EXAMPLES, COUNTER-EXAMPLES AND APPLICATIONS TO THE THEORY OF OPERATOR TRANSFORMATIONS

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Received December 28, 1976.
Presented by Prof. DR. T. FREY

This note is closely connected to some previous works ([1] [2]), and also the notations are the same as in them. This paper consists of four parts. First a necessary (but applicable) condition for semi-groups of endomorphisms will be given, the second part contains some counter-examples and a new version of extention of operator transformations. In the third section applications will be presented. The last part is devoted to the applications of the semi-group theory and the Cauchy-problem.

1.

Let Φ be a linear operator transformation with $\mathcal{D}(\Phi)\subset M$. Since M and Q are isomorphic and the isomorphism is continuous, there is a linear subspace $Q_0\subset\mathcal{D}(\tilde{\Phi})$, and the mapping $\tilde{\Phi}$ corresponding to Φ which maps Q_0 into Q. If Φ is an isomorphism, then $\tilde{\Phi}$ is, too. When $a\in Q_0$, then $\tilde{\Phi}(a)=\{\varphi_n(p)\}$ is a field sequence, and if Φ depends on a parameter, so does $\tilde{\Phi}$; $\tilde{\Phi}=\tilde{\Phi}^\alpha$ and $\tilde{\Phi}^\alpha(a)=\{\varphi_n(p,\alpha)\}$. If Φ^α depends continuously on the parameter, then $\{\varphi_n(p,\alpha)\}$ represents a continuous operator function. In the special case when Φ^α is a semi-group (or a group — see [1]), $\tilde{\Phi}^\circ(a)=$ $=\{a_n(p)\}=\{\varphi_n(p,0)\}$ and $\tilde{\Phi}^{\alpha+\beta}(a)=\{\varphi_n(p,\alpha+\beta)\}$. Assume that Φ^α is a continuous endomorphism of M for each α, then by representation theorem (see [2], [9])

$$\Phi^\alpha(\varphi)=\int_0^\infty \exp(-\lambda\Phi^\alpha(s))\varphi(\lambda)\,d\lambda \tag{1.1}$$

for each $\varphi\in C^0$. From here one can see at once that it is "enough" to know the operator $\Phi^\alpha(s)$. Let us assume that $\tilde{\Phi}^\alpha(p)$ can be representated by a single meromorphic function (i.e., $\tilde{\Phi}^\alpha(p)$ has an n-independent representation) for each $\alpha\geqq 0$. If Φ^α is a continuous semi-group (every Φ^α is an endomorphism but $\{\Phi^\alpha\}$ forms a semi-group only on $\mathcal{D}(\Phi^\alpha)$), then

$$\frac{d\Phi^\alpha(\varphi)}{d\alpha}=A_0\Phi^\alpha(\varphi) \tag{1.2}$$

for each $\varphi \in \mathcal{D}(\Phi^\alpha)$. (See [1].) Let $\tilde{\Phi}^\alpha(\varphi)=\{\varphi_n(\omega(p,\alpha))\}$ be a field sequence representation of $\Phi^\alpha(\varphi)$ in $\operatorname{Re}(p)\geqq\sigma_0>0$. Hence we have that

$$\frac{d\varphi_n(\omega(p,\alpha))}{d\alpha}=[\tilde{A}_0\{\varphi_n(\omega(p,\alpha))\}]_n \tag{1.3}$$

where $[.]_n$ is the n-*th* member of the field sequence. Since $\tilde{A}_0$ is linear, it follows that $[\tilde{A}_0\{\varphi_n(\omega(p,\alpha))\}]_n=\tilde{A}_0[\varphi_n(\omega(p,\alpha))]$; where $\tilde{A}_0$ is the correspondence of A_0 on the Ditkin-Berg model. If $\omega(p,\alpha)$ has the derivative with respect to α, then from (1.3)

