the syntactical approach to the probability (used till now) assigned to the formulae.

Consider a consistent set of sentences Δ and all its possible models.

The measure (probability) assigned to each closed formula by (classical) semantical way is meant as real assignment of range $[0, 1]$ on the set of the closed formulae with the following requirements. (Since there is a one to one natural correspondence between certain classes of models and the formulae, it is sufficient to define our assignment on these classes of models.)

Measure 1 is assigned to the class of models Δ (i.e. also to formulae valid in this class of models); the sum of the measures to disjoint classes of models (i.e. the sum of the measures of the disjunction terms to the disjunction of logically inconsistent sentences with respect to Δ). Further, if the class of model of a sentence contains that of another sentence (with respect to Δ) then the probability of the first sentence is greater than or equal to that of the second one. (This property is already implied by the former requirements.)

Since these classes of models of the closed formulae form *a class* (or set) *algebra* on the class of models Δ (see below), *the measure above assigned to these classes of models or formulae can be considered as defined by the probability defined on this class algebra.*

Our definition is in accordance with the fact (also referring to the syntactical approach to the probability) that the class algebra above *is isomorphic to the relative Lindenbaum algebra of all sentences with respect to Δ*. Therefore the semantical approach to the concept of probability assigned to the formulae is on accordance with the definition of the probability above built upon the syntactical system.

The semantical interpretation of the probability assigned to the closed formulae can be approached even in a more general and very important way.

We start from *an arbitrary class of models* among those corresponding to the type of a given language (i.e. our model class may differ from the class of models of a given consistent set of sentences).

Probability assigned to closed formulae is heuristically meant as the probability of choosing from these classes of models at random a model which satisfies the given formula.[3] One can give an exact definition in a way similar

[3] To use a terminology closer to the theory of probability, if the given model class represents the "possible, realizable models", and our model results at random, by what probability we can consider the corresponding formula as true in advance.

to the case above (when the set of models Δ was the basic class of models). Here we have to take into account that those subclasses of the given class of models which consist of models satisfying the corresponding sentence of the language also form a class algebra. In the latter case *the relative Lindenbaum algebra of all sentences of the language with respect to the theory of the given class of models* is the "language-image" of the foregoing class algebra (see: also below).

(Remark: In Section 1 probability was assigned to formulae in connection with the syntactical concept of theory. Its semantical equivalent is as follows:

The theory of a class of models is known to mean the set of sentences which are true for each element of this class of models; i.e. a theory induces a correspondence between a subset of the formulae and $\{0, 1\}$. As a probabilistic generalization of the semantical concept of theory a number is assigned to *every sentence*; again 1 to the formulae belonging to the theory of the model class, 0 to their negations, and an arbitrary probability to the remaining sentences. As already mentioned because of isomorphism, this correspondence can be considered as a probability distribution on the relative Lindenbaum algebra with respect to the given theory — in accordance with the syntactical generalization of the concept of theory.)

Continuing the ordinary logical semantical approach to the probability assigned to the formulae *the generalization of the concept of inference* can be interpreted as a real number $c(\varphi, \psi)$ expressing the probability of choosing a model at random satisfying φ from the models satisfying ψ. Its numerical relation ($c(\varphi, \psi)$) to the ordinary probability m introduced in this Section can be explained easily on the relation $m(\varphi \wedge \psi) = c(\varphi, \psi) \cdot m(\psi)$.

Indeed, the problem of choosing a model satisfying both ψ and φ from a given class of models is solved by checking first the truth of ψ; then that of φ, when ψ was true. The left-hand side of our equation is the probability of the success of the simultaneous check, the right-hand side is the product of the probability of the success of the first check by the conditional success of the second one. Therefore $c(\varphi, \psi)$ can be defined as in Section 1.

(*Remark*: One can also explain by semantical way the fact that while $\psi \models \varphi$ and $\models \psi \to \varphi$ are equivalent in the ordinary logic $c(\varphi, \psi)$ generally differs from $m(\psi \to \varphi)$.

Until now the meaning of the probability assigned to the closed formulae has been dealt with. Probability was, however, also assigned to the open formulae through the syntactical system, hence it calls for a semantical explanation. The following heuristic explanation has an exact form, too [18]:

Let us interpret *the probability assigned to an open formula as the probability of choosing a model satisfying a formula if it was chosen from a prefixed class of models and the valuations of the individual variables were chosen at random in this model.*

Thus, while for closed formulae it is sufficient to define the probability on a class algebra corresponding to the prefixed set of sentences, for open formula we need yet define for each model, the probability of valuation of individual variables satisfying the given formula has still to be defined. The latter is a probability family. Namely, it holds (see [18] for details), that for every probability distribution defined on the open formulae, there exists a probability family such that, jointly with the restriction of the original probability to the closed formulae, it determines uniquely the extension to the open formulae. Its reverse also holds: The probability defined on the closed formulae and the probability family determine uniquely an extension to the open formulae. (That is, the probability assigned to the open formulae may in fact have the following meaning: What is the probability of receiving a valuation satisfying a formula by choosing a model from the prefixed class of models at random by a distribution which extends the distribution defined on the closed formulae and choosing the valuation of the individual variables on a model at random by another distribution?)

