SOME REMARKS ON INTEGRAL-TRANSFORM AND OPERATOR METHODS

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Starting from the Fourier series, this paper first introduces the Fourier transform; building on this latter, it then presents the two-sided Laplace transform. The next subject is the one-sided Laplace transform. Transfer functions and frequency functions are given. Various inverse transformation procedures are outlined and several forms of convolution integrals are reviewed. The distinction between the initial values figuring in Laplace-transform formulas and differential equations is pointed out: the first refer (with the exception of the limit theorem) to the instant $t=0-$, whereas the latter usually refer to the instant $t=0+$. Means of conversion between the two cases are outlined. The Laplace transform is applied not only to time-invariant but also to time-variable linear systems.

In the sequel, the fundamentals of Mikusinski's operational calculus are outlined. The connection with the Laplace transform is pointed out.

Classical operational calculus and its relationship to the Carson-Wagner transform are also discussed.

The Fourier series

Single-valued periodic functions can be resolved into a sum of harmonic functions provided the number of maxima, minima and discontinuities within a period T is finite and if the integral of the function over said period T is finite. The Fourier series of such a function $f(t)=f(t+T)$ is

$$f(t)=A_0+\sum_{k=1}^{\infty}(A_k\cos k\omega_0 t+B_k\sin k\omega_0 t) \tag{1}$$

where $\omega_0=2\pi/T$ is the angular frequency of the fundamental. The coefficients are

$$A_0=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt$$

$$A_k=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos k\omega_0 t \qquad (k\neq 0) \tag{2}$$

$$B_k=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin k\omega_0 t.$$

At any instant t, the Fourier series tends to the value $f[t]$ of the function $f(t)$, if $f(t)$ is continuous in the neighbourhood of that instant. If there is a finite discontinuity, then the series tends to the value $(f[t-]+f[t+])/2$ that is, to the arithmetic mean of the values assumed on the two sides of the discontinuity.

By the Dirichlet-Jordan theorem, the Fourier series is uniformly convergent for any t provided the function $f(t)$ satisfies certain conditions and if it remains finite, that is, if it is a function of bounded variation.

Clearly, the Fourier series can be written also in the form

$$f(t)=A_0+\sum_{k=1}^{\infty} c_k\cos(k\omega_0 t+\varphi_k) \tag{3}$$

where

$$c_k=\sqrt{A_k^2+B_k^2} \quad \text{and} \quad \varphi_k=\tan^{-1}(B_k/A_k).$$

Using Euler's theorem, Eq. (3) can be brought to the so-called complex form

$$f(t)=\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \tag{4}$$

where

$$C_k=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jk\omega_0 t}\,\mathrm{d}t \tag{5}$$

and

$$C_0=A_0, \quad C_k=\frac{c_k}{2}e^{j\varphi_k} \qquad (k\neq 0). \tag{6}$$

Hence, $C_k(k \neq 0)$ is complex. Its magnitude is one-half of the amplitude c_k of the kth harmonic, and its phase equals the phase φ_k of the kth harmonic referred to a cosine wave. Since C_k uniquely defines the function $f(t)$ in terms of frequency-domain concepts, C_k is called the discrete-frequency spectrum of the function $f(t)$.

The Fourier series is used to describe steady-state (quasistationary) outputs generated by periodic inputs.

The Fourier integral

The complex form (4) and the frequency spectrum (6) of the Fourier series may be modified by introducing the notation $\omega = k\omega_0$ and $\Delta\omega = 2\pi/T = \omega_0$ with the results

$$f(t) = \sum_{k=-\infty}^{\infty} \frac{C_k}{\Delta\omega} e^{j\omega t} \Delta\omega \tag{7}$$

and

$$\frac{C_k}{\Delta\omega} = \frac{1}{T\Delta\omega} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-j\omega t}\, dt = \frac{1}{2\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{j\omega t}\, dt. \tag{8}$$

The transition from periodic to nonperiodic functions is effected by letting T approach infinity, $T \to \infty$. This entails $\Delta\omega \to 0$, and the discrete frequency spectrum passes into a continuous one. As a result, $C_k \to 0$, but $C_k/\Delta\omega$ will in general tend to a value other than zero. Let the complex limit of $C_k/\Delta\omega$ be denoted $C(j\omega)$. Taking the limit $T \to \infty$ and $\Delta\omega \to d\omega$, (7) and (8) are converted into

$$f(t) = \int_{-\infty}^{\infty} C(j\omega) e^{j\omega t}\, d\omega \tag{9}$$

and

$$C(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt. \tag{10}$$

Introducing the notation $F(j\omega) = 2\pi C(j\omega)$ yields the usual control-theory forms for the Fourier integral and its inverse:

$$\mathcal{F}[f(t)] \triangleq F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \tag{11}$$

$$\mathcal{F}^{-1}[F(j\omega)] \triangleq f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega. \tag{12}$$