$$\varphi_n'(\omega(p,\alpha))\cdot\omega_\alpha'(p,\alpha)=\tilde{A}_0(\varphi_n(\omega(p,\alpha))) \tag{1.4}$$

follows. At $\alpha=0$:

$$\varphi_n'(\omega(p,0))\cdot\omega_\alpha'(p,0)=\tilde{A}_0(\varphi_n(\omega(p,0))),$$

and by virtue of $\Phi^0=I$:

$$\varphi_n'(p)\cdot\omega_\alpha'(p,0)=\tilde{A}_0(\varphi_n(p)).$$

We can conclude that $A_0=\omega_\alpha'(s,0)D$, where $\omega_\alpha'(s,0)$ is the operator from M which has the correspondence $\omega_\alpha'(p,0)$ in Q and D is the operation of algebraic derivation.

Let us consider the operator function $\Phi^\alpha(\exp(-\lambda s))$. $\Phi^\alpha(\exp(-\lambda s))=$ $=\exp(-\lambda\Phi^\alpha(s))=\exp(-\lambda\cdot\omega(p,\alpha))$. Using the previous results and assumptions:

$$-\lambda\exp(-\lambda\cdot\omega(p,\alpha))\frac{\partial\omega(p,\alpha)}{\partial\alpha}=\frac{\partial\omega(p,\alpha)}{\partial\alpha}\bigg|_{\alpha=0}\frac{d}{dp}(\exp(-\lambda\cdot\omega(p,\alpha)),$$

hence

$$\frac{\partial\omega(p,\alpha)}{\partial\alpha}=\omega_\alpha'(p,0)\frac{\partial\omega(p,\alpha)}{\partial\alpha}. \tag{1.5}$$

Summarizing, we obtained

Theorem I. Let $\{\Phi^\alpha|\alpha\geqq0\}$ be a continuous semi-group of continuous endomorphisms on $D(\Phi^\alpha)\subset M$. Assume $s\in D(\Phi^\alpha)$ and $\tilde{\Phi}^\alpha(p)=\omega(p,\alpha)$ is continuously differentiable with respect to α. Then the infinitesimal generator of the semi-group is $\omega_\alpha'(s,0)D$ where $\omega_\alpha'(s,0)$ is the operator from M which is representated by $\omega_\alpha'(p,0)$ in the Ditkin–Berg model, moreover $\omega(p,\alpha)$ satisfies equation (1.5) with the initial data $\omega(p,0)=p$.

An example: Let $\omega(p)=(p^r+\beta)^{\frac{1}{r}}$, where $r>0$ and β is real. In this formula $p^r=\exp(r\cdot\log|p|+r\cdot i\cdot\operatorname{arc}p)$ with $0\leqq\arg p\leqq2\pi$. Since $\omega(p)=p+0(|p|)$ as $\operatorname{Re}(p)\to\infty$ in $\operatorname{Re}(p)\geqq\sigma_0>0$, therefore $\omega(p)$ represents a continuous endomorphism — a substitution mapping (see [2]) —. It is easy to see that the transformations ${}^r\Phi^\beta=\Phi\sqrt[r]{p^r+\beta}$ form a group when β is running over the reals and r is fixed.

Now it will be shown that ${}^r\Phi^\beta$ maps D'_+ into D'_+.* Indeed, if $x\in D'_+$, then $x=\{\chi_n(p)\}$, where

$$\chi_{n+1}(p)=\chi_n(p)+p^{l_n}\cdot 0\,(\exp(\gamma-n)p), \tag{1.6}$$

as $\mathrm{Re}(p)\to\infty$, for $\mathrm{Re}(p)\geqq c>0$, $n=0, 1, \ldots$, where $\chi_0(p)\equiv 0$, γ is a real number independent of n, l_n is a positive integer and $\chi_n(p)$ is analytic in $\mathrm{Re}(p)\geqq c>0$ for each n. (See [5].) If $l_n=l_{n_0}$ for $n\geqq n_{n_0}$ then x is a distribution in finite order. Since ${}^r\Phi^\beta(x)=\{\chi_n(\sqrt[r]{p^r+\beta})\}$, we obtain from (1.6) that

