CALCULATION OF GUIDED WAVES BY EXPANSION IN POWERS OF THE FREQUENCY

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Introduction

Guided waves propagating in an inhomogeneous dielectric and varying sinusoidally with time are treated in this paper. Both the dielectric and the conductor bounding the wave field are supposed to be lossless. Waveguides are investigated in which neither the geometry of the arrangement nor the properties of the media vary in the direction of the propagation and the permittivity and permeability of the dielectric are constant, but different in each region of the cross section.

The functions describing the propagation coefficient and the field strengths are obtained as series in the frequency. STEVENSON ([1], [2]) applied a similar method to the solution of scattering problems. As against the problem treated by Stevenson the determination of the propagation coefficient of waveguides requires the solution of an eigenvalue problem, the application of the method is much more complicated here than in scattering problems.

From among the so-called quasi-TE, TM and TEM modes appearing in waveguides only the quasi-TE and TM modes are treated in this paper. Quasi-TEM mode can only appear if in addition to the bounding conductor there is another conductor in the wave field, e.g. at microstrip transmission lines. For this reason the method of calculating the quasi-TEM mode differs considerably from that of the other modes, exceeding the scope of this paper.

The equations of the eigenvalue problem and the boundary conditions

Let the z axis of the co-ordinate system be parallel to the direction of the propagation and let the unit vector in this direction be denoted by **k**. The complex value of the electric and magnetic field in the m-th region of the cross section A_m, where the permittivity and the permeability are the constant values ε_m and μ_m, resp.,(see Fig. 1) can be written as:

$$\mathbf{E}_m = (\mathbf{e}_{Tm} + \mathbf{e}_{zm}) \exp(-pz) \tag{1}$$

$$\mathbf{H}_m = (\mathbf{h}_{Tm} + \mathbf{h}_{zm}) \exp(-pz)\,. \tag{2}$$

Here p denotes the propagation coefficient, the vectors $\mathbf{e}_{Tm}$ and $\mathbf{h}_{Tm}$ are perpendicular to the z axis, the vectors $\mathbf{e}_{zm}$ and $\mathbf{h}_{zm}$ are parallel to it, and all the four vectors depend only on the two co-ordinates in the cross section. The three other vectors can be expressed in terms of the vector $\mathbf{e}_{Tm}$ as:

$$\mathbf{e}_{zm} = \frac{\mathbf{k}}{p} \operatorname{div} \mathbf{e}_{Tm}$$

$$\mathbf{h}_{zm} = -\frac{1}{j\omega\mu_m} \operatorname{curl} \mathbf{e}_{Tm} \tag{4}$$

$$\mathbf{h}_{Tm} = \frac{1}{j\omega\mu_m} \mathbf{k} \times \left(p\mathbf{e}_{Tm} + \frac{1}{p} \operatorname{grad} \operatorname{div} \mathbf{e}_{Tm} \right) =$$

$$= \frac{\mathbf{k}}{p} \times \left(j\omega\varepsilon_m \mathbf{e}_{Tm} + \frac{1}{j\omega\mu_m} \operatorname{curl} \operatorname{curl} \mathbf{e}_{Tm} \right) . \tag{5}$$

The vector $\mathbf{e}_{Tm}$ itself can be determined from the vectorial wave equation

$$\Delta \mathbf{e}_{Tm} + (p^2 + \varepsilon_m\mu_m\omega^2)\, \mathbf{e}_{Tm} = 0 \tag{6}$$

by considering the boundary conditions. Along the surface of the perfect conductor bounding the wave field the elctric field is normal to it, and along the surfaces dividing the dielectric into parts with different properties of media the tangential component of both the electric and the magnetic field are equal on the two sides of the surface. In consideration of Eqs (3) to (5), along the contour of the perfect conductor

$$\mathbf{n}_m \times \mathbf{e}_{Tm} = 0 \tag{7}$$

$$\operatorname{div} \mathbf{e}_{Tm} = 0 , \tag{8}$$

and along the contour separating the m-th and k-th region of the dielectric

$$\mathbf{n}_{mk} \times \mathbf{e}_{Tm} = \mathbf{n}_{mk} \times \mathbf{e}_{Tk} \tag{9}$$

$$\varepsilon_m\, \mathbf{n}_{mk}\, \mathbf{e}_{Tm} = \varepsilon_k\, \mathbf{n}_{mk}\, \mathbf{e}_{Tk} \tag{10}$$

$$\operatorname{div} \mathbf{e}_{Tm} = \operatorname{div} \mathbf{e}_{Tk} \tag{11}$$

$$\frac{1}{\mu_m} \operatorname{curl} \mathbf{e}_{Tm} = \frac{1}{\mu_k} \operatorname{curl} \mathbf{e}_{Tk} . \tag{12}$$

In these relationsships $\mathbf{n}_m$ and $\mathbf{n}_{mk}$ denote a unit vector normal to the contour and pointing inside the mth region (see Fig. 1).

Equation (6) and boundary conditions (7) to (12) define an eigenvalue problem concerning the vectorial function $\mathbf{e}_{Tm}$, in which the square of the angular frequency ω acts as parameter. The functions $p^2(\omega^2)$ and $\mathbf{e}_{Tm}(\mathbf{r},\omega^2)$ are obtained as solution of this eigenvalue problem. For the total uniqueness of the vectors $\mathbf{e}_{Tm}(\mathbf{r}, \omega^2)$ a quantity defining the intensity of the wave, e.g. the transferred power must be given also as a function of frequency.

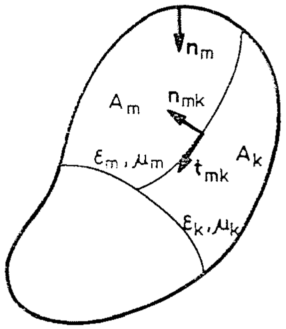

Fig. 1

It will be shown how the Taylor's series in ω^2 of the functions $p^2(\omega^2)$ and $\mathbf{e}_{Tm}(\mathbf{r}, \omega^2)$ can be determined, the square of the cut-off frequency ω_0 being chosen as the centre of expansion. The presented method can be generalized so that the centre of the expansion is an arbitrary value, but the calculation is simpler if the centre is at ω_0^2. Namely, every mode turns into TM, TE or TEM mode at the cut-off frequency [3]. The dimensionless quantity

$$w = \frac{\omega L}{c} \tag{13}$$

is suitably introduced for the expansion, where c is light velocity in vacuum, and L is a constant of length dimension, which can be identified most simply by some characteristics geometry of the waveguide. After introducing the quantity

$$w_0 = \frac{\omega_0 L}{c} \tag{14}$$

connected with the cut-off frequency (to be determined later) the Taylor's series are suitably written in the form:

$$p^2 = L^{-2} \sum_{i=1}^{\infty} a_i (w^2 - w_0^2)^i \tag{15}$$

$$\mathbf{e}_{Tm} = \sum_{i=0}^{\infty} \mathbf{e}_{mi} (w^2 - w_0^2)^i \tag{16}$$

Here the vectors $\mathbf{e}_{mi}$ depend only on the two cross-sectional co-ordinates.