$F(j\omega)$ is the continuous frequency spectrum. $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse. The function $f(t)$ must be single-valued also in this case; the number of its maxima, minima and discontinuities must be finite; moreover, $f(t)$ must be absolutely integrable:

$$\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t < M.$$

The Fourier transform of the weighting function is called the frequency response. It is also called the a.c. steady-state system function or frequency function. The frequency response of a transfer element or that of a system is, then,

$$G(j\omega)=\int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,\mathrm{d}t; \quad W(j\omega)=\int_{-\infty}^{\infty} w(t)e^{-j\omega t}\,\mathrm{d}t. \tag{13}$$

If for a certain element $U(j\omega)$ and $X(j\omega)$ are the Fourier transforms of the input $u(t)$ and the output $y(t)$, respectively, then it can be shown that

$$Y(j\omega)=G(j\omega)U(j\omega) \tag{14}$$

and

$$y(t)=\mathcal{F}^{-1}[G(j\omega)U(j\omega)]. \tag{15}$$

The frequency response proper is

$$G(j\omega)=\frac{Y(j\omega)}{U(j\omega)}. \tag{16}$$

We shall revert to these relationships further below.

Example 1. Let us consider the Fourier transform of the unit step function $1(t)$. This function is not absolutely integrable: hence, the integral (12) does not converge for this function.
On the other hand,

$$\mathcal{F}[e^{-\alpha t}1(t)]=\int_{0}^{\infty} e^{-\alpha t}e^{-j\omega t}\,\mathrm{d}t=\frac{1}{\alpha+j\omega}, \qquad (\alpha\neq 0).$$

If $\alpha\to 0$, the exponential function tends to the unit step at any positive time, and the above Fourier transform tends to $1/j\omega$. This is not, however, the Fourier transform of the unit step $1(t)$, since by Eq. (12):

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{j\omega t}}{j\omega}\,\mathrm{d}\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\cos\omega t}{j\omega}+\frac{\sin\omega t}{\omega}\right]\mathrm{d}\omega=$$

$$=\frac{1}{\pi}\int_{0}^{\infty}\frac{\sin\omega t}{\omega}\,\mathrm{d}\omega=\begin{cases}-\frac{1}{2} & \text{if} \quad t<0\\ +\frac{1}{2} & \text{if} \quad t>0\end{cases}$$

since

$$Si(\Omega t)\to\pi/2, \quad (t>0), \quad \text{and} \quad Si(\Omega t)\to-\pi/2 \quad (t<0) \quad \text{if} \quad \Omega\to\infty.$$

This and similar examples reveal that some of the simplest functions do not have Fourier transforms. It is precisely the presence of the damping factor $\exp(-at)$ that motivates the introduction of the Laplace transform.

The two-sided Laplace transform

The problems of convergence inherent in the Fourier transform can be overcome by introducing the factor $\exp(-\sigma-j\omega t)$ instead of $\exp(-j\omega t)$ in Eq. (11). Putting $s=\sigma+j\omega$, we can derive from Eqs. (11) and (12) the two-sided Laplace transform

$$\mathcal{L}[f(t)]\triangleq F(s)=\int_{-\infty}^{\infty} f(t)e^{-st}\,dt \tag{17}$$

and its inverse

$$\mathcal{L}^{-1}[F(s)]\triangleq f(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{ts}\,ds. \tag{18}$$

The choice of a suitable value for $\sigma=\mathrm{Re}\,s$ is of importance.

If it is possible to put $\sigma=0$, then $s=j\omega$ and the two-sided Laplace merges into the Fourier transform.

Example 2. By Eq. (17), the Laplace transform of the function defined as

$$f(t)=\begin{cases} e^{at} & \text{for} \quad t>0 \\ 0 & \text{for} \quad t<0 \end{cases}$$

is

$$F(s)=\int_0^{\infty} e^{at}e^{-st}=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}=\frac{1}{s-a}$$

provided $\mathrm{Re}\,(s-a)=\sigma-a>0$. The pole of $F(s)$ is $p_1=a$, and the region of convergence is defined by the inequality $\sigma>a$. For real and positive a, $f(t)$ has no Fourier transform.

If, on the other hand,

$$f(t)=\begin{cases} -e^{at} & \text{for } t<0 \\ 0 & \text{for } t>0 \end{cases}$$

then

$$F(s)=-\int_{-\infty}^{0} e^{at}e^{-st}\,dt=\left[\frac{e^{-(s-a)t}}{(s-a)}\right]_{-\infty}^{0}=\frac{1}{s-a}$$

provided $\mathrm{Re}(s-a)=\sigma-a<0$. The pole of $F(s)$ is $p_1=a$ once more, and the region of convergence is defined by the inequality $\sigma<0$. If a is real and positive, then $\sigma=0$ yields the Fourier transform $F(j\omega)=1/(j\omega-a)$.

This example reveals how two different functions can possess formally identical Laplace transforms, all the difference being in regions of convergence. The unique definition of any transform requires, then, the precise specification of a region of convergence. The Fourier transform is always characterized by $\sigma=0$, and hence, no two time functions can have the same Fourier transform.