$$\chi_{n+1}(\sqrt[r]{p^r+\beta})=\chi_n(\sqrt[r]{p^r+\beta})+(p^r+\beta)^{\frac{l_n}{r}}0(e^{(\gamma-n)\sqrt[r]{p^r+\beta}}), \tag{1.7}$$

as $\mathrm{Re}(p)\to\infty$. Because $r>0$ and $\omega(p)=p+0(|p|)$, from (1.7) we have

$$\chi_{n+1}(\sqrt[r]{p'+\beta})=\chi_n(\sqrt[r]{p^r+\beta})+p^{l_n}0(e^{(\gamma-n)p}). \tag{1.8}$$

Since a representation with (1.6) is necessary and sufficient for $x\in M$ to be in D'_+, (1.8) proves the assertion.

It must be noted that our proof works for any continuous endomorphism Φ for which $\Phi(s)=\omega(p)=cp+0(|p|)$, $c>0$, as $\mathrm{Re}(p)\to\infty$.

Let us fix x from D'_+ and also $r>0$ be fixed. Then ${}^r\Phi^\beta(x)=g(\beta)\in C_\infty[(-\infty,\infty)]M$. Indeed, by (1.8):

$$\frac{\chi_n(\sqrt[r]{p^r+\beta})}{e^{\sqrt{p}}}+0(e^{-np})=\chi_{n+1}(\sqrt[r]{p^r+\beta})$$

as $\mathrm{Re}(p)\to\infty$ for each n, hence $\{e^{-\sqrt{p}}\chi_n(\sqrt[r]{p^r+\beta})\}$ defines a continuous operator function, and in fact this function is differentiable in any order with respect to β. (Since if $x\in D'_+$ then $e^{-\sqrt{s}}\cdot x\in C^0$, see [5].)

It is easy to check whether the infinitesimal generator of the group ${}^r\Phi^\beta$ is $\frac{1}{r}l^{r-1}D$. A familiar argument shows that $\{{}^r\Phi^\beta\}$ is a strongly continuous group on D'_+. (See [1], [10]). Let us mention two special cases of $r=1$ and $r=2$;

$${}^1\Phi^\beta(a)=T^{-\beta}(a)=\{e^{-\beta t}a(t)\}, \tag{1.9}$$

$${}^2\Phi^\beta(a)=\left\{\int_0^t J_0(\sqrt{\beta(t^2-\lambda^2)})a(\lambda)\,d\lambda\right\}\sqrt{p^2+\beta} \tag{1.10}$$

* *Footnote:* D'_+ is the set of distributions with half-line support. For the embedding of D'_+ into M see [5].

if $a \in C^0$. Since, for $i\beta=\gamma$, $J_0(\beta(\sqrt{t^2-\lambda^2}))=I_0(\sqrt{\gamma(t^2-\lambda^2)})$ (1.10) holds for each real β. The endomorphisms ${}^r\Phi^\beta$ are bijective as it can easily be checked, and ${}^r\Phi^\beta\ {}^r\Phi^{-\beta}$ are inverse transformations for fixed r and β.

Let us consider the operator function differential equation

$$y^{(n)}(\lambda)=\left(\frac{1}{r}\,l^{r-1}D\right)^n(y(\lambda)) \tag{1.11}$$

with initial data $y^{(k)}(0)=y_k$, $k=0, 1, \ldots, (n-1)$. By theorem 4.8 of [1], we have obtained that (1.11) has unique solution in $B^{(n)}(M)$ — see for the definition [1] — if $y_k \in D'_+$ (See for application (3.8)).

2.