If the above series and the relationship $\omega = wc/L$ are substituted into Eq. (6), from the coefficients of the powers of $(w^2 - w_0^2)$ the following equations arise:

$$\Delta \mathbf{e}_{m0} + w_0^2 L^{-2} \varepsilon_{rm} \mu_{rm} \mathbf{e}_{m0} = 0 \tag{17}$$

$$\Delta \mathbf{e}_{mi} + w_0^2 L^{-2} \varepsilon_{rm} \mu_{rm} \mathbf{e}_{mi} = -L^{-2} \left(\varepsilon_{rm} \mu_{rm} \mathbf{e}_{m,i-1} + \sum_{j=1}^{i} a_j \mathbf{e}_{m,i-j} \right)$$
$$i = 1, 2, \ldots \tag{18}$$

where ε_{rm} and μ_{rm} denote the relative permittivity and permeability, resp., valid in the region A_m. As the vectors $\mathbf{e}_{Tm}$ must satisfy the boundary conditions at all frequencies, each vector $\mathbf{e}_{mi}$ is required to satisfy separately the boundary conditions (7) to (12).

Eq. (17) and the boundary conditions define an eigenvalue problem, the solution of which gives the functions $\mathbf{e}_{mi}$ and the eigenvalue w_0^2, and the cut-off frequency is also obtained from the latter. In knowledge of the functions $\mathbf{e}_{m0}$ and the eigenvalue w_0^2 the functions $\mathbf{e}_{mi}$ can be determined successively from Eq. (18) and the boundary conditions by recursion procedure. In calculations first the divergence and the curl of the functions $\mathbf{e}_{mi}$ are possibly determined and from these the functions $\mathbf{e}_{mi}$. Introducing the notations

$$u_{mi} = \operatorname{div} \mathbf{e}_{mi} \tag{19}$$

$$v_{mi} \mathbf{k} = \operatorname{curl} \mathbf{e}_{mi} \tag{20}$$

from Eqs (17) and (18) the vectors $\mathbf{e}_{mi}$ become:

$$\mathbf{e}_{m0} = \frac{L^2}{w_0^2 \varepsilon_{rm} \mu_{rm}} (\operatorname{grad} v_{m0} \times \mathbf{k} - \operatorname{grad} u_{mo}) \tag{21}$$

$$\mathbf{e}_{mi} = \frac{1}{w_0^2} \left[\frac{1}{\varepsilon_{rm} \mu_{rm}} \left(L^2 \operatorname{grad} v_{mi} \times \mathbf{k} - L^2 \operatorname{grad} u_{mi} - \sum_{j=1}^{i} a_j \mathbf{e}_{m,i-j} \right) - \mathbf{e}_{m,i-1} \right]$$
$$i = 1, 2, \ldots \tag{22}$$

Taking the divergence and curl of both sides of Eqs (17) and (18) yields for the functions u_{mi} and v_{mi}:

$$\Delta u_{m0} + w_0^2 L^{-2} \varepsilon_{rm} \mu_{rm} u_{m0} = 0 \tag{23}$$

$$\Delta v_{m0} + w_0^2 L^{-2} \varepsilon_{rm} \mu_{rm} v_{m0} = 0 \tag{24}$$

$$\Delta u_{mi} + w_0^2 L^{-2} \varepsilon_{rm} \mu_{rm} u_{mi} = -L^{-2} \left(\varepsilon_{rm} \mu_{rm} u_{m,i-1} + \sum_{j=1}^{i} a_j u_{m,i-j} \right) \tag{25}$$

$$\Delta v_{mi} + w_0^2 L^{-2} \varepsilon_{rm} \mu_{rm} v_{mi} = -L^{-2} \left(\varepsilon_{rm} \mu_{rm} v_{m,i-1} + \sum_{j=1}^{i} a_j u_{m,i-j} \right) . \tag{26}$$

According to Eq. (8), along the contour of the perfect conductors

$$u_{mi} = 0 . \tag{27}$$

Taking this and Eqs (21) and (22) into account, according to the boundary condition (7), along the contour of the perfect conductors

$$\frac{\partial v_{mi}}{\partial n_m} = 0 . \tag{28}$$

According to Eqs (11) and (12), along the contour separating the regions A_m and A_k of the dielectric:

$$u_{mi} = u_{ki} \tag{29}$$

$$\frac{1}{\mu_m} v_{mi} = \frac{1}{\mu_k} v_{ki} \tag{30}$$

Taking Eqs. (21), (22), (29) and (30) into account, from (9) and (10) along the same contour boundary conditions will be:

$$\frac{1}{\mu_m}\left(L^2 \frac{\partial u_{mi}}{\partial n_{mk}} + \sum_{j=1}^{i} a_j \, \mathbf{n}_{mk} \, \mathbf{e}_{m,i-j}\right) = \frac{1}{\mu_k}\left(L^2 \frac{\partial u_{ki}}{\partial n_{mk}} + \sum_{j=1}^{i} a_j \, \mathbf{n}_{mk} \, \mathbf{e}_{k,i-j}\right) \tag{31}$$

$$\begin{aligned} &\frac{1}{\varepsilon_m \mu_m}\left(L^2 \frac{\partial v_{mi}}{\partial n_{mk}} + L^2 \frac{\partial u_{mi}}{\partial t_{mk}} + \sum_{j=1}^{i} a_j \, \mathbf{t}_{mk} \, \mathbf{e}_{m,i-j}\right) = \\ &= \frac{1}{\varepsilon_k \mu_k}\left(L^2 \frac{\partial v_{ki}}{\partial n_{mk}} + L^2 \frac{\partial u_{mi}}{\partial t_{mk}} + \sum_{j=1}^{i} a_j \, \mathbf{t}_{mk} \, \mathbf{e}_{k,i-j}\right), \end{aligned} \tag{32}$$

introducing notation $\mathbf{t}_{mk} = \mathbf{k} \times \mathbf{n}_{mk}$ (see Fig. 1).

It follows from the symmetry of Maxwell's equations that equations entirely similar to the previous ones can be written on the basis of the magnetic field instead of the electric field. These equations, excepted the equivalents of Eqs (7), (8), (27) and (28) result from the previous ones by interchanging the letter symbols:

$$\mathbf{e}_{Tm} \leftrightarrow \mathbf{h}_{Tm} \qquad \mathbf{e}_{zm} \leftrightarrow \mathbf{h}_{zm} \qquad \mathbf{e}_{mi} \leftrightarrow \mathbf{h}_{mi} \tag{33}$$
$$\varepsilon_m \leftrightarrow -\mu_m$$

While the electric field is normal to the surface of the perfect conductor, the magnetic field is parallel to it. So (7), (8), (27) and (28) will be replaced by the following boundary conditions along the contour of the perfect conductor:

$$\mathbf{n}_m \, \mathbf{h}_{Tm} = 0 \tag{34}$$

$$\operatorname{curl} \, \mathbf{h}_{Tm} = 0 \tag{35}$$

$$\frac{\partial u_{mi}}{\partial n_m} = 0 \tag{36}$$

$$v_{mi} = 0 \tag{37}$$

Determination of the cut-off frequency and the different modes

Quasi-TM modes

Equation (23) and the boundary conditions (27), (29) and (31) define a boundary value problem yielding the function u_{m0} and the eigenvalue w_0^2

and the cut-off frequency ω_0. It is proved in Appendix I that the eigenvalues ω_0^2 are non-negative real, as it could be expected.