The one-sided Laplace transform

Control theory makes frequent use of signals (or variables) that are identically zero for all negative values of time, called switching signals, or, in less rigorous parlance, positive-time functions. In the discussion of these, the two-sided Laplace transform may be replaced by the one-sided one:

$$\mathcal{L}[f(t)]=F(s)=\int_{-0}^{\infty} f(t)e^{-st}\,\mathrm{d}t. \tag{19}$$

Formally, the expression of the inverse transform remains unchanged:

$$\mathcal{L}^{-1}[F(s)]=f(t)=\frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{ts}\,\mathrm{d}s. \tag{20}$$

Also here, integration is to be performed so that σ should be included in the region of convergence.

Note how simply the Laplace transform solves the problem of generalized functions (distributions). The transform of the generalized function $\delta^{(n)}(t)$ is s^n $(n=\ldots-2,-1,0,1,2,\ldots)$; furthermore, $\delta^{(-k)}(t)=t^{k-1}/(k-1)!(t\geqq 0)$. The number $k=-n$ is called the type (or characteristic) number of the typical signal.

The conditions of the existence and uniqueness of the Laplace transform are laid down in numerous theorems. The most fundamental ones of these are the following. For $F(s)$ to exist, the function $f(t)$ must be defined for every instant $t\geqq 0$ except a countable set of instants. Any piecewise continuous function of exponential order possesses a Laplace transform. (A function $f(t)$ is called of exponential order if there exists a real number k such that

$$\lim_{t\to\infty} f(t)e^{-kt}=0. \tag{21}$$

The functions e^{t^2} and t^t are not of exponential order.) For any Laplace-transformable function $f(t)$, the integral (19) is absolutely convergent over a certain region $\sigma>\sigma_0$ where σ_0 is called the convergence abscissa.

Two time functions will have identical Laplace transforms if and only if they coincide at every instant $t \geqq 0$, except a countable set of instants. Disregarding these latter instants, we may state that there exists a one-to-one correspondence between $f(t)$ and $F(s)$. The time function $f(t)$ furnished by the inverse Laplace transform is zero for negative time ($t<0$) and, except for an occasional generalized function or singularity function (like the step function), it is continuous in the interval $t \geqq 0$.

The so-called switching signals encountered in control engineering are often discontinuous at $t=0$; moreover, the impulse function and its derivatives may also occur at this instant. To account for these circumstances it is indicated to start the Laplace transform and any other integration that is required at $0-$ (whereas the initial-value theorem furnishes the right-hand limit $f(0+)$ of the function). In the theorems of differentiation, on the other hand, the value of the function should invariably be taken at the instant $0-$ because, clearly, it is so and only so that the Laplace transform of the unit impulse becomes

$$\mathcal{L}[\delta(t)]=\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\,1(t)\right]=s\frac{1}{s}-1[0-]=1-0=1 \tag{22}$$

whereas the right-hand limit would furnish the erroneous result

$$\mathcal{L}[\delta(t)]=\mathcal{L}\left[\frac{\mathrm{d}}{\mathrm{d}t}\,1(t)\right]=s\frac{1}{s}-1[0+]=1-1=0$$

This convention raises however, a certain difficulty in connection with the solution of nth-order differential equations, as the releveant procedures are usually based on the right-hand limits $y(0+)$, $\dot{y}(0+)$, ... If the forcing function $u(t)$, or its derivatives occurring in the differential equation, contain no generalized function such as an impulse, then one may put $y[0-]=y[0+]$, $\dot{y}[0-]=\dot{y}[0+]$, ... as in that case neither the output, nor its first $n-1$ derivatives will be discontinuous. If, on the other hand, $u(t)$ and its derivatives involved are generalized functions, then some may be resorted to, with due attention to the fact that the initial conditions defined there, $y[0+]$, $\dot{y}[0+]$, ... are essentially $\Delta y[0+]$, $\Delta\dot{y}[0+]$, ..., since in deriving the weighting function there we have started from the initial conditions $y[0-]$, $\dot{y}[0-]$... The initial conditions required for applying the Laplace transform to the differential equation can then be obtained from the stated conditions $y[0+]$, $\dot{y}[0+]$, ... using the relationships

$$y[0-]=y[0+]-\Delta y[0+],\ \dot{y}[0-]=\dot{y}[0+]-\Delta\dot{y}[0+],\ \ldots \tag{23}$$

In another procedure, the differential equation is first Laplace-transformed with zero initial values, and the right-hand initial conditions $\mathcal{L}[y(t)]$, $\mathcal{L}[\dot{y}(t)]$, ... are determined subsequently, using Initial-value Theorem. This procedure furnishes precisely the values $\Delta y(0+)$, $\Delta\dot{y}[0+]$, ...