If $\omega(p)=o(|p|)$ as $\mathrm{Re}(p)\to\infty$ in $\mathrm{Re}(p)>\sigma_0$ and here $\mathrm{Re}(\omega(p))\geqq\sigma_0\geqq 0$, then the $\omega(p)$-substitution transformation does not exist for each $x\in Q$ in general. There is an operator $\omega(p)=o(|p|)$ and $x\in Q$ such that by the formal $\omega(p)$-substitution $x(\omega(p))$, does not present an operator, although x has an n-independent representation. We encounter this case, for example, when $\omega(p)=p^{-1}$, which corresponds to the Hankel transformation (see [7]); let now $x=p^{-2}\exp(-p^{-2})$ (it is a function from C^0), then $x(\omega(p))=p^2\exp(-p^2)\notin(MR)\subset Q$ (see [5]). Similar examples can be constructed for $\omega(p)=o(|p|^\alpha)$, $\alpha\geqq 0$.

There are examples when $w(p)$ defines an isomorphism of (MR), i.e. for each $\sigma_0>0$ there is $\sigma_1>0$ such that $\mathrm{Re}(\omega(p))>\sigma_0$ for $\mathrm{Re}(p)>\sigma_1$, but it cannot be extended to the whole. It can be shown that if $w(p)=\sqrt{p}$, then $f(t)=\exp(\exp t^2)$ cannot be $w(p)$-t transformed. (See [10].)

Now we shall show another possibility of making extension of the above kind of transformations. Let $\mathcal{M}$ be the set of all complex functions f which are defined on some right half plane, $\mathrm{Re}(p)>\sigma_f$, and meromorphic there. $\mathcal{H}$ will stand for the set of all holomorphic functions defined on a right half plane. If f_1 and f_2 belong to $\mathcal{M}$, then $f_1=f_2$ iff there exists a right half plane where $f_1(p)\equiv f_2(p)$. By the theorem of Mittag–Leffler and Weierstrass any $f\in\mathcal{M}$ can be written in a ratio of two functions from $\mathcal{H}$. We say that $f_n(p)\in\mathcal{H}$ tends to $f(p)\in\mathcal{H}$ if there exists a right half plane Ω such that $f_n(p), f(p)$ are holomorphic there and $f_n(p)\to f(p)$ uniformly on any compact subset of Ω. Let C_0^∞ denote the set of perfect functions, i.e.: $r(t)\in C_0^\infty$ if $r(t)$ is differentiable in any order, $r(t)=0(e^{ct})$ for some $c>0$ as $t\to\infty$, $r^{(k)}(t)\in C^0$ (for $k=0, 1, 2, \ldots$) and $r^{(k)}(0)=0$ for each k. It can be proved that the quotient field* Q_0 of C_0^∞ is isomorphic to (MR). (See [11], [12]). If $x\in C_0^\infty$, then $\bar{x}(p)$ denotes its Laplace transform.

* with respect to the convolution product.

Let $x \in Q_0$, and $x=\frac{\{r(t)\}}{\{q(t)\}}$, then $\bar{x}(p)=\frac{\bar{r}(p)}{\bar{q}(p)} \in (MR)$. The sequence $x_n \in Q_0$ is called *fundamental sequence* if there is a representation of x_n, $x_n=\frac{r_n}{q_n}$ such that $\bar{r}_n, \bar{q}_n \in \mathcal{H}$, $\lim_{n\to\infty} \bar{r}_n(p)=\bar{r}(p)$ and $\lim_{n\to\infty} \bar{q}_n(p)=\bar{q}(p)$ in $\mathcal{H}$. Obviously $\bar{x}(p)=\frac{\bar{r}(p)}{\bar{q}(p)} \in \mathcal{M}$. The function $\bar{x}(p)$ is said to be the *Laplace transform of the fundamental sequence* $\{x_n\}$ and $\mathcal{L}(\{x_n\})= =\bar{x}(p)$. Two fundamental sequences are said to be *equivalent* if they have the same Laplace transform. The set of the equivalence classes will be denoted by $\mathcal{A}$ and the elements of $\mathcal{A}$ are called *hyper-functions*. The following theorem holds (see [11]): *Theorem II. Let* $A=\{x_n\} \in \mathcal{A}$. *Then the mapping* $\mathcal{L}: \mathcal{A} \to \mathcal{M}$ *defined by* $\mathcal{L}(A)=A(z)= =\mathcal{L}(\{x_n\})$ *is an isomorphism between* $\mathcal{A}$ *and* $\mathcal{M}$. *If* $r \in C_0^\infty$ *then* $I=\left\{\frac{r}{r}\right\} \in \mathcal{A}$ *is the operator of identity and for* $x \in Q_0$ $X=\left\{x \cdot \frac{r}{r}\right\} \in \mathcal{A}$.