For the sake of clearness the further treatment is restricted to the case of simple eigenvalue, though the presented method can be generalized with no difficulty to the case of multiple eigenvalue. (Multiple eigenvalue occours, if the cut-off frequencies of two or more quasi-TM modes happen to have the same value.) If the eigenvalue is simple, the functions u_{m0} are defined unequivocally up to a constant factor. This factor can be chosen arbitrarily, unless the power transferred by the waveguide is given as a function of the frequency. In knowledge of functions u_{m0} and eigenvalue w_0^2, functions v_{m0} can be determined as solution of the simple boundary value problem defined by Eq. (24) and the boundary conditions (28), (30) and (32). Thereafter vectors $\mathbf{e}_{m0}$ can be calculated by means of (21). If the product $\varepsilon\mu$ is constant along the whole cross section, then $v_{m0} = 0$, i.e. the vector $\mathbf{e}_{m0}$ is seen to be irrational.

Unless functions u_{m0} are identically zero, vector $\mathbf{h}_{Tm}$ grows beyond any bound according to (5) when the frequency tends to the cut-off frequency ω_0, because the propagation coefficient converges to zero. But $\mathbf{h}_{zm}$ remains bounded according to (4), thus the magnetic field is perpendicular to the z axis at the cut-off frequency. On the other hand it results from Eq. (3) that the electric field is parallel to the z axis at the cut-off frequency. Accordingly these modes can be identified with the so-called quasi-TM (qTM) modes.

Quasi-TE modes

The equation and the boundary conditions relative to functions u_{m0} are satisfied by the function idetically zero. For these modes the eigenvalue w_0^2 can be determined from the eigenvalue problem relative to the functions v_{m0} defined by Eq. (24) and boundary conditions (28), (30) and (32). By analogy to the idea in Appendix I, the eigenvalues w_0^2 of this boundary value problem can be proved to be all non-negative real.

If $u_{m0} = 0$, div $\mathbf{e}_{Tm}$ converges to zero in order of magnitude of $(\omega^2 - \omega_0^2)$ and p in order of magnitude of $\sqrt{\omega^2 - \omega_0^2}$, when the frequency tends to the cut-off frequency. $\mathbf{e}_{zm}$ converges to zero according to (3), thus the electric field is perpendicular to the z axis at the cut-off frequency. On the other hand, according to Eqs (4) and (5), the magnetic field is parallel to the z axis at the cut-off frequency. Accordingly these modes can be identified with the so-called quasi-TE (qTE) modes.

The qTE modes are simpler to calculate on the basis of vectors $\mathbf{h}_{Tm}$ than $\mathbf{e}_{Tm}$. Then the eigenvalue w_0^2 has to be determined from the eigenvalue problem relative to vectors $\mathbf{h}_{m0}$, using equations derived and calculation process as described above.

Quasi-TEM modes

The equations and boundary conditions relative to functions u_{m0} and v_{m0} are satisfied by the function identically zero, i.e. modes are possible for wich

$$\operatorname{div} \mathbf{e}_{m0} = 0 \tag{38}$$

$$\operatorname{curl} \mathbf{e}_{m0} = 0 \tag{39}$$

According to Eq. (17) the value w_0^2 and so the cut-off frequency must be zero because else the vector $\mathbf{e}_{m0}$ would be zero. Vector $\mathbf{e}_{m0}$ has not to be determined now from an eigenvalue problem, but from a simple boundary value problem defined by Eqs (38) and (39) and the proper boundary conditions. This problem has the trivial solution $\mathbf{e}_{m0} = 0$ only if the cross section of the dielectric is a singly connected region. So now the modes investigated can only occur in arrangements which contain further conductors in addition to that bounding the wave field.

It can be proved by means of Eqs (3) to (5) that in these modes both the electric and the magnetic field are perpendicular to the z axis at zero frequency, which is at the same time the cut-off frequency. Accordingly they can be identified with the so-called quasi-TEM (qTEM) modes. Apparently their calculation differs from that of the other modes, this is why they will be discussed in a separate paper.

Calculation of qTM modes

It has been presented so far how the cut-off frequency ω_0 and the vector functions $\mathbf{e}_{m0}$ can be determined. Now it will be discussed how coefficients of the series of p^2 and $\mathbf{e}_{Tm}$ can be computed for the qTM modes by recursion procedure starting from the functions $\mathbf{e}_{m0}$.

The i-th step of the recursion procedure involves determining functions u_{mi}, v_{mi} and $\mathbf{e}_{mi}$ and the coefficient a_i. The function u_{mi} can be computed from Eq. (25) and boundary conditions (27), (29) and (31). The homogeneous equivalent of this boundary value problem has a non-trivial solution by definition of the eigenvalue w_0^2. So according to Fredholm's alternative the equation given in Appendix II must be satisfied for the existence of a solution. The coefficient a_i can be computed from this equation. Introducing the notation

$$g_i = \sum_m \frac{1}{\mu_{rm}} \left[\int\limits_{A_m} u_{m0}\, u_{mi}\, \mathrm{d}A + \int\limits_{l'_m} u_{m0}\, \mathbf{n}_m\, \mathbf{e}_{mi}\, \mathrm{d}l \right] \tag{40}$$

where l'_m denotes the part of the boundary curve of the region A_m inside the dielectric, and $\mathbf{n}_m$ a unit vector normal to the curve l'_m and pointing inside the region A_m, coefficient a_i is, according to Appendix II:

$$a_i = -\frac{1}{g_0}\left[\sum_m \varepsilon_{rm} \int\limits_{A_m} u_{m0}\ u_{m,i-1}\,\mathrm{d}A + \sum_{j=1}^{i-1} a_j\, g_{i-j}\right] . \tag{41}$$

Of course the sum vanishes in the case $i = 1$.

With this value for the coefficient a_i Eq. (25) has a solution that satisfies the boundary conditions (27), (29) and (31), but this solution is not unique. If u^*_{mi} denotes a particular solution, the general solution has the form

$$u_{mi} = u^*_{mi} + Cu_{m0} \tag{42}$$

where C is an undefined constant. Its value can be chosen arbitrarily unless the power transferred by the wave guide is given as a function of the frequency.

Now by solving the simple boundary value problem in Eq. (26) and the boundary conditions (28), (30) and (32), functions v_{mi} can be determined. After this the vectors $\mathbf{e}_{mi}$ have to be computed by means of (22), ending the i-th step of the recursion procedure.