Example 3. Laplace-transforming the differential equation $\ddot{y}+(a+1)\dot{y}+ay=$ $=A[\dot{u}+au]$ with zero initial conditions, we get

$$\mathcal{L}[y(t)]=Y(s)=A\frac{s+a}{s^2+(a+1)s+a}.$$

Now by the limit theorem

$$\Delta y[0+]=A$$

and by the rule of differentiation

$$\mathcal{L}[\dot{y}(t)]=\dot{Y}(s)=sY(s)-\Delta y[0+]=-A\frac{s+a}{s^2+(a+1)s+a}.$$

The repeated application of the limit theorem now yields

$$\Delta\dot{y}[0+]=-A.$$

These results are seen to agree with the above ones. Assuming now for instance that $y[0+]=A$, $\dot{y}[0+]=-A$, and using (23), we obtain the two initial conditions desired as $y[0-]=0$; $\dot{y}[0-]=0$.

The inverse transformation

Eq. (20) is virtually never used in practice to establish time functions belonging to the one-sided Laplace transform: expressions based on the theorem of residues are used instead.

Assume $F(s)$ to be a rational fraction of the form $F_z(s)/F_p(s)$ with the degree of the polynomial in the numerator one less than the degree of the one in the denominator. (If this is not the case, one can always perform a division beforehand. The result of the division is a polynomial in non-negative powers of s, e.g. $c_0+c_1s+c_2s^2+\ldots$, to which the generalized time function $c_0\delta(t)+c_1\dot{\delta}(t)+c_2\ddot{\delta}(t)+\ldots$ belongs.)

1. For multiple poles (and $t\geqq 0$),

$$f(t)=\sum_{\alpha=1}^{P}\lim_{s\to p_\alpha}\frac{1}{(k_\alpha-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}s^{k_\alpha-1}}(s-p_\alpha)^{k_\alpha}F(s)e^{st} \tag{24}$$

where P is the number of poles (each set of multiple poles taken as a single pole), k_α is the multiplicity of pole p_α, and

$$\sum_{\alpha=1}^{P}k_\alpha=n$$

is the degree of the denominator.

For simple poles $P=n$ (and $t\geqq 0$),

$$f(t)=\sum_{\alpha=1}^{n}\lim_{s\to p}(s-p_\alpha)F(s)e^{st}. \tag{25}$$

2. Another tool useful in such cases is the so-called expansion theorem

$$f(t)=\sum_{\alpha=1}^{n}\frac{F_z(p_\alpha)}{F_p'(p\alpha)}e^{p_\alpha t}, \qquad (t\geqq 0) \tag{26}$$

where

$$F_p'(p_\alpha)=\mathrm{d}F_p(p)/\mathrm{d}s|_{s=p_\alpha}, \quad F_z(p_\alpha)=F_z(s)|_{s=p_\alpha}.$$

For multiple poles, the expansion theorem reads

$$f(t)=\sum_{\alpha=1}^{P}\sum_{i=1}^{k_\alpha}\frac{R_{\alpha i}}{(k_\alpha-i)!}t^{(k_\alpha-i)}e^{p_\alpha t}, \qquad (t\geqq 0) \tag{27}$$

where

$$R_{\alpha i}=\frac{1}{(i-1)!}\left[\frac{\mathrm{d}^{i-1}}{\mathrm{d}s^{i-1}}(s-p_\alpha)^{k_\alpha}F(s)\right], \qquad (i=1, 2, \ldots, k_\alpha). \tag{28}$$

3. A further available means is partial-fraction expansion (making use of the fact that $j=k_\alpha-i+1$);

$$F(s)=\sum_{\alpha=1}^{P}\sum_{j=1}^{k_\alpha}\frac{R_{\alpha j}}{(s-p_\alpha)^j} \tag{29}$$

where

$$R_{\alpha j}=\lim_{s\to p_\alpha}\frac{1}{(k_\alpha-j)!}\frac{\mathrm{d}^{k_\alpha-j}}{\mathrm{d}s^{k_\alpha-j}}[(s-p_\alpha)^{k_\alpha}F(s)], \qquad (j=1, 2, \ldots, k_\alpha). \tag{30}$$

Once the partial fractions are established, a comprehensive tabulation can be resorted to.

Let us add that these methods are applicable with slight modifications even if $F(s)$ is of the form $F_z(s)e^{-sT}/F_p(s)$. In this case, the first method furnishes the time function for all instants $t\geqq T$; in the second one, t is to be replaced by $t-T$ and $(t\geqq 0)$ by $(t\geqq T)$; in the third one, the partial fraction expansion is to be performed for $F_z(s)/F_p(s)$ and Theorem 3 is to be applied afterwords.