In the beginning of this section we saw transformations which cannot be extended to Q_0, i.e. there exists $x \in Q_0$ such that the image of x — by the formal $\omega(p)$-substitution — is a meromorphic function not belonging to (MR). It can be seen that $x(\omega(p)) \in \mathcal{M}$, i.e. it is a hyper-function. One can show that if $x=r/q$, r, $q \in C_0^\infty$ and if

$$r_n=\begin{cases} r(t) & \text{if } 0 \leq t < n, \\ 0 & \text{if } \quad t > n. \end{cases}$$

and

$$q_n=\begin{cases} q(t) & \text{if } 0 \leq t < n \\ 0 & \text{if } \quad t > n \end{cases},$$

then

$$\Phi^\omega(x)=\bar{x}(\omega(p))=\frac{\bar{r}(\omega(p))}{\bar{q}(\omega(p))}$$

and $\lim_{\to\infty} \bar{r}_n(\omega(p))=\bar{r}(\omega(p))$, $\lim_{\to\infty} \bar{q}(\omega(p))=\bar{q}(\omega(p))$ in $\mathcal{H}$.

An example. Consider the integral equation

$$F(t)=G(t)+\lambda \int_0^\infty \frac{\cos 2\sqrt{tx}}{\sqrt{t}} F(x)\, dx \tag{2.1}$$

where $G(t)$ is a given function. In the operator term (2.1) becomes

$$F=G+\lambda \int_0^\infty \sqrt{\frac{\pi}{s}} \cdot \exp\left(-\frac{x}{s}\right) F(x)\, dx. \tag{2.2}$$

Introducing the transformation $\Phi^{\frac{1}{s}}$

$$F=G+\lambda\sqrt{\frac{\pi}{s}}\,\Phi^{\frac{1}{s}}(F). \tag{2.9}$$

Since $\Phi^{\frac{1}{s}}(\Phi^{\frac{1}{s}}(F))=F$, we obtain

$$\Phi^{\frac{1}{s}}(F)=\Phi^{\frac{1}{s}}(G)+\lambda\sqrt{\pi}\Phi^{\frac{1}{s}}\left(\frac{1}{s}\right)\cdot F, \tag{2.4}$$

Therefore

$$F=\frac{1}{1-\pi\lambda^2}\left[G+\lambda\sqrt{\frac{\pi}{s}}\,\Phi^{\frac{1}{s}}(G)\right],$$

hence in case $\lambda\neq\pm\frac{1}{\sqrt{\pi}}$

$$F(t)=\frac{1}{1-\pi\lambda^2}\left[G(t)+\lambda\int_0^\infty\frac{\cos 2\sqrt{tx}}{\sqrt{t}}\,G(x)\,\mathrm{d}x\right] \tag{2.5}$$

if the integral exists. Take $\{G(x)\}=\{1\}=l$, then the solution F is not a function, and

$$F=\frac{1}{1-\pi\lambda^2}\,(l+\lambda\sqrt{\pi s}),$$

is a distribution in finite order. Take, now, $\{G(x)\}=s^{-2}\exp(-s^{-2})$, then the solution of (2.1) is a hyper-function, since $F(p)\in\mathcal{M}$,

$$F=\frac{1}{1-\pi\lambda^2}\,(s^{-2}\exp(-s^{-2})+\lambda s\sqrt{\pi s}\exp(-s^2)),$$

but $F(p)\notin(MR)$.