If the product $\varepsilon\mu$ is constant throughout the cross section, the relationship (41) yields for the coefficient a_1:

$$a_1 = -\varepsilon_r\,\mu_r\,. \tag{43}$$

The right-hand sides of Eqs (25) and (26) are seen to be identically zero for every value of i, so both functions u_{mi} and v_{mi} can be chosen identically zero, furthermore all the coefficients a_i but a_1 are zero. Thus, the square of the propagation coefficient:

$$p^2 = -\varepsilon\mu(\omega^2 - \omega_0^2)\,. \tag{44}$$

Calculation of qTE modes

The qTE modes can be calculated by recursion procedure as described if the calculation is based on the magnetic instead of the electric field strength. So, starting from functions $\mathbf{h}_{m0}$, the coefficients of the series of p^2 and $\mathbf{h}_{Tm}$ can be determined by recursion procedure.

The recursion procedure and its formulae are obtained from those for the qTM modes by interchanging the letter symbols according to (33). The only difference is to substitute boundary conditions (36) and (37) for (27) and (28) in determining functions u_{mi} and v_{mi}.

On the region of convergence of the series

The region of convergence of the series raises some theoretical questions. Unfortunately the results of this investigation permit to determine the numerical value of the radius of convergence in very simple cases alone.

The parameter ω^2 in Eq. (6) can be regarded as a complex variable. Then the eigenvalue problem described by the equation defines a function $p^2(\omega^2)$ of a complex variable with an infinity of values. The branches of the function are connected with several modes. The position of the branch points on the branch relative to the investigated mode determines the region of convergence of the series.

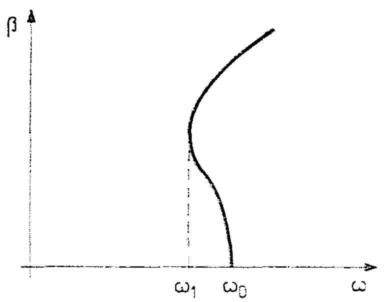

Fig. 2

The branch points on the real axis are related to the phenomenon of the backward waves. The dispersion curve of certain waveguides has the character shown in Fig. 2 (see e.g. [4], [5]). Dispersion curves of two different modes are really seen in the figure. The dispersion curve of one of these modes is sloping between frequencies ω_1 and ω_0, i.e. the phase velocity and the group velocity have opposite directions. This is the phenomenon of the backward waves. Over the frequency ω_0 the propagation coefficient is real (not seen in the figure), then it turns again purely imaginary with increasing frequency. The propagation coefficient of the other mode is purely imaginary over the frequency ω_1, and its absolute value increases with the frequency. Approaching the frequency ω_1 from above, both modes converge to the same wave pattern, i.e. two different solutions of the eigenvalue problem turn into a single solution at the frequency ω_1. With frequency decreasing below the value ω_1, two solutions appear again, with eigenvalues p^2 not purely real any more, and of course the two eigenvalues are conjugate to each other. Now the function $p^2(\omega^2)$ is seen to have a branch point at ω_1^2.

The function $p^2(\omega^2)$ has real branch points solely for certain arrangements, but complex branch points may occur in all arrangements. These determine the radius of convergence of the series, more exactly that one among the singular points lying on the branch of the function $p^2(\omega^2)$ related to the investigated mode which is nearest to the centre of the series development.

Examination of the derivative $dp^2/d\omega^2$ permits to locate the branch points. Appendix III shows this derivative to be expressible as:

$$\frac{dp^2}{d\omega^2} = -\frac{\sum_m \varepsilon_m \mu_m \int_{A_m} (\mathbf{e}_{Tm} \times \mathbf{h}_{Tm})\, d\mathbf{A}}{\sum_m \int_{A_m} (\mathbf{e}_{Tm} \times \mathbf{h}_{Tm})\, d\mathbf{A}} \,. \tag{45}$$

There is a branch point for the value ω^2 where the denominator in the right-hand side of (45) becomes zero, while the numerator is not zero. If also the numerator is zero, the limit value of the derivative must be examined. If such a limit value exists, the function is differentiable at the given point, and there is no branch point. Generally it is so if ω equals the cut-off frequency. If it could be determined where the denominator in the right-hand side of (45) becomes zero, determination of the radius of convergence would cause no problem. Unfortunately, except for very simple arrangements, finding these points seems to be unpromising.

If $\mathbf{e}_{Tm}$ and $\mathbf{h}_{Tm}$ are real, the power transferred by the waveguide stands in the denominator of (45). If the phenomenon of backward waves occurs, this power is negative because the direction of the energy flow is opposite to the direction of wave propagation, and the transferred power is zero at both boundary points of the range of frequency where the phenomenon occurs. One of the two boundary points is the cut-off frequency, it is, however, no branch point, as it was mentioned. But the other boundary point is a branch point. If $\mathbf{e}_{Tm}$ and $\mathbf{h}_{Tm}$ are real, the right side of (45) is the weighted arithmetic mean of the product $-\varepsilon\mu$, with the powers flowing in several layers of the dielectric as weighting factors.

On the application of the method

For simple arrangements the coefficients of the series can be analytically determined. As an illustration the case of the rectangular waveguide with inhomogeneous dielectric will be investigated in the last part of the paper. Cases imposing numerical evaluation will be treated in a subsequent paper. Here only a rough comparison will be presented between the usual numerical method to determine the dispersion curve and the method described in this paper.

First, let us consider how the phase factor can be determined at a given frequency by the usual method. On this purpose let the eigenvalue problem defined by Eq. (6) or an equivalent problem relative to a two-dimensional vector be solved. If this eigenvalue problem requires to discretize the differential equations, and on this purpose N grid-points are chosen in the cross section, the eigenvalue relative to the investigated mode has to be selected among the eigenvalues of a matrix of size $2N \times 2N$. The eigenvalue found, the approximate values of vector $\mathbf{e}_{Tm}$ or $\mathbf{h}_{Tm}$ in the grid-points can be computed by solving a system of linear equations with $2N$ unknowns. Determination of the dispersion characteristic requires multiple repetitions of the procedure at frequency values chosen suitably densely.

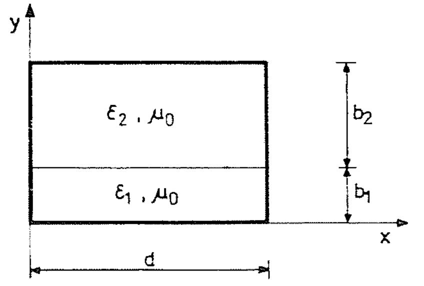

Fig. 3

Using the method in this paper for determining the Taylor's polynomial approximating the function $p^2(\omega^2)$, an eigenvalue problem has to be solved but once for determining the cut-off frequency. This eigenvalue problem is connected to a scalar function, so if there are N grid-points, the eigenvalue of a matrix of size $N \times N$ rather than $2N \times 2N$ must be determined. The eigenvalue problem once solved, only systems of linear equations with N unknowns has to be solved, and integrals to be evaluated numerically.

Rectangular waveguide with inhomogeneous dielectric

The described procedure will be adopted to the rectangular waveguide with two dielectrics of permitivities ε_1 and ε_2 in Fig. 3. Prache [6] and Yegorov [7] investigated this arrangement most minutely in the usual way. Their works contain among others the dispersion equation, the numerical solution of which gives the dispersion characteristic of a concrete arrangement. Applying the presented procedure to this arrangement yields the results below.

qTM modes

The qTM modes can be identified with the so-called LE_{mn} modes for which m is non-zero.