The methods illustrated will apply also to the two-sided inverse Laplace transform, but in establishing the negative-time function, (a) instead of the poles p_α, lying to the left of the domain of regularity, the poles p_β lying to the right of that domain are to be taken, (b) $F(s)$ is to be replaced by $-F(s)$ in all formulae, and (c) the time function will apply to the interval $t<0$ (or $t<T$, whichever is relevant). Thus for instance, for multiple poles, so-called pseudo-positive-time and pseudo-negative-time functions are obtained abutting at the instant $t=T$ rather than $t=0$, in the form

$$f_{(+)}(t)=\sum_{\alpha=1}^{P}\lim_{s\to p_\alpha}\frac{1}{(k_\alpha-1)!}\frac{\mathrm{d}^{k_\alpha-1}}{\mathrm{d}s^{k_\alpha-1}}[(s-p_\alpha)^{k_\alpha}F(s)e^{st}]; \qquad T\leqq t \tag{31}$$

and

$$f_{(-)}(t)=-\sum_{\beta=1}^{P_\beta}\lim_{s\to p_\alpha}\frac{1}{(k_\beta-1)!}\frac{\mathrm{d}^{k_\beta-1}}{\mathrm{d}s^{k_\beta-1}}[s-p_\beta)^{k_\beta}F(s)e^{st}]; \qquad t<T, \qquad (32)$$

respectively. The rest of the inverse transformation formulas may be generalized in a similar fashion.

Example 4. To complete our discussion of the Laplace transform, let us consider a simple example. Let a current $i(0-)=y(0-)$ initially flow in a reactor of inductance L. Let us apply a voltage impulse $u(t)=\psi\delta(t)$ to the reactor at the instant $t=0$. What will the transient process of the current be?

The differential equation of the reactor is

$$L\frac{\mathrm{d}y(t)}{\mathrm{d}t}=u(t).$$

Integration yields

$$y[t]=y[0-]+\frac{1}{L}\int_{-0}^{[T]}z(t)\,\mathrm{d}t=y[0-]+\frac{\Psi}{L}$$

and further

$$y(t)=\left[y[0-]+\frac{\Psi}{L}\right]1(t).$$

The same results will be obtained also by solving the homogeneous differential equation with the initial condition $y[0+]=y[0-]+\psi/L$.

On the other hand, the Laplace transform of the differential equation of the reactor

$$L[sY(s)-y[0-]]=U(s)$$

and rearranging it, we get

$$Y(s)=\frac{y[0-]}{s}+\frac{\Psi}{Ls}$$

The inverse transform now yields precisely the above result. The transient process of the voltage across capacitor C under the influence of a current impulse $\mathrm{i}(t)=Q\delta(t)$ can be studied in a similar manner.

Application of the Laplace transform to time-variable systems

The Laplace transform method can also be extended to time-variable systems. Some special differential equations such as the Euler or Bessel equations can be solved also by means of the Laplace transform. But in a general way the problem will lead to a convolution integral with complex-valued functions, and the transform $Y(s)$ will

be given by a complicated integral equation, not easier to solve than the original differential equation.

The Laplace transform will, however, help us to make certain theoretical points.

The transfer function of a time-variable element is

$$G(t, s)=\mathcal{L}_{\vartheta}[g_*(t, \vartheta)] \tag{33}$$

with the transform applied to the variable ϑ, and t taking the role of a parameter:

$$G(t, s)=\int_0^{\infty} g_*(t, \vartheta)e^{-s\vartheta}\,\mathrm{d}\vartheta=\int_0^{\infty} g(t, t-\vartheta)e^{-s\vartheta}\,\mathrm{d}\vartheta=$$

$$=\int_{-\infty}^{\tau} g(t, \tau)e^{-s(t-\tau)}\,\mathrm{d}\tau \neq \mathcal{L}_{\tau}[g(t, \tau)].$$

Conversely,

$$g_*(t, \vartheta)=\mathcal{L}_{\vartheta}^{-1}[G(t, s)]=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} G(t, s)e^{j\vartheta s}\,\mathrm{d}s. \tag{34}$$

In can further be shown that for the differential equation

$$A(p, t)y(t)=B(p, t)u(t)$$

we have

$$A(p, t)[G(t, s)e^{st}]=B(p, t)e^{st} \tag{35}$$

and

$$y(t)=\mathcal{L}^{-1}[G(t, s)U(s)]=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} G(t, s)U(s)e^{ts}\,\mathrm{d}s. \tag{36}$$

However, this does not imply that the expression in the brackets is $Y(s)=\mathcal{L}[y(t)]$,

$$Y(s) \neq G(t, s)U(s)$$

since a time-variable and a time-invariant expression cannot be equal as a rule. The inverse transform would, properly speaking, first introduce ϑ into Eq. (36); ϑ would have to be replaced by t subsequently (since the output $y(t)$ is generated by the input $u(\vartheta)|_{\vartheta=t}$). Eq. (36) performs all this in one step.

Equation (36) will furnish the output if the transfer function is known. It is precisely the establishing of the transfer function that is the hardest part of the entire procedure. Once the weighting function is known, then the transfer function may be derived using Eq. (33).

Another procedure is based on Eq. (35). With reference to Leibniz' rule concerning the kth derivative of a product function,

$$p^k[G(t, s)e^{st}] = \sum_{i=0}^{k} \binom{k}{i} [p^{k-i}G(t, s)][p^i e^{st}] =$$

$$= e^{st} \sum_{i=0}^{k} \binom{k}{i} p^{k-i}s^i G(t, s) = e^{st}(p+s)^k G(t, s).$$

Applying this to Eq. (35) and reducing the fraction by e^{st}, we get $G(t, s)$ in explicit form:

$$A(p+s, t)G(t, s) = B(s, t). \tag{37}$$

Often, however, the calculation of $G(t, s)$ is complicated enough to demand the use of approximation methods. These may be fairly simple if the time variation of the parameters is either slow, or of small amplitude about a mean value.

