ON SEMI-GROUPS OF OPERATOR TRANSFORMATIONS

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Introductory remarks

In Heaviside's operational calculus, in particular in the application of this operational calculus to partial differential equations, difficulties arise as a result of the accurrence of certain operators whose meaning is not obvious. Therefore it became necessary to develop a mathematical theory that will justify the process. One of these theories, the theory of *convolution quotient* is due to the Polish mathematician *Jan Mikusinski*. His theory provides a satisfactory basis for the operational calculus, and it can successfully be applied to ordinary and partial differential equations with constant coefficients, to difference equations, integral equations, and also in some other fields. *E. Gesztelyi* [6] studied the integral representation of linear transformations of the operator field and proved that every continuous operator transformation which is continuous his sense can be realized on the set of continuous functions as an integral transformation. But to decide whether an operator transformation is continuous in *Gesztelyi's* sense is either very difficult or cannot be carried out at all. The case when the operator transformation is multiplicative, i.e. is an endomorphism, is not any simpler.

1. Definitions and notations

Here we give some notations which will be used througouht the paper. Let C^λ denote the set of all complex-valued functions of a real variable t which vanish if $t<\lambda$ and are continuous if $t \geqq \lambda$. The set of locally integrable functions with left sided bounded support is denoted by CU. The quotion field with respect to the convolution product

$$fg(t)=\int_0^t f(t-x)g(x)\,\mathrm{d}x \qquad (\text{for } f, g \in C^0) \tag{1.1}$$

is called operator field and denoted by M. In general we follow the terminology of *Mikusinski*'s book [11], definitions and notions which are not included in, or are different from, [11] are as follows:

1.1. *Definition.* Let $f(\lambda)$ be an operator-valued function on a real variable λ running over the interval Λ. We call $f(\lambda)$ continuous if $f(\lambda)$ has a representation $f(\lambda)=a\{f(\lambda, t)\}$ where $f(\lambda, t)$ is continuous on $\Lambda\times[0, \infty)$; in this case $\{f(\lambda, t)\}\in C(\Lambda)C^0$ and $f(\lambda)\in \in C(\Lambda)M$. The function $f(\lambda)$ is said to be differentiable if there is a representation $f(\lambda)=a\{f(\lambda, t)\}$ such that $\dfrac{\partial f(\lambda, t)}{\partial\lambda}$ exists and belongs to C^0 for each $\lambda\in\Lambda$; then we say that $f(\lambda)\in C_1(\Lambda)M$ and

$$f'(\lambda)=a\left\{\frac{\partial f(\lambda, t)}{\partial\lambda}\right\} \tag{1.2}$$

$C_n(\Lambda)M$ can be defined similarly. $f(z)$ is said to be an analytic operator function in domain S if $f(z)$ can be expressed by $a\{f(z, t)\}$, where $f(z, t)\in C^0$ for all complex $z\in S$, and $f(z, t)$ is an analytic function with respect to the variable z. In this case $f(z)\in A(S)M$.

1.2. *Definition.* An operator transformation $F: M_1\to M$ (M_1 is a subspace of M) is said to be M_1-continuous if the operator function $F(g(\lambda))$ is continuous; $F(g(\lambda))\in V(\Lambda)M$, whenever the operator function $g(\lambda)\in C(\Lambda)M$ and has values in M_1.

1.3. *Definition.* An operator transformation $F: M_1\to M$ is called weakly (or sequentially) continuous if $p_n\overset{M}{\to}p$ in M_1 always implies $F(p_n)\to F(p)$, where $\overset{M}{\to}$ means the usual convergence of M (see [5], [11]).

Two kinds of the integral of operator functions will be used; one of them has been defined by Mikusinski (see [11]) and the other type is due to *Gesztelyi* (see [7]).

1.4. *Definition.* $M\times M$ is defined as the linear space of all ordered pairs (x, y), $x, y\in M$ with the usual definition of addition and scalar multiplication and with the convergence structure defined in M, i.e. (x_n, y_n) tends to (x, y) iff

$$x_n\overset{M}{\to}x, \qquad y_n\overset{M}{\to}y \qquad \text{(see [1]).}$$

Since M is not a topological space with respect to the usual I-type convergence (see [4], [12]), $M\times M$ is not a topological space either.

1.5. *Definition.* T is a linear mapping defined on $D(T)\subset M$. The graph $G(T)$ of T is the set $\{[x, T(x)]/x\in D(T)\}$. Since T is linear, $G(T)$ is a subspace of $M\times M$. If the graph $G(T)$ of T is closed in $M_1\times M_1\subset M\times M$, then T is said to be closed in M_1, briefly T is closed.