The constant L in the definition of the parameter w is chosen as $L = b_1 + b_2 = b$. Determination of the eigenvalue $w_0^2 = \omega_0^2 b^2/c^2$ requires to solve the system of transcendental equations

$$\varepsilon_{r2}\, k_1^2 - \varepsilon_{r1}\, k_2^2 = (\varepsilon_{r2} - \varepsilon_{r1})\, \pi^2 s^2 m^2$$
$$k_1 \operatorname{ctg} r_1 k_1 + k_2 \operatorname{ctg} r_2 k_2 = 0$$

using notations $r_1 = b_1/b$, $r_2 = b_2/b$ and $s = b/d$, m being an arbitrary positive integer. With knowledge of k_1 and k_2 the eigenvalue w_0^2 is given by the relationship:

$$w_0^2 = \frac{1}{\varepsilon_{r1}} (k_1^2 + \pi^2 s^2 m^2) = \frac{1}{\varepsilon_{r2}} (k_2^2 + \pi^2 s^2 m^2) .$$

For the qTM modes the vectors $\mathbf{e}_{1i}$ and $\mathbf{e}_{2i}$ are parallel to the x axis. Their magnitudes are:

$$e_{1ix} = \sum_{j=0}^{i} A_{ij} \cos \frac{\pi m x}{d} f_j(y, k_1) \qquad 0 < y < b_1$$

$$e_{2ix} = \sum_{j=0}^{i} B_{ij} \cos \frac{\pi m x}{d} f_j(b - y, k_2) \quad b_1 < y < b ,$$

where the following notation has been introduced:

$$f_j(z, k) = \begin{cases} (z/b)^j \sin (kz/b) , & \text{if } j \text{ is even.} \\ (z/b)^j \cos (kz/b) , & \text{if } j \text{ is odd.} \end{cases}$$

The coefficients A_{ij} can be determined from the recurrence formula

$$A_{ij} = \frac{(-1)^{j-1}}{2jk_1} \left(\varepsilon_{r1} A_{i-1,j-1} + \sum_{k=1}^{i-j+1} a_k A_{i-k,j-1} + j(j+1) A_{i,j+1} \right) .$$
$$i = 1, 2, \ldots \qquad j = i, i-1, \ldots 1$$

Of course the coefficient $A_{i,i+1}$ appearing in the case $j = i$ must be zero. The coefficients B_{ij} can be determined by the same formulae, except that ε_{r2} and k_2 replace ε_{r1} and k_1 . One of the coefficients A_{i0} and B_{i0} may be chosen arbitrarily, the other can be computed from the equation:

$$\sum_{j=0}^{i} A_{ij} f_j(b_1, k_1) = \sum_{j=0}^{i} B_{ij} f_j(b_2, k_2) . \qquad i = 0, 1, \ldots$$

With knowledge of the coefficients A_{ij} and B_{ij} the quantities q_i defined by (40) are:

$$g_i = \sum_{j=0}^{i} (A_{00} A_{ij} F_{1j} + B_{00} B_{ij} F_{2j}) \qquad i = 0, 1, \ldots$$

where:

$$F_{mi} = \frac{d}{2} \int_0^{b_m} \sin (k_m y/b) f_i(y, k_m)\, dy \qquad m = 1, 2$$

So the value of a_{i+1} is given by the formula

$$a_{i+1} = -\frac{1}{g_0}\left[\sum_{j=0}^{i}(\varepsilon_{r1} A_{00} A_{ij} F_{1j} + \varepsilon_{r2} B_{00} B_{ij} F_{2j} + \sum_{j=1}^{i} a_j g_{i-j}\right]$$

$$i = 0, 1, \ldots$$

and now the procedure continues with computing the coefficients $A_{i+1,j}$ and $B_{i+1,j}$.

The radius of convergence can be determined approximately by investigating the derivative $dp^2/d\omega^2$. Applying the results of (7) the evaluation of (45) leads to the relationship

$$\frac{dp^2}{d\omega^2} = -\mu_0 \times$$

$$\times \frac{\varepsilon_1(2q_1b_1 - \sin 2q_1b_1)\, q_2 \sin^2 q_2b_2 + \varepsilon_2(2q_2b_2 - \sin 2q_2b_2)\, q_1 \sin^2 q_1b_1}{(2q_1b_1 - \sin 2q_1b_1)\, q_2 \sin^2 q_2b_2 + (2q_2b_2 - \sin 2q_2b_2)\, q_1 \sin^2 q_1b_1},$$

where q_1 and q_2 denote the solution of the following system of transcendental equations:

$$q_1 \operatorname{ctg} q_1b_1 + q_2 \operatorname{ctg} q_2b_2 = 0$$

$$q_1^2 - q_2^2 = (\varepsilon_1 - \varepsilon_2)\, \mu_0\omega^2 .$$

The differential coefficient $dp^2/d\omega^2$ does not exist for the values q_1 and q_2 which satisfy the system of equations

$$(2q_1b_1 - \sin 2q_1b_1)q_2 \sin^2 q_2b_2 + (2q_2b_2 - \sin 2q_2b_2)q_1 \sin^2 q_1b_1 = 0$$

$$q_1 \operatorname{ctg} q_1b_1 + q_2 \operatorname{ctg} q_2b_2 = 0 .$$

All pairs of values $q_1 = 0$ and q_2 satisfying the equation $1 + b_1q_2 \operatorname{ctg} q_2b_2 = 0$ and all pairs of values $q_2 = 0$ and q_1 satisfying the equation $1 + b_2q_1 \operatorname{ctg} q_1b_1 = 0$ are solutions of this system of equations. In these cases both the numerator and the denominator in the formula of $dp^2/d\omega^2$ are zero, but it is easy to see that the limit value exists, so these pairs of values do not represent singular points of the function $p^2(\omega)^2$. Let us introduce notations $x = q_1b_1$, $y = q_2b_2$ and $r = b_1/b_2$ and rewrite the system of equations in the following form:

$$(2y - \sin 2y)x \sin^2 x + r(2x - \sin 2x)y \sin^2 y = 0$$

$$x \operatorname{ctg} x + r \operatorname{ctg} y = 0 .$$

The approximate solution of this system of equations is treated in Appendix IV. The singular points of the function, $p^2(\omega^2)$ can be determined from the solutions for which $xy \neq 0$, by means of the relationship:

$$\omega^2 = \frac{(1+r)^2}{(\varepsilon_1 - \varepsilon_2)\, \mu r^2 b^2}(x^2 - ry^2) .$$

For the determination of the radius of convergence those must be considered which lie on the branch of the function $p^2(\omega^2)$ connected with the investigated mode. Among these singular points the one nearest to the point ω_0^2 determines the radius convergence of.