3.

In this section some applications will be given. Let us consider the integral equation

$$f(t)=\int_0^t J_0(\sqrt{2u(t-u)})g(u)\,\mathrm{d}u \tag{3.1}$$

where $f(t)$ is a given function. Using the endomorphism $\Phi^{p+\frac{1}{p}}$, (3.1) can be written in operator term as follows:

$$\Phi^{p+\frac{1}{p}}(g)=p\{f(t)\}.$$

Since $\Phi^{p+\frac{1}{p}}$ is invertible on M, the inverse transformation is $\Phi^{\omega(p)}$ where

$$\omega(p)=\frac{1}{2}(p+\sqrt{p^2-4}),$$

and thus we get the solution

$$g=\frac{1}{2}(p+\sqrt{p^2-4})f\left(\frac{1}{2}(p+\sqrt{p^2-4})\right)$$

in operator term. Let us study the solution, assuming $f'(t)\in C^0$ and $f(0)=0$. Then, by $p\{f(t)\}=f'+f(0)$, we obtain

$$g=(p^2-4)\left\{\int_0^t f'(u)\int_{\frac{1}{2}u}^{t-\frac{1}{2}u} I_0\left(2\left(t-\frac{1}{2}u-v\right)\right)I_0\left(2\sqrt{v^2-\left(\frac{1}{2}u\right)^2}\right)dv\,du\right\}=$$

$$=\left\{\frac{d^2}{dt^2}\int_0^t f'(u)\int_{\frac{1}{2}u}^{t-\frac{1}{2}u} I_0\left(2\left(t-\frac{1}{2}u-v\right)\right)I_0\left(2\sqrt{v^2-\left(\frac{1}{2}u\right)^2}\right)dv\,du\right\}-$$

$$-4\left\{\int_0^t f'(u)\int_{\frac{1}{2}u}^{t-\frac{1}{2}u} I_0\left(2\left(t-\frac{1}{2}u-v\right)\right)I_0\left(2\sqrt{v^2-\left(\frac{1}{2}u\right)^2}\right)dv\,du\right\}\in C^0.$$

If $f'(t)\in C^0$, but $f(0)\neq 0$, then the solution $g=\Phi^{\omega}(f')+f(0)$ is a distribution wihch is not a function.

The kernel function of equation (3.1) is a zero order Bessel function in first kind, if we use the transformation $\Phi^{p+\frac{1}{p}}$, and then we can solve the equation similarly:

$$f(t)=\int_0^t\left(\frac{t-v}{v}\right)^{\frac{1}{2}\nu}J_\nu(2\sqrt{v(t-v)})g(v)\,dv \tag{3.2}$$

where J_ν is the ν-th order Bessel function in first kind, with $\mathrm{Re}(\nu)>-1$. Indeed, using the relation

$$\frac{e^{-v\left(p+\frac{1}{p}\right)}}{p^{\nu+1}}=\begin{cases}0 & \text{if}\quad 0\leqq t<v\\ \left(\dfrac{t-v}{v}\right)^{\frac{1}{2}\nu}J_\nu(2\sqrt{v(t-v)}) & \text{if}\quad t>v\end{cases}$$

(3.2) can be written in operator term:

$$\Phi^{p+\frac{1}{p}}(g)=p^{\nu+1}\{f(t)\}.$$

The same method as it was treated for (3.1) can be applied for (3.2).