One can give the "semantical images" of the entire Lindenbaum algebras (containing the open formulae too) — in parallel to the relation between the Lindenbaum algebras of the closed formulae and the class algebras of the corresponding classes of models [21].

3. Probability logic

This Section outlines the construction and interpretation of the probabilistic generalization of the ordinary first-order logic as a logical system.

Our aim was to build upon the classical logic. Section 1 outlined the generalization of a few concepts of the syntactical system of the classical logic in a relative form with respect to a theory, referring to the corresponding concepts of the rule of inference, theory, concept of consequence, to the probabilistic analogue on the fact that a relative logic can be build up on the concept of the theory as well as on the concept of the relative consequence, involving the following question: Is there an analogue of the axiomatizability of a theory, or of the fact that certain consistent assertions uniquely determine a theory in the classical logic?

The following remark is closely connected with the latter question: while the deducibility of assertions determined the deducibility of sentential combinations in the classical logic, the probabilities of assertions do not determine in general the probabilities of the sentential combinations in the probabilistic logic. There exists, however, a simple condition under which the probabilities of all the quantifier-free closed formulae determine the probabilities of the closed formulae with quantifier, and the determining relation is simple, too. It is also possible to state interesting relationships between the probabilities of closed and of open formulae [10].

The correspondence between a problem given in the theory of probability in common parlance and an exact logical theorem (with its hypotheses and consequence) in the theory of inference by probability can be illustrated by an example [3] using the concepts of the probabilistic logic introduced above.

Section 2 referred to the classic semantical background of the probabilistic generalizations of the concepts of the classic syntactical system mentioned just above (but not to the probabilistic logical semantics to be discussed below).

Now, *the semantical part of the probabilistic logic* will be considered. The simplest way to obtain the concept of the so called "probability model" (in a different terminology: probability system) is to generalize the Boolean algebra models [15] with identity. The "relations" in this latter are known to be defined as functions (of finite arguments) from the universum into the complete Boolean algebra. (Of course, the corresponding Boolean algebra may vary from model to model.)

If the Boolean algebra representing the range of the truth values is the two-element algebra, we are back at the concept of the ordinary relational system.

Now the probabilistic generalization is as follows: Probability on the Boolean algebras representing the truth values is defined. The Boolean models thus generalized will be referred to as "*probability models*". The valuation of the formulae of the ordinary first-order language on these "probability models" is perfectly similar to the valuations in Boolean models (relying on Tarski's idea) except the following modification: now real numbers are assigned to the formulae by the probabilities assigned to the Boolean elements. To explain these definitions and to answer the question: — *Why to assign real numbers to the formulae probability?* — We mention the following arguments: In the case of an infinite language a probability model in fact induces a probability on the basic Lindenbaum algebra of the language extended with symbols corresponding to the elements of the basic set of the model [4]. In this connection, the probability model and the valuation seem to be quite natural probabilistic generalizations of the classical relational systems. One can, however, construct a model and a concept of valuation well known from the so-called Continuous Model Theory [5]: Also this construction assigns real numbers between zero and one to the formulae of the first-order predicate language, and the logic (defined by this assignment) is also the generalization of the first order one; nonetheless the real assignment to these formulae has nothing to do with any concept of probability.

[4] σ-additivity from the Boolean algebra of the range and positivity from the probability are required — which jointly provide the completeness of the Boolean algebra.

Thus the reference to the attribute "probability" of the model above is correct as a first approximation.

In the ordinary logic a set of consistent sentences always has a model, moreover, if it has an infinite model then it has a countable one (Löwenheim—Skolem theorem).

Its equivalent also holds in the probabilistic logic. The theorem will be generalized in the form concerning consistent theories. The probability on the basic Lindenbaum algebra will be considered rather than that on the relative Lindenbaum algebra. (The latter is the probabilistic generalization of the concept of the theory.) As mentioned, this replacement is feasible.

Let us state the following : *A probability model is a model of the probability defined on the basic Lindenbaum algebra of the language* if the probabilities on the former and the latter are the same on the closed formulae of the language.

Now we have the following *theorem*: Every probability defined on the formulae of the language has a probability model, moreover it always has a countable model, too. It has even such special models as a symmetrical model or one which has a strict identity [24].

The last sentence involved two concepts:

A probability model is referred to as *symmetrical* if the probability of the formulae are "invariant to all the permutations of the constants of the language".

A model is referred to as *having strict identity* if it assigns zero probability to the formula $a = b$ provided $a \neq b$.