As numerical example let an arrangement be considered where $d = 2.3b$, $b_1 = 0.2b$, $\varepsilon_{r1} = 10$ and $\varepsilon_{r2} = 1$. For the qTM mode of the smallest cut-off frequency, i.e. for the LE_{11} mode the eigenvalue w_0^2 was found to be $w_0^2 = 6.1691$, in correlation with the cut-off frequency $\omega_0 = 2.4838c/b$. The following values were obtained for the first six coefficients a_i:

$$a_1 = -2.6948 \qquad a_2 = -0.19991 \qquad a_3 = -0.013380$$

$$a_4 = 2.1452 \cdot 10^{-4} \qquad a_5 = 1.4630 \cdot 10^{-4} \qquad a_6 = 8.7376 \cdot 10^{-6} .$$

In Fig. 4 the continuous line represents the exact dispersion curve, and the dotted lines represent its approximation by Taylor's polynomials of first, second, third and fifth degree. The radius of convergence of Taylor's series

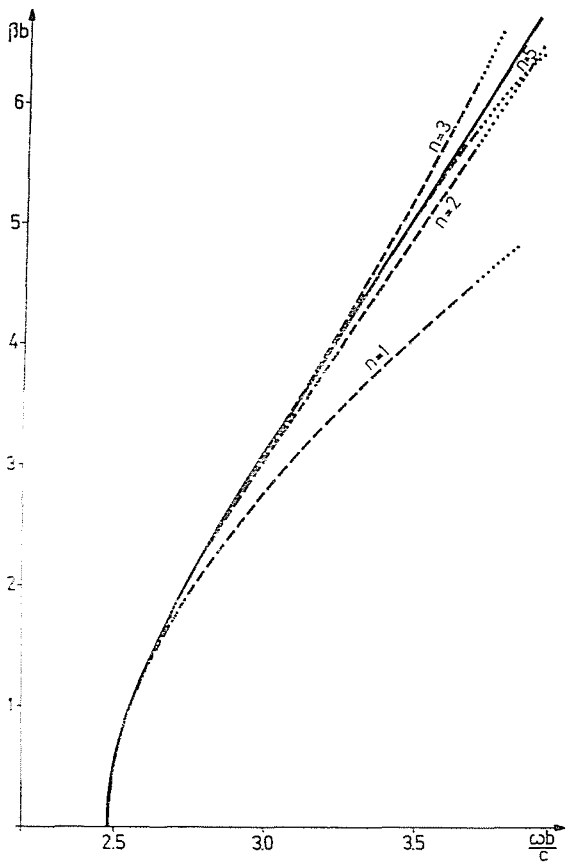

Fig. 4

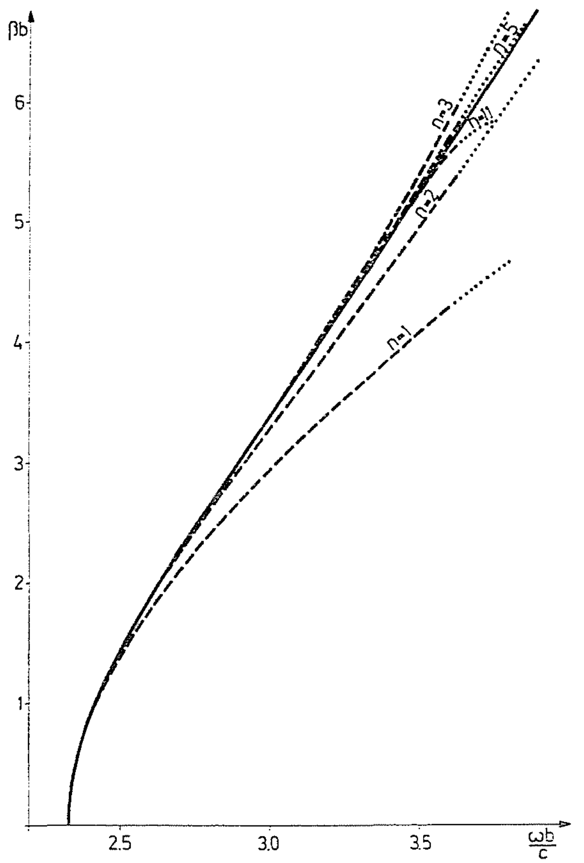

Fig. 5

was determined approximately on the basis of Appendix IV. The estimation described there gave that the singular point nearest to the value ω_0^2 is $\omega_1^2 = (7.15 - j7.54)\ (c/b)^2$. From this it follows that the radius of convergence of Taylor's series in ω^2 is $R = 7.60\ (c/b)^2$, from what it follows that the series is convergent to the value $\omega = 3.71c/b$. It correlates with this that e.g. the approximation of tenth degree not plotted in the figure is worse over the value $\omega = 3.7c/b$ than the one of fifth degree.

qTE modes

The cut-off frequency of the qTE modes can be determined from the system of transcendental equations

$$\varepsilon_{r2}k_1^2 - \varepsilon_{r1}k_2^2 = (\varepsilon_{r1} - \varepsilon_{r2})\ \pi^2 s^2 m^2$$

$$\varepsilon_{r2}k_1\ \mathrm{tg}\ r_1 k_1 + \varepsilon_{r1}k_2\ \mathrm{tg}\ r_2 k_2 = 0\ ,$$

where r_1, r_2 and s represent the same quantities as previously, and m is a non-negative integer. Knowing k_1 and k_2, the cut-off frequency can be calculated the same way as for the qTM modes.

In the case of $m = 0$ the field strenghts do not depend on x, and the vectors $\mathbf{h}_{T1}$ and $\mathbf{h}_{T2}$ are parallel to the y axis. This is why the electric field is parallel to the x axis, so these modes are true TE modes and can be identified with the LE_{0n} modes. If m is positive integer, the vectors $\mathbf{h}_{T1}$ and $\mathbf{h}_{T2}$ are parallel to the x axis, so these modes can be identified with the LM_{mn} modes. On the basis of the analogy previously described, these modes can be computed similarly to the qTM modes. The calculation of the modes belonging to the index $m = 0$ differs from this because m cannot be zero for the qTM modes. Formally substituting $m = 0$ in the results found for the qTM modes, the relationships describing the qTE modes belonging to the index $m = 0$ are directly got.

Instead of further details the numerical results are given for the mode LE_{01} of the arrangement already investigated. The cut-off frequency of this mode is $\omega_0 = 2.3322c/b$. The following values were got as the first twelve coefficients a_i:

$$
\begin{array}{lll}
a_1 = -2.4245 & a_2 = -0.17046 & a_3 = -0.013306 \\
a_4 = -2.3516 \cdot 10^{-4} & a_5 = 9.7536 \cdot 10^{-5} & a_6 = 1.2329 \cdot 10^{-5} \\
a_7 = -4.2496 \cdot 10^{-8} & a_8 = -1.8891 \cdot 10^{-7} & a_9 = -2.0196 \cdot 10^{-8} \\
a_{10} = 9.1837 \cdot 10^{-10} & a_{11} = 4.6234 \cdot 10^{-10} & a_{12} = 3.7221 \cdot 10^{-11}
\end{array}
$$

In Fig. 5 the exact dispersion curve is plotted in continuous line, and its approximations by Taylor's polynomials of several degrees in dotted line. Taylor's series is convergent approximately to the value $\omega = 3.63c/b$ according to calculations similar to previous ones and not given here in detail. The curve of Taylor's polynomial of eleventh degree, plotted in the figure, is in accordance with this.