Applying the transformation $\Phi^{\sqrt{p^2+\nu^2}}$ we could solve

$$\int_0^t J_0(v(t^2-u^2)^{-\frac{1}{2}})g(u)\,du=f(t), \tag{3.3}$$

$$\int_0^t I_0(v(t^2-u^2)^{-\frac{1}{2}})g(u)\,\mathrm{d}u=f(t). \tag{3.4}$$

Having different representations of $\exp(-u(p^2+v)^{\frac{1}{2}})$ we can solve integral equations similar to (3.3) and (3.4).

Consider the following integro-differential equation

$$\frac{\partial y(v,t)}{\partial v}+\frac{1}{2}\int_0^t xy(v,x)\,\mathrm{d}x=0. \tag{3.5}$$

The equation can be written in operator form as follows:

$$y'(v)=\tfrac{1}{2}lD(y(v)). \tag{3.6}$$

Since $\frac{1}{2}lD$ is the infinitesimal generator of the transformation semi-group $\Phi^{\sqrt{p^2+v}}$, having the initial value at $v=0$, $Y_0=y(0)$, the solution of (3.6) in $D'_+\subset D(\Phi)$ is

$$y(v)=\Phi^{\sqrt{p^2+v}}(Y_0)=(p^2+v)^{\frac{1}{2}}\left\{\int_0^t J_0((v(t^2-u^2))^{\frac{1}{2}})Y_0(u)\,\mathrm{d}u\right\},$$

assuming $Y_0\in L_{\mathrm{loc}}$. By an easy computation, from (3.5) one can get

$$\frac{\partial^2}{\partial v\partial t}(y(v,t))+t\frac{1}{2}y(v,t)=0. \tag{3.7}$$

Put $y(0,t)=a(t)$, $y'_v(v,0)=z(v)$ or $y(v,0)=z_1(v)$. Assume that $Y(v)$ is a solution of (3.6) with $\{Y(0,t)\}=Y(0)=\{a(t)\}$, then the solution of (3.7) is $y(v)=Y(v)+z_1'(v)$. Indeed, integrating (3.7) with respect to t from 0 to t

$$y'_v(v,t)-y'_v(v,0)+\frac{1}{2}\int_0^t xy(v,x)\,\mathrm{d}x=0$$

follows and by $y'_v(v,0)=z_1'(v)$ we have that

$$Y'(v)=\frac{1}{2}lD(Y(v))-\frac{1}{2}lD(z_1'(v))=\frac{1}{2}lD(Y(v)),$$

since

$$\frac{1}{2}lD(z_1'(v))=z_1'(v)\frac{1}{2}lD(1)=0.$$

If y_0 is assumed to be Laplace-transformable $y(0)=y_0=\{y_0(t)\}$, $y(0)=y^{\wedge}(p)$ then the solution is

$$y(v,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}y^{\wedge}((p^2+v)^{\frac{1}{2}})\mathrm{d}p.$$

Here we should take care of certain conditions related to the differentiability of $y(v, t)$, but it can be found by a familiar argument.

Similar investigations can be made for the integro-differential equation

$$y'_v(v, t) = -\frac{1}{2}\int_0^t \frac{(t-x)^{r-1}}{\Gamma(r)} y(v, x)x\, dx \tag{3.8}$$

for $r>0$, $-\infty<v<\infty$ and $t\geqq 0$. If r is not an integer, (3.8) cannot be reduced to a partial differential equation. The solution of (3.8) — compared with the equation (1.11) — is

$$y(v, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt} y_0^{\wedge}((p^r+v)^{1/r})\, dp,$$

which is Laplace-transformable, whenever $y_0 \in C^0$; if y_0 is not Laplace-transformable, then the solution exists also but theorem IV of [2] must be used. If $y_0 \in D_+$, then the solution is similarly given by the same formula (see [2], theorem IV, (2.11)).

4.