Otherwise, the syntactical equivalents of these specifications have already been tackled with.

Remark 1: A $0-1$ measure defined by an ordinary complete and consistent theory has a symmetrical probability model and another one having a strict identity.

Remark 2: For the so-called infinite languages it does not hold that the probability defined on a basic Lindenbaum algebra always has a probability model.

Similarly to considering an ordinary relational system as a special probability model (looking for their place in the set of all the probability models), *other important probability models of special types may also be studied.* While the probability is maximally concentrated on the numbers 0 and 1 for the classical relational systems it is diametrically opposite for the symmetrical systems mentioned above, for which the probability is distributed very uniformly on the formulae.

Concerning the systems having strict identity only the following remark is made: There are a lot of very important probability system (e.g. the equivalents of the ultraproducts) which have not strict identities. Consequently, "to have strict identity" may be considered as an individual property.

As for as the most fundamental semantical constructions of the model theory are concerned, the theory has succeeded in finding their *probabilistic logical equivalents* for their majority, but did not succeed for minority. For example: one can give the probabilistic logical equivalents of the following concepts: *submodel, elementary submodel, elementary equivalence, ultraproduct,* but it is impossible for the *direct product* to give a counter-example.

Considering the corresponding concepts of the concepts of classical syntactical systems in the probabilistic logic as outlined above, the concept of axiom, theory, relative consequence (of two argument) are seen to have been, by an essential degree, though retaining the usual meaning of the formula.

Next step in the further probabilistic generalization of the ordinary logic (relying on the previous results) is to introduce *a generalized concept of the formula, the so-called "probability assertion"*. This step is considered as final in the generalizations of this paper. In this generalization it was aimed at getting such a logic *whose general status is even more close to that of the classical one.*

As a matter of fact, the concept of "probability assertion" corresponds to that of the formula of the sentential logic. The introduction of this concept of the formula is analogous to the procedure by which the syntactics in the Boolean models can be considered (among others) as means to symbolize the operations on the Boolean algebras representing the range of the truth values hence, to manipulate the Boolean elements and the truth values in this sense. *Similarly to the syntactics of every logical system*, aiming at describing a definite discipline, for the probability models the truth values are probabilities, i.e. real numbers between 0 and 1. Therefore a (*n* auxiliary) language appropriate for the manipulation of the real numbers is needed. By definition, *a "probability assertion" of a first order language* is meant as $(N+1)$-tuple $\vartheta = \langle \psi, \varphi_1, \varphi_2, \ldots \varphi_N \rangle$ where ψ is a quantifier-free formula with N free variables of a so-called algebraic language, and $\varphi_1, \varphi_2, \ldots, \varphi_N$ are arbitrary closed formulae of the original language $\mathfrak{S}$.[5] (An *algebraic language* contains countable individual variable, the binary predicate constant $\leq$, the individual constants $0, +1, -1$, the function symbols of two-argument $+$ and $-$, and the logical constants $\wedge, \vee, \neg, \forall, \exists$.)

[5] Thus this definition is not a recursive one.

The assertion ϑ *is said to hold on a probability model* (of an appropriate type) if the N-tuple $m(\psi_1), m(\psi_2), \ldots m(\psi_N)$ satisfies ψ on the ordinary model $\langle \mathrm{Re}, \leq, +, \cdot, 0, +1, -1 \rangle$ (where m denotes the probability defined on the given model and Re is the set of the real numbers).

Now the corresponding semantical concepts are ready to be introduced in the logic belonging to this probability model. For example: we say that a probability assertion ϑ *is a consequence of a set of probability assertions* Σ; if ϑ holds on every such probability model where all the elements of Σ hold. (Thus, a terminology of this logic has been created, close to that of the classical one; the "formulae" are merely true or not in a model like in the ordinary logic, in contrary to the probability generalizations of the ordinary logic up to now, where the formulae in their ordinary meaning hold only with a certain probability. Thus one can construct further terminologies similar to that of the semantics of the ordinary logic.)

Contrary to the semantics, of the logic discussed in this Section it is somewhat difficult to find the properties corresponding to the tautology or the deducibility, though an experiment has been made to solve this problem [24].

Scott and Krauss [24] also provide the answer to the question: What is the connection between the semantics and syntactics outlined above and what are mathematical devices applicable for its examination.

Summary

A generalization of the classic first-order logic "into probabilistic direction" with the devices of modern mathematics is presented. The generalization of the concepts belonging to the classical model theory has only recourse to exact mathematical tools.

The discussion involves the possibilities of the probabilistic generalizations of the axiomatic method, and the concept of logical theory with probability, the equivalents of rules of inference in such a theory; the concepts of "inference by probability" and "inference by a concrete probability", the meaning of "inference in a random model"; what is the background of the conditional probability in the "mathematical logic": the construction of an exact logic discipline on such concepts analogous to the theory of probability.

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