Appendix I

It will be proven that the eigenvalues of the boundary value problem defined by Eq. (23) and boundary conditions (27), (29) and (31) are all non-negative real.

Let Eq. (23) be multiplied by the conjugate of the function u_{m0}/μ_{rm} and integrated over the region A_m of the cross section:

$$\frac{1}{\mu_{rm}} \int_{A_m} \Delta u_{m0}\, u_{m0}^*\, \mathrm{d}A + w_0^2\, L^{-2} \varepsilon_{rm} \int_{A_m} u_{m0}\, u_{m0}^*\, \mathrm{d}A = 0 \,.$$

Transforming the first integral by means of one of Green's theorems, the following equation arises:

$$w_0^2\, \varepsilon_{rm} L^{-2} \int\limits_{A_m} u_{m0}\, u_{m0}^*\, \mathrm{d}A = \frac{1}{\mu_{rm}} \int\limits_{A_m} \text{grand}\; u_{m0}\; \text{grad}\; u_{m0}^*\; \mathrm{d}A + \int\limits_{l_m} u_{m0}^* \frac{1}{\mu_{rm}} \frac{\partial u_{m0}}{\partial n_m}\, \mathrm{d}l\,,$$

where l_m denotes the boundary curve of the region A_m, and $\mathbf{n}_m$ denotes a unit vector normal to it and pointing inside the region. Writing this equation for every region of the cross section, and summarizing them, the line integrals cancel out along the separating contours because of the boundary conditions (29) and (31) and give zero along the contour of the conductor because of (27). So after ordering the following expression is obtained for w_0^2:

$$w_0^2 = \left(\sum_m \varepsilon_{rm} \int\limits_{A_m} u_{m0}\; u_{m0}^*\; \mathrm{d}A\right)^{-1} L^2 \sum_m \frac{1}{\mu_{rm}} \int\limits_{A_m} \text{grad}\; u_{m0}\; \text{grad}\; u_{m0}^*\; \mathrm{d}A$$

It is seen that w_0^2 can only be non-negative real, if the values ε_{rm} and μ_{rm} are all positive real.

Appendix II

In this Appendix the condition will be derived that must be satisfied by the right-hand side of Eq. (25) and the inhomogeneous part of the boundary condition (31) so that the boundary value problem relative to the function u_{mi} should have a solution.

Multiplying Eq. (23) by u_{mi} and (25) by $(-u_{m0})$, adding them and integrating over the region A_m:

$$\int\limits_{A_m} (u_{mi}\, \Delta u_{m0} - u_{m0}\, \Delta u_{\mathrm{mi}})\; \mathrm{d}A = L^{-2} \int\limits_{A_m} u_{m0} \left(\varepsilon_{rm}\, \mu_{rm}\, u_{m,i-1} + \sum_{j=1}^{i} a_j\; u_{m,i-j}\right) \mathrm{d}A.$$

The left-hand side of the equation can be transformed by means of one of Green's theorems. Writing such an equation for every region of the cross section dividing them by μ_{rm} and summing up, yields:

$$\sum_m \int\limits_{l_m} \frac{1}{\mu_{rm}} \left(u_{m0} \frac{\partial u_{mi}}{\partial n_m} - u_{mi} \frac{\partial u_{m0}}{\partial n_m}\right) \mathrm{d}A = L^{-2} \sum_m \int\limits_{A_m} u_{m0} \times$$

$$\times \left(\varepsilon_{rm}\, u_{m,i-1} + \frac{1}{\mu_{rm}} \sum_{j=1}^{i} a_j\; u_{m,i-j}\right) \mathrm{d}A\,,$$

where l_m denotes the boundary curve of the region A_m and $\mathbf{n}_m$ the unit vector normal to the contour and pointing inside the region. The line integrals give zero along the contour of the conductor because of the boundary condition

(27). Integrals of the second term of the integrand are cancelled for two contiguous regions because of the boundary conditions (29) and (31). The other part of the integral can be transformed by means of the same boundary conditions, giving the equation

$$\sum_m \left[\int_{A_m} u_{m0} \left(\varepsilon_{rm}\, u_{m,i-1} + \frac{1}{\mu_{rm}} \sum_{j=1}^{i} a_j\, u_{m,i-j} \right) \mathrm{d}A + \frac{1}{\mu_{rm}} \sum_{j=1}^{i} a_j \int_{l'_m} u_{m0}\, \mathbf{n}_m\, \mathbf{e}_{m,i-j}\, \mathrm{d}l \right] = 0,$$

where l'_m denotes the part of the boundary curve of the region A_m in the dielectric, i.e. not including the part on the contour of the conductors. This equation must be true, if the equation (25) is to have a solution satisfying the boundary conditions (27), (29) and (31).

Appendix III

A relationship will be derived, giving the value of the differential coefficient $\mathrm{d}p^2/\mathrm{d}\omega^2$.

The solution of the boundary value problem relative to Eq. (6) is considered in the points Ω^2 belonging to a small neighbourhood of the point ω^2:

$$\Delta \mathbf{e}_{Tm}\ (\Omega^2) + [p^2(\Omega^2) + \varepsilon_m \mu_m \Omega^2] \mathbf{e}_{Tm}(\Omega^2) = 0\ . \tag{I}$$

The solution of the boundary value problem is chosen so that the vectors $\mathbf{e}_{Tm}(\Omega^2)$ are continuous in dependence on Ω^2. For this reason, if the eigenvalue $p^2(\omega^2)$ is multiple, the vector $\mathbf{e}_{Tm}(\omega^2)$ cannot be an arbitrary solution of the boundary value problem, but such one which corresponds to a mode. Consider Eq. (I) in the points $\Omega^2 = \omega^2$ and $\Omega^2 = \omega^2 + \delta\omega^2$ and substract one of these equations from the other. Introducing the notations

$$\delta \mathbf{e}_{Tm} = \mathbf{e}_{Tm}(\Omega^2) - \mathbf{e}_{Tm}(\omega^2)$$

$$\delta p^2 = p^2(\Omega^2) - p^2(\omega^2)$$

the following equation is obtained:

$$\Delta \delta \mathbf{e}_{Tm} + [p^2(\omega^2) + \varepsilon_m \mu_m \omega^2] \delta \mathbf{e}_{Tm} + (\delta p^2 + \varepsilon_m \mu_m \delta\omega^2)\, \mathbf{e}_{Tm}(\Omega^2) = 0\ . \tag{II}$$

Let $\mathbf{h}_{Tm}$ be a solution of the boundary value problem relative to the equation

$$\Delta \mathbf{h}_{Tm} + [p^2(\omega^2) + \varepsilon_m \mu_m \omega^2]\ \mathbf{h}_{Tm} = 0\ . \tag{III}$$

Multiplying vectorially Eq. (II) by the vector $\mathbf{h}_{Tm}$ and Eq. (III) by the vector $\delta \mathbf{e}_{Tm}$, and adding them:

$$\mathbf{h}_{Tm} \times \Delta \delta \mathbf{e}_{Tm} + \delta \mathbf{e}_{Tm} \times \Delta \mathbf{h}_{Tm} + (\delta p^2 + \varepsilon_m \mu_m \delta\omega^2)\, \mathbf{h}_{Tm} \times \mathbf{e}_{Tm}(\Omega^2) = 0\ .$$