One of the most familiar operations in classical analysis having the semi-group property is the Riemann-Lioville integral of fractional order. In the classical case we have the following: let X be the Lebesgue space L([0, 1]), and let $\mathrm{Re}(v)>0$ and form

$$J^v(f) = \frac{1}{\Gamma(v)}\int_0^t (t-u)^{v-1} f(u)\, du. \tag{4.1}$$

It is well-known that $J_v(f) \in L([0, 1])$, and has the semi-group property in the righ half-plane, we also have

$$J_v = \sum_{n=0}^{\infty} (J-I)^n \binom{v}{n}, \qquad J = J^1$$

and a theorem of Hille shows that $f(t) \in \mathcal{D}(A)$ if there is an α, $0\leqq\alpha\leqq 1$ such that $J^\alpha(f)$ is an absolutely continuous function of t. For such an $f(t)$ we have

$$A(f) = d/dt \int_0^t K_\alpha(t-u) f_\alpha(u)\, du, \tag{4.2}$$

where $K_\alpha(t) = 1/\Gamma(1-\alpha)t^{-\alpha}(\log t - \Psi(1-\alpha))$, $f_\alpha(t) = J^\alpha(f)$ and $\Psi(.)$. is the logarithmic derivative of the gamma function.

Using the operator calculus, this semi-group can be defined on L_{loc} by

$$J^{\alpha}(f)=l^{\alpha}f \tag{4.3}$$

where α is real and $l=\{1\}$. By the use of the result of Boehme [3] it can be extended to the whole complex plane and it is an analytic strongly continuous group with infinitesimal generator $A=\ln l=s\{\ln t\}+C=s\{\ln \gamma t\}$, where $C=0.577\ldots$ is Euler's constant and $\gamma=eC$. This semi-group, too, has the basis relation, for $\mathrm{Re}(\alpha)>0$ and $f\in C^0$,

$$\{d/dtJ^{\alpha+1}(f)(t)\}=J^{\alpha}(f), \tag{4.4}$$

and the resolvent formula can be given by

$$R(\lambda, J)(f)=\lambda^{-1}f(t)+\lambda^{-2}\int_0^t \exp((t-u)\lambda^{-1})f(u)\,du \tag{4.5}$$

for all $f\in L_{loc}$, or for $x\in M$

$$R(\lambda, J)(x)=\lambda^{-1}x+\lambda^{-2}\{\exp(t/\lambda)\}x. \tag{4.6}$$

Therefore the solution of the singular integro-differential equation

$$y'(\lambda, t)=\mathrm{d}/\mathrm{d}t\int_0^t \ln(\gamma(t-u))y(\lambda, u)\,du \tag{4.7}$$

is given by

$$\{y(\lambda, t)\}=l^{\lambda}y_0, \tag{4.8}$$

where $y_0=y(0)=\{y(0, t)\}$. To investigate the properties of the solution we might use the results of Boehme [3].

Finally we show another example of the Cauchy-problem. Consider the equation

$$y^{(n)}(v)=(D+sD)^n(y(v)),\quad Y_i=y^{(i)}(0)\quad (i=0, 1, \ldots (n-1)) \tag{4.9}$$

where D is the operation of algebraic derivation. It is not too difficult to prove that $D+sD$ is a bounded transformation on D'_+, and using (1.6) one can prove that for any $x\in D'_+$ there are $g_x\in C^0$ and $0\neq q_x\in C^0$ such that

$$\| q_x(D+sD)^n(x)\|_{\Omega}\leqq \| ng_x\|_{\Omega},$$

herefore

$$y(v)=\sum_{k=0}^{n-1}\sum_{m=0}^{\infty}\frac{v^{mn+k}}{(mn+k)!}(D+sD)^{mn}(Y_k) \tag{4.10}$$

is the unique solution of the Cauchy-problem in $C_n(C)D'_+$. (See [1].)

Summary

In the study of the operational calculus the notions of the linear operator transformations play a very important role and have proved very useful. This paper deals with semi-groups of endomorphisms, gives a sufficient condition for an operator to be the infinitesimal generator of a semigroup. Several examples, and applications of this subject can be found in this paper. Also some connections between distribution and operator transformation have been discovered here.

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