T will stand for the set of all weakly continuous operator transformations. The notation $\|f\|_\Omega$ for $f\in C^0$ means

$$\|f\|_\Omega=\max_{t\leq\Omega}|f(t)| \qquad (\Omega\geq 0).$$

2. Notion of the semi-group of transformations, foundations

The set Γ of linear operator transformations is called semi-group if the following conditions hold:

(a) $\gamma=\{T_\alpha/\alpha\geqq 0,\quad T_\alpha\in T\}$;
(b) $D(T_\alpha)\supset D(\gamma)$, for every α and $D(\gamma)$ is a linear subspace of M;
(c) $T_{\alpha+\beta}=T_\alpha T_\beta$ on $D(\Gamma)$ for $\alpha, \beta\geqq 0$;
(d) $T_0=I$ (I is the identity);
(e) For each fixed $x\in D(\Gamma)$, $T_\alpha(x)$ is a continuous operator function with respect to α in any finite interval $[\alpha_1;\ \alpha_2]\subset[0;\ \infty]$, i.e.; $T_\alpha(x)\in C([\alpha_1;\ \alpha_2])M$.

2.1. *Proposition.* For each $x\in D(\Gamma)$, $T_\alpha(x)$ is integrable (in the sense of *Mikusinski* [11]) in any finite interval $[\alpha_1;\ \alpha_2]$.

Let us define the operator transformation A_η by

$$A_\eta=\frac{1}{\eta}(T_\eta-I). \tag{2.1}$$

$A_\eta\in T$ and $D(A_\eta)\supset D(\Gamma)$ for each $\eta>0$. By (e), for each fixed $x\in D(\Gamma)$ the operator function $A_\eta(x)\in C([\delta, \beta])M$ for $\delta, \beta>0$. An operator is denoted by $x_{\alpha,\beta}$ when it is of the form

$$x_{\alpha,\beta}=\int_\alpha^\beta T_\eta(x)\,d\eta. \tag{2.2}$$

By 2.1 the set of such kind of operators is not empty. Since $A_\eta\in T$,

$$A_\eta(x_{\alpha,\beta})=\frac{1}{\eta}[T_\eta-I](x_{\alpha,\beta})=\frac{1}{\eta}[T_\eta-I]\left(\int_\alpha^\beta T_\gamma(x)\,d\gamma\right)=$$
$$=\frac{1}{\eta}\int_\alpha^\beta[T_\eta-I](T_\gamma(x))\,d\gamma=\frac{1}{\eta}\left[\int_\alpha^\beta(T_{\eta+\gamma}-T_\gamma)(x)\,d\gamma\right]=$$
$$=\frac{1}{\eta}\int_\beta^{\beta+\eta}T_\gamma(x)\,d\gamma-\frac{1}{\eta}\int_\alpha^{\alpha+\eta}T_\gamma(x)\,d\gamma;$$

on the right-hand side both integrals exist and there is a common function g_x from C^0 such that $T_\tau(x)=g_x^{-1}\{f(\tau, t)\}$, where $f(\tau, t)$ is a continuous function in the domain $[\alpha, \beta+\eta]\times[0, \infty)$. Therefore using the mean value theorem for integrals:

$$A_\eta(x_{\alpha,\beta})=T_{\Theta 1}(x)-T_{\Theta 2}(x), \tag{2.3}$$

where $\beta\leqq\Theta_1\leqq\beta+\eta$ and $\alpha\leqq\Theta_2\leqq\alpha+\eta$. Thus when $\eta\to 0$ by (e)

$$A_\eta(x_{\alpha,\beta})\to T_\beta(x)-T_\alpha(x). \tag{2.4}$$

As mentioned already, $A_\eta(x)$ is a continuous operator function in $(0; 1)$; therefore there exists a nonzero function $g_x \in C^0$ such that $g_x^{-1}\{f(\eta, t)\} = A_\eta(x)$, where $f(\eta, t)$ is continuous on the set $(0; 1) \times [0; \infty)$. Assuming that $\lim_{\eta \to 0} f(\eta, t) = a(t)$ does exist, then the value of the operator transformation A_0 is defined by