It can be proven simply, but somewhat lengthily that if the vectors **a** and **b** lie in the xy plane and do not depend upon the co-ordinate z, then

$$\mathbf{a} \times \Delta\mathbf{b} + \mathbf{b} \times \Delta\mathbf{a} = \mathbf{k}\ \mathrm{div}\ [\mathbf{a}(\mathbf{k}\ \mathrm{curl}\ \mathbf{b})] + \mathbf{k}\ \mathrm{div}\ [\mathbf{b}(\mathbf{k}\ \mathrm{curl}\ \mathbf{a})] - $$
$$- \mathrm{curl}\ (\mathbf{a}\ \mathrm{div}\ \mathbf{b}) - \mathrm{curl}\ (\mathbf{b}\ \mathrm{div}\ \mathbf{a})\ .$$

Applying this identity to the previous equation, and integrating the equation over the region A_m, the following relationship is obtained by means of Gauss' and Stokes' theorems:

$$-\int_{l_m} [\mathbf{n}_m\mathbf{h}_{Tm}(\mathbf{k}\ \mathrm{curl}\ \delta\mathbf{e}_{Tm}) + \mathbf{n}_m\delta\mathbf{e}_{Tm}(\mathbf{k}\ \mathrm{curl}\ \mathbf{h}_{Tm}) + \mathbf{t}_m\mathbf{h}_{Tm}\ \mathrm{div}\ \delta\mathbf{e}_{Tm} +$$
$$+ \mathbf{t}_m\delta\mathbf{e}_{Tm}\ \mathrm{div}\ \mathbf{h}_{Tm}]\ dl + \int_{A_m} [(\delta p^2 + \varepsilon_m\mu_m\delta\omega^2)\ \mathbf{h}_{Tm} \times \mathbf{e}_{Tm}(\Omega^2)]\ \mathrm{d}\mathbf{A} = 0\ ,$$

where $\mathbf{t}_m = \mathbf{n}_m \times \mathbf{k}$. Let this equation be written for every region A_m of the cross section and add them up. The function $\delta\mathbf{e}_{Tm}$ must satisfy the boundary conditions (7) to (12), the function $\mathbf{h}_{Tm}$ the boundary conditions (34) and (35) and the ones analogous to Eqs (9) to (12). The line integrals are seen to give zero in the sum of the equations, and so the following relationship is obtained after ordering:

$$\frac{\delta p^2}{\delta\omega^2} = -\frac{\sum\limits_m \varepsilon_m\,\mu_m \int\limits_{A_m} [\mathbf{e}_{Tm}(\Omega^2) \times \mathbf{h}_{Tm}]\ \mathrm{d}\mathbf{A}}{\sum\limits_m \int\limits_{A_m} (\mathbf{e}_{Tm}(\Omega^2) \times \mathbf{h}_{Tm})\ \mathrm{d}\mathbf{A}}\ .$$

Let now the value Ω^2 converge to ω^2, and denoting the vector $\mathbf{e}_{Tm}(\omega^2)$ simply by $\mathbf{e}_{Tm}$, the derivative of the function $p^2(\omega^2)$ with respect to ω^2 is obtained as:

$$\frac{\mathrm{d}p^2}{\mathrm{d}\omega^2} = -\frac{\sum\limits_m \varepsilon_m\,\mu_m \int\limits_{A_m} (\mathbf{e}_{Tm} \times \mathbf{h}_{Tm})\ \mathrm{d}\mathbf{A}}{\sum\limits_m \int\limits_{A_m} (\mathbf{e}_{Tm} \times \mathbf{h}_{Tm})\ \mathrm{d}\mathbf{A}}\ ,$$

where the denominator is supposed to be non-zero. If the numerator is zero and the denominator is not, the function is not differentiable at the given point. If both are zero, the limiting value can exist in the foregoing sense. In this case the function is differentiable in the investigated point.

Appendix IV

A method for the approximate solution of the following system of equation will be presented:

$$(2y - \sin 2y)x \sin^2 x + r(2x - \sin 2x)y \sin^2 y = 0$$
$$x \operatorname{ctg} x + ry \operatorname{ctg} y = 0\ .$$

Solutions where x or y is zero are uninteresting, and it is enough to look for solutions where y lies in the first quarter of the complex plane. First the limiting case $r = 0$ is treated, the solutions of which are denoted by X and Y. They satisfy the equations

$$\operatorname{ctg} X = 0$$
$$2Y - \sin 2Y = 0$$

This yields the values for X

$$X_m = \left(m + \frac{1}{2}\right)\pi,$$

where m is an arbitrary integer. Ten roots of the other equation, which have the smallest absolute values are given here:

$Y_1 = 3.749 + j1.384$ $\quad Y_2 = 6.950 + j1.676$ $\quad Y_3 = 10.119 + j1.858$

$Y_4 = 13.277 + j1.992$ $\quad Y_5 = 16.430 + j2.097$ $\quad Y_6 = 19.580 + j2.184$

$Y_7 = 22.727 + j2.258$ $\quad Y_8 = 25.874 + j2.184$ $\quad Y_9 = 29.020 + j2.379$

$Y_{10} = 32.164 + j2.430$.

The other roots can be calculated with a relative error less than 10^{-3} by means of the asymptotical formula

$$Y_n \approx \left(n + \frac{1}{4}\right)\pi + \frac{1}{2}\operatorname{Ln}(4n + 1)\,\pi\,.$$

Now the solutions of the system of equations are investigated in dependence on the parameter r. Let $x_{mn}(r)$ and $y_{mn}(r)$ denote the pair of solutions for which $x_{mn}(0) = X_m$ and $y_{mn}(0) = Y_n$. The functions $x_{mn}(r)$ and $y_{mn}(r)$ can be determined e.g. numerically if the two equations are differentiated with respect to r, and the system of differential equations so obtained for the functions $x_{mn}(r)$ and $y_{mn}(r)$ is numerically integrated starting from the initial values X_m and Y_m. If the roots are needed for a value $r > 1$, of course the reciprocal of r is introduced as a new parameter, the calculation is based upon. As rough estimation the functions $x_{mn}(r)$ and $y_{mn}(r)$ can be approximated by Taylor's polynomial of first degree:

$$x_{mn} \approx X_m + r\,\frac{1 + \sqrt{1 - 4Y_n^2}}{2X_m}$$

$$y_{mn} \approx \left(1 - \frac{r}{2}\right) Y_n\,,$$

where into the expression of x_{mn} the value of the square root found in the right half plane has to be substituted.

Summary

A method has been given to determine the Taylor's series expansion of the dispersion function of waveguides with inhomogeneous dielectric about the cut-off frequency. The Taylor's series of the transversal component of the electric or magnetic field is got also as a result of the presented recursion procedure. The radius of convergence has been determined from the branch points of the square of the propagation coefficient on the complex plane. The rectangular waveguide with inhomogeneous dielectric has been treated as an example and compared with published results.

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