Example 6. Let the differential equation of the transfer element be

$$t^2\ddot{y} + (a+2)t\dot{y} + ay = u$$

with a an integer greater than unity. It can be shown that the weighting function (or impulse response) is

$$g(t, \tau) = \frac{1}{a-1} \quad \frac{t^{a-1} - \tau^{a-1}}{t^a}, \qquad 0 \leqq \tau \leqq t.$$

Hence

$$g(t, \vartheta) = g(t, t-\vartheta) = \frac{t^{a-1} - (t-\vartheta)^{a-1}}{(a-1)t^a}, \quad 0 \leqq \vartheta \leqq t$$

that is,

$$g(t, \vartheta) = \frac{1}{(a-1)t^a} \sum_{i=1}^{a-1} \binom{a-1}{i} (-1)^{i+1} \vartheta^i t^{a-1-i}.$$

By Eq. (33), the transfer function is

$$G(t, s) = \sum_{i=1}^{a-1} \binom{a-2}{i-1} \frac{(-1)^{i-1}(i-1)!}{t^{i+1}s^{i+1}}.$$

This rather complicated expression can be shown to satisfy the equation based on Eq. (37)

$$[t^2(p+s)^2 + (a+2)t(p+s) + a]\, G(t, s) = 1(t).$$

Let for simplicity $u(t) = \delta(t)$: then, by (36),

$$y(t) = \sum_{i=1}^{a-1} \binom{a-2}{i-1} \frac{(-1)^{i-1}}{it} \quad \frac{1}{a-1} \sum_{i=1}^{a-1} \binom{a-1}{i} \frac{(-1)^{i-1}}{t} = \frac{1}{(a-1)t}$$

and this agrees, of course, with the weighting function $g_*(t, t) = g(t, 0)$.

Fundamentals of Mikusinski's operational calculus

Mikusinski's operational calculus is one of the mathematically rigorous formulations of Heaviside's classical operational calculus.

In the following, the distinction between a function and the value of that function at a particular instant t will be of a special importance. The first will be denoted f *or* $f(t)$; the latter will be denoted $f[t]$.

Let us consider the set G of continuous functions $f, g, \ldots$ defined over the interval $0 \leqq t < \infty$. If the addition of functions $f+g$ and the product of a function f by a number c are defined in the conventional manner, and the relationship

$$(f*g)[t]=\int_0^{[t]} f(\vartheta)g(t-\vartheta)\,\mathrm{d}\vartheta \tag{38}$$

is used to introduce a convolution "product" $f*g$ (convolution for short),

$$f*g=(f*g)(t)=\int_0^{t} f(\vartheta)g(t-\vartheta)\,\mathrm{d}\vartheta \tag{39}$$

then the set of functions G may be said to constitute an (algebraic) ring. A ring is a set wherein addition and multiplication are defined, analogous in their properties to the equivalent operations on integers: addition is commutative, associative and inversible (that is, subtraction is defined); multiplication is associative; and finally, for addition and multiplication the distributive law is valid. Any of the above-defined operations performed on elements of a ring will furnish functions that are themselves elements of the ring. Clearly, if $f \in G$ and $g \in G$, then $f*g \in G$. A further property is the commutativity of convolution:

$$f*g=g*f$$

that is, G is a commutative ring, also called a convolution ring.

The equation $f*g \equiv 0$ can be shown to imply that either f or g or both are identically zero (Titchmarsh's theorem). The ring G has, then, no divisor of zero. Hence, by one of the fundamental theorems of abstract algebra, it may be uniquely extended into a quotient field—this means that it is possible to define an operation of division, a convolution quotient. (This extension is similar to the one involved in deriving rational numbers from integers.) Let H denote the quotient field, that is, the set of all convolution quotients. (A field is an algebraic structure—a set—in which the four basic operations are defined, and the usual identities concerning these operations hold.) Denoting the elements of the quotient field H by f/g, the so-called Mikusinski operator is obtained. This operator is identical with a function h if and only if the convolution integral equation $g*h=f$ has a continuous solution h.

The operators thus defined include—as we shall see—the impulse (Dirac delta) defined for any $t \geqq 0$ as well as any of its derivatives and also the unit-step function.