$$A_0(x) = g_x^{-1} a; \tag{2.5}$$

hence

$$A_0 = \lim_{\eta \to 0} \frac{1}{\eta} [T_\eta - I].$$

The domain of A_0 (the set of all operators x belonging to $D(\Gamma)$ for which (2.5) is well defined) is denoted by $D(A_0)$. (2.5) unambigouosly defines $A_0(x)$, when it exists. Indeed, let us assume that $A_\eta(x) = g_x^{-1}\{f(\eta, t)\}$ and $A_\eta(x) = \hat{g}_x^{-1}\{\hat{f}(\eta, t)\}$ are different representations of $A_\eta(x)$ in $\eta \in (0; 1)$ for which $\lim_{\eta \to 0} f(\eta, t) = a(t)$ and $\lim_{\eta \to 0} \hat{f}(\eta, t) = \hat{a}(t)$. Then by the continuity of the convolution and by virtue of

$$\hat{g}_x\{f(\eta, t)\} = g_x\{\hat{f}(\eta, t)\}$$

$g_x^{-1} a = \hat{g}_x^{-1} \hat{a}$ follows.

The operator transformation A_0 is said to be the *infinitesimal generator* of the semi-group Γ.

2.2. *Theorem.* The operator transformations T_η and A_0 commute on $D(A_0)$, moreover for each $x \in D(A_0)$ the operator function $T_\eta(x)$ can be differentiated with respect to η and satisfies

$$\frac{\mathrm{d}T_\eta(x)}{\mathrm{d}\eta} = A_0 T_\eta(x) = T_\eta A_0(x) \tag{2.6}$$

for each $\eta \geqq 0$.

Proof: If $x \in D(A_0)$, then, by the definition of A_0 and (c),

$$\frac{1}{\eta}[T_{\xi+\eta}(x) - T_\xi(x)] = \frac{1}{\eta}[T_\xi(T_\eta(x) - I(x))] = T_\xi\left(\frac{1}{\eta}(T_\eta - I)(x)\right) = \tag{2.7}$$

$$= \frac{1}{\eta}(T_\eta - I) T_\xi(x) \to T_\xi A_0(x) = A_0 T_\xi(x),$$

where the limit should be taken as in the definition of A_0 (by (c): $A_0(x) \in D(\Gamma)$). Consequently, $T\xi(x)$ is a differentiable operator function, i.e., there exists a representation in $[\xi - \eta, \xi + \eta]$ (where $\eta > 0$ arbitrary, but $\xi - \eta \geqq 0$) for that

$$\frac{\mathrm{d}T_\xi(x)}{\mathrm{d}\xi} = a^{-1}\left\{\frac{\partial f(\xi, t)}{\partial \xi}\right\} = a^{-1}\left\{\lim_{\eta \to 0} \frac{f(\xi+\eta, t) - f(\xi, t)}{\eta}\right\} = T_\xi A_0(x).$$

It has only to be proved that (2.6) holds for the left-side derivative, too. By (2.5) and (c), for $\eta < 0$,

$$\frac{1}{\eta}[T_{\xi+\eta} - T_\xi] = \frac{1}{\eta} T_{\xi+\eta}[I - T_{-\eta}] = \frac{1}{-\eta} T_{\xi-(-\eta)}[T_{-\eta} - I] \to T_\xi A_0 = A_0 T_\xi;$$

for each $x \in D(A_0)$ whenever $\xi + \eta \geqq 0$.

From the above proof and (2.5) one can easily get that.

2.3. *Proposition.* If $x \in D(A_0)$, then $T\xi(x) \in D(A_0)$ for each $\xi \geqq 0$ and $T\xi(x)$ is differantiable with respect to ξ in any order.

2.4. *Lemma.* If $x \in D(A_0)$, then

$$T_\xi(x) - x = \int_0^\xi A, T_\eta(x)\, d\eta. \tag{2.8}$$

Proof. From (2.5) and 2.2 it follows that

$$\frac{d}{d\xi}(T_\xi(x)) = g_x^{-1}\left\{\frac{\partial f(\xi, t)}{\partial \xi}\right\},$$

where $T_\xi(x) = g_x^{-1}\{f(\xi, t)\}$. Multiplying both sides of (2.5) by g_x and integrating from 0 to ξ, then

$$\left\{\int_0^\xi \frac{\partial f(\xi, t)}{\partial \xi}\, d\xi\right\} = \int_0^\xi g_x T_\xi A_0(x)\, d\xi$$

holds; and taking into consideration (d), from

$$\{f(\xi, t) - f(0, t)\} = g_x \int_0^\xi T_\xi A_0(x)\, d\xi$$

(2.8) follows.

The next two theorems show the main character of the operator transformation A_0.

2.5. *Theorem.* $D(A_0)$ is dense in $D(\Gamma)$.

Proof. For arbitrary $x \in D(\Gamma)$ by 2.1:

$$A_\eta\left(\int_0^\xi T_\xi(x)\, d\xi\right) = \frac{1}{\eta}\int_0^\xi [T_\eta - I] T_\xi(x)\, d\xi = \frac{1}{\eta}\int_\eta^{\xi+\eta} T_\xi(x)\, d\xi - \frac{1}{\eta}\int_0^\xi T_\xi(x)\, d\xi =$$

$$= \frac{1}{\eta}\int_0^\eta T_\mu(T_\xi - I)(x)\, d\mu \to [T_\xi - I](x);$$

therefore, for each $x \in D(\Gamma)$,

$$\int_0^\xi T_\xi(x)\, d\xi \in D(A_0).$$

Using the mean-value theorem for integrals and property (*e*),

$$\frac{1}{\xi}\int_0^{\xi} T_\mu(x)\,\mathrm{d}\mu \to x,$$

which proves the theorem, since $D(A_0)$ is a linear subspace of $D(\Gamma)$.

2.6. *Theorem.* A_0 is closed in $D(\Gamma)$.

Proof. A_0 is closed if $G(A_0)$ is closed in $D(\gamma)\times D(\gamma)$, i.e.; if $f_n\in D(A_0)$ and $f_n\to f\in D(\Gamma)$ (in I.-type convergence) such that $A_0(f_n)\to g$ in M, then $f\in D(A_0)$ and $g=A_0(f)$. Put f_n in (2.8), then by theorem 2.2

$$T_\xi(f)-f=\int_0^{\xi} T_\mu(g)\,\mathrm{d}\mu$$

follows, since $T_\mu\in\tau$. Therefore

$$A_\eta(f)=\frac{1}{\eta}\int_0^{\eta} T_\mu(g)\,\mathrm{d}\mu\to g,$$

whenever $\eta\to 0$, because of (e), i.e.; $f\in D(A_0)$ and $A_0(f)=g$.

2.7. Let us define *the powers of the infinitesimal generator*. The operator transformation A_0^r will be defined inductively, as follows:

$$A_0^0=I,\ A_0^1=A_0 \text{ and for } r=2,3,\dots$$

(2.9) $$D(A_0^r)=\{x/x\in D(A_0^{r-1}) \quad\text{and}\quad A_0^{r-1}(x)\in D(A_0)\}$$

and for $x\in D(A_0^r)$

(2.10) $$A_0^r(x)=\lim_{\eta\to 0} A_\eta(A_0^{r-1}(x))=A_0(A_0^{r-1}(x)).$$

2.8. *Theorem.* (a) $D(A_0^r)$ is a linear subspace in $D(\Gamma)$, and A_0^r is a linear operator transformation.

(b) If $x\in D(A_0^r)$, then $T_\tau(x)\in D(A_0^r)$ for all $\tau\geqq 0$ and

(2.11) $$\frac{\mathrm{d}^r}{\mathrm{d}\xi^r}(T_\xi(x))=A_0^r T_\xi(x)=T_\xi A_0^r(x);$$

(2.12) $$T_\xi(x)-\sum_{k=0}^{r-1}\frac{\xi^k}{k!}A_0^k(x)=\frac{1}{(r-1)!}\int_0^{\xi}(\xi-u)^{r-1}T_u A_0(x)\,\mathrm{d}u.$$

(c) For each n both $D(A_0^n)$ and $\bigcap_n D(A_0^n)$ are dense in $D(\Gamma)$, moreover A_0^n is closed in $D(\Gamma)$.

Proof. (a) follows from the definition. (b) is a generalization of 2.3, and 1.3. (2.11) can be proved by induction, using (2.6). Integrating both sides of (2.11) from 0 to ξ we obtain for each $x\in D(A_0^k)$:

$$T_\xi A_0^{k-1}(x)-A_0^{k-1}(x)=\int\limits_0^\xi T_\xi A_0^k(x)\,\mathrm{d}\xi. \tag{2.13}$$

By repeated integration of (2.13) and using its reductive character relation (2.12) follows.