Let us list below the rules concerning these operators. Let $f \in G$, $g \in G$, $f_1 \in G$, $g_1 \in G$; then,

(a) $$\frac{f}{g} = \frac{f_1}{g_1} \quad \text{if and only if} \quad f * g_1 = f_1 * g \tag{40}$$

(b) $$\frac{f}{g} = \frac{f_1}{g_1} = \frac{f * g_1 + f_1 * g}{g * g_1} \tag{41}$$

(c) $$c\frac{f}{g} = \frac{cf}{g} \quad (c \text{ a number}) \tag{42}$$

(d) $$\frac{f}{g} * \frac{f_1}{g_1} = \frac{f * f_1}{g * g_1} \tag{43}$$

(e) $$\frac{f/g}{f_1/g_1} = \frac{f * g_1}{g * f_1} = \frac{f * g_1}{f_1 * g} \tag{44}$$

(f) $$\frac{f * g}{g} = f \qquad (g \not\equiv 0). \tag{45}$$

This last rule embeds ring G into a quotient field H for any g. The operator

$$\frac{g}{g} = \delta \qquad \forall g \in G, \qquad g \neq 0 \tag{46}$$

is independent of the choice of g. The operator δ is the unity element of the quotient field, since

$$\delta * g = g * \delta = g, \ \forall g \in G. \tag{47}$$

It is therefore called also unity operator, in addition to its more conventional names of Dirac delta and unit impulse. By the sifting property of operator δ,

$$(\delta * f)[t] = (f * \delta)[t] = \int_0^{[t]} \delta(\vartheta) f(t - \vartheta) \, \mathrm{d}\vartheta =$$

$$= \int_0^{[t]} f(\tau) \delta(t - \tau) \, \mathrm{d}\tau = f[t] \tag{48}$$

which yields an expression similar to Eq. (47):

$$\delta * f = f * \delta = f. \tag{49}$$

It is noteworthy that

$$cf = c\delta * f. \tag{50}$$

Multiplication by a number c can, then, be reduced to convolution by the operator $c\delta$.

Let $1(t)\in G$ denote the unit-step function. (The notation $l=l(t)$ is also often employed but we shall retain $1=1(t)$.) By

$$1*f=\int_0^t 1(\vartheta)f(t-\vartheta)\,\mathrm{d}\vartheta=\int_0^t f(\tau)\,\mathrm{d}\tau$$

and

$$f*1=\int_0^t f(\tau)1(t-\tau)\,\mathrm{d}\tau=\int_0^t f(\tau)\,\mathrm{d}\tau \tag{51}$$

the function $1*f=f*1$ furnishes the definite integral of function t as a function of the upper limit t. In this sense, $1=1(t)$ is the integral operator. Clearly,

$$1*1=\int_0^t 1(\vartheta)1(t-\vartheta)\,\mathrm{d}\vartheta=t \tag{52}$$

yields the function $t\in G$. Denoting the convolution product $1*1$ by 1^2, we get in general

$$1^n=\frac{1}{(n-1)!}\,t^{n-1}=\frac{1}{\Gamma(n)}\,t^{n-1}. \tag{53}$$

The inverse element within the field H of the integral operator 1 is

$$s=1^{-1}=\frac{g}{g*1}=\frac{g}{1*g}, \qquad g\in G,\ g\not\equiv 0. \tag{54}$$

Hence

$$g*1*s=s*1*g=g \tag{55}$$

or, stated in another way,

$$1*s=s*1=\delta \tag{56}$$

that is,

$$s=\frac{\delta}{1}. \tag{57}$$

It would seem obvious to interpret the operator s as a differential operator. However, this would not be quite correct. Let $f\in G$ be a function of continuous derivative: hence, $\dot{f}\in G$ is an integrable function, and the value of f is

$$f[t]=f[0]+\int_0^{[t]}\dot{f}(\tau)\,\mathrm{d}\tau. \tag{58}$$

We shall hereinafter identify 0 with $0-$. Introducing the integral operator, and passing from function values to functions, we get

$$f=f[0]1+1*\dot{f}. \tag{59}$$

Multiplying by the operator s and making use of Eqs. (56) and (49), we now obtain

$$\delta*f=f[0]\delta+\dot{f}=\dot{f}+f[0]\delta. \tag{60}$$

The product $s*f$ will provide the derivative function if and only if the value of f at $t=0$ is $f[0]=0$, that is, if the function is not discontinuous there. If it is, then the derivative is to be complemented with the operator $f[0]\delta$. The convolution $s*f$ will not, then, always provide an ordinary function; on the contrary, the result may include a term with a generalized function.

As a generalization of the above relationship it can be shown that, if the function f is differentiable n times, and if its nth derivative $f^{(n)}$ exists and is integrable, then

$$s^n*f=f^{(n)}+f^{(n-1)}[0]\delta+f^{(n-2)}[0]s+\ldots+f[0]s^{n-1} \tag{61}$$

where s^n*f will in general include generalized-function components (δ clearly plays the role of s^0 here). On the other hand, considering any (not necessarily differentiable) function f or convolution quotient, we find that s^n*f is itself a convolution quotient. It is usual to call the generalized function s^n*f the generalized nth derivative of f.

Example 7. Let $f(t)=e^{at}$. Then, by Eq. (60)

$$s*e^{at}=ae^{at}+\delta$$

whence

$$e^{at}=\frac{\delta}{s-a\delta}=(s-a\delta)^{-1}.$$

The operator $\delta/(s-a\delta)=(s-a\delta)^{-1}$ then coincides with the function exp at. Similarly,

$$\overset{--}{(s-a\delta)^{-n}}=\frac{t^{n-1}}{(n-1)!}e^{at}.$$

Example 8. Let us solve the differential equation

$$\ddot{x}(t)+a^2x(t)=f(t).$$

By Eq. (61)

$$\ddot{x}=s^2*x-\dot{x}[0]\delta-x[0]s.$$

Hence

$$(s^2+a^2\delta)*x=f+x[0]s+\dot{x}[0]\delta$$

and

$$x=\frac{\delta}{s^2+a^2\delta}*f+\frac{x[0]s+x[0]\delta}{s^2+a^2\delta}.$$

Since

$$\frac{\delta}{s^2+a^2}=\frac{1}{a}\sin at$$

$$\frac{s}{s^2+a^2}=\cos at$$

and further

$$\frac{f}{s^2+a^2\delta}=\frac{1}{a}\int_0^t f(\vartheta)\sin a(t-\vartheta)\,\mathrm{d}\vartheta$$

he required solution is the function

$$x(t)=x[0]\cos at+x[0]\frac{1}{a}\sin at+\frac{1}{a}\int_0^t f(\vartheta)\sin a(t-\vartheta)\,\mathrm{d}\vartheta.$$

For instance, for the impulse $f(t)=\delta(t)$

$$x(t)=x[0]\cos at+(\dot{x}[0]+1)\frac{1}{a}\sin at.$$

Remark. The results of Mikusinski's operational calculus are conspicuously similar to those of the Laplace transform. This similarity can be increased further by writing the convolution product $f*g$ in the form fg, and attributing, owing to $cf=$ $=c\delta*f=c\delta f$, the value 1 to the operator δ. In other words, multiplication shall mean convolution if the factors are two functions, and ordinary multiplication if one or both of the factors are a number. The similarity to the Laplace transform is not accidental, but a consequence of the fact that, for any integrable function f, the integral

$$\int_0^\infty e^{-st}f(t)\,\mathrm{d}t$$

exists and is equal to f. Here s is an operator (and not a complex number): Remarkably, it is not necessary here to make any assumption as to the rate of increase of f (that is, Eq. (21) can be dropped).

Supplementary remarks

In classical operational calculus, the differential operator d/dt is replaced by the symbol p and it is regarded from then on as a simple algebraic quantity. For instance for time-invariant systems

$$y(t)=\frac{B(p)}{A(p)}u(t). \tag{62}$$

If the input is $u(t)=1(t)$, and the output is assumed to be of the form $y(t)1(t)$, then the form

$$y(t)=\frac{B(p)}{A(p)} \tag{63}$$

will result. For zero initial conditions referring to the instant $t=0-$, the time function can be obtained e.g. for single poles, by Heaviside's expansion theorem:

$$y(t)=\frac{B(0)}{A(0)}+\sum_{i=1}^{n}\frac{B(p_i)}{p_iA'(p_i)}e^{p_it} \tag{64}$$

where p_i is the root of the characteristic equation $A(p)=0$ and, furthermore, $A'(p_i)=$ $=\mathrm{d}A(p)/\mathrm{d}p|_{p=p_i}$.

The deviation from the expansion theorem (26) of the Laplace transform is worthy of notice. It is understandable inasmuch as the inverse Laplace transform should be derived by $F(s)=B(s)/sA(s)$, since $1/s$ is the transform of $1(t)$. Of course, the time functions coincide.

Owing to the circumstances just pointed out, Heaviside's operational calculus is more closely related to the Carson-Wagner transform given by

$$\tilde{F}(p)=\mathcal{C}[f(t)]=p\int_0^{\infty}f(t)e^{-pt}\,\mathrm{d}t \tag{65}$$

and

$$f(t)=\mathcal{C}^{-1}[\tilde{F}(p)]=\frac{2}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{\tilde{F}(p)}{p}e^{tp}\,\mathrm{d}p \tag{66}$$

than to the Laplace transform.

The rules of transformation from the Laplace to the Carson-Wagner transform and vice versa can be expressed as

$$F(s)=\frac{\tilde{F}(p)}{p}\bigg|_{p=s};\qquad \tilde{F}(p)=sF(s)|_{s=p}. \tag{67}$$

Let us point out finally that in distribution theory it is usual in certain cases to introduce the so-called multitudes of distributions. In such cases, the operator s and its

powers are replaced by the operator δ and its powers. For instance, the operational impedance $Z(s)=Ls+R+1/Cs$ is replaced by the impedance operator $Z=L\delta'+R\delta+1/C$, where $I'=\delta$. For details, the reader is referred to literature.

Summary

This paper contains basic information about the Fourier transform, the two-sided and one-sided Laplace transforms, Mikusinski's operational calculus and Heaviside's classical operational calculus, about the Carson–Wagner transform and finally about multitudes of distributions and operators. The similarities and differences between the relevant concepts and methods are pointed out.

A few illustrative examples show some pecularities of the methods treated.

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