In order to prove (c) let us consider $C_{00}^\infty(R^+)$ to be the class of infinite differentiablely functions with compact support defined on $R^+=\{t/0<t<\infty\}$. If $\varphi\in C_{00}^\infty(R^+)$, then for every integer r $\varphi^{(r)}\in C_{00}^\infty(R^+)$, moreover the mapping $\varphi(\lambda)T_\lambda(f)$ is a continuous operator function with domain R^+ for each $f\in D(\Gamma)$. Let $D(\Gamma)_{00}$ be the subset of $D(\Gamma)$ for which $g\in D(\Gamma)_{00}$ if there exist $f\in D(\Gamma)$ and $\varphi\in C_{00}^\infty(R^+)$ such that

$$g=\int\limits_0^\infty \varphi(\lambda)T_\lambda(f)\,\mathrm{d}\lambda. \tag{2.14}$$

Since $T_\lambda\in t$ and the support of φ is compact for each $f\in D(\Gamma)$ and $\varphi\in C_{00}^\infty(R^+)$ the integral (2.14) does exist and $g\in D(\Gamma)$. Obviously, $D(\Gamma)_{00}$ is a linear space. First we show that $D(\Gamma)_{00}\subset D(A_0^r), r=1,2,\dots$. For sufficiently small $\tau\geqq 0$ supp $\varphi(\mu-\tau)\subset(0,\infty)$, therefore

$$A_\lambda(g)=\frac{1}{\tau}\int\limits_0^\infty \varphi(\mu)[T_{\mu+\tau}-T_\mu](f)\,\mathrm{d}\mu=\frac{1}{\tau}\int\limits_0^\infty(\varphi(\mu-\tau)-\varphi(\mu))T_\mu(f)\,\mathrm{d}\mu\to$$

$$\to-\int\limits_0^\infty \varphi'(\mu)T_\mu(f)\,\mathrm{d}\mu=A_0(g),$$

where the limit should be taken as in the definition of A_0 it was treated. Repeating the above argument for all $g\in D(A_0^r)$ $(r=1,2,\dots)$, we get that

$$A_0^r(g)=(-1)^r\int\limits_0^\infty \varphi^{(r)}(\mu)T_\mu(f)\,\mathrm{d}\mu,$$

hence

$$D(\Gamma)_{00}\subset\bigcap_{n=1}^\infty D(A_0^n).$$

Now it will be proved that $D(\Gamma)_{00}$ is dense in $D(\Gamma)$. Let $f\in D(\Gamma)$ and be fixed. $\varphi_n\in C_{00}^\infty(R^+)$ is such a function:

$$\varphi_n(u)=\begin{cases} n & \text{if } u\in\left(\frac{1}{n}+\frac{1}{n^2},\ \frac{2}{n}-\frac{1}{n^2}\right)=I_n;\\ \text{arbitrary but } |\varphi_n(u)|\leqq n & \text{if } u\in\left(\frac{1}{n},\ \frac{2}{n}\right)-I_n;\\ 0 & \text{otherwise.}\end{cases}$$

Then
$$g_n=\int_0^\infty \varphi_n(u)T_u(f)\,du\in D(\Gamma)_{00}$$
and $g_n\to f$ in M. Indeed, $T_u(f)\in C([0,1])M$, therefore there exists $q\neq 0, q\in C^0$ such that $T_u(f)=q^{-1}\{f(u,t)\}$, where $f(u,t)$ is continuous in $[0,1]\times[0,\infty)$. We shall prove that $qg_n\to qf$ in C^0. If ε and Ω are arbitrary positive numbers, then

$$\|qg_n-qf\|_\Omega=\left\|\int_{n^{-1}}^{2n^{-1}}\varphi_n(u)f(u,t)\,du-qf(t)\right\|_\Omega\leqq$$
$$\leqq n^{-2}[\max_{I_n^1}(|\varphi_n(u)|\cdot\|f(u,t)\|_\Omega)+\max_{I_n^3}(|\varphi_n(u)|\cdot\|f(u,t)\|_\Omega)+$$
$$+\left\|\int_{n^{-1}+n^{-2}}^{2n^{-1}-n^{-2}}(f(u,t)\varphi_n(u)-(n^{-1}-2n^{-2})^{-1}qf(t))\,du\right\|_\Omega\leqq$$
$$\leqq\varepsilon+(n^{-1}-2n^{-2})\|f(u_n,t)n-qf(t)\|_\Omega\leqq 2\varepsilon,$$

since $n^{-1}+n^{-2}\leqq u_n\leqq 2n^{-1}-n^{-2}$ and $f(u_n,t)\to qf(t)$ almost uniformly by (e).

Hence $D(\Gamma)_{00}$ is dense in $D(\Gamma)$ and, as a consequence of the statement proved previously, we have that both $D(A_0^r)$ and $\bigcap_n D(A_0^n)$ are dense in $D(\Gamma)$. To prove that A_0^r is closed we might use the method used in the proof of theorem 2.6.

The proof of theorem 2.8 is complete.

Remark. The formula (2.12), being similar to the classical Taylor series of the exponential function, is called Taylor-formula.

When X is a Banach algebra and B is a bounded operator acting on X into X, then the operator function (in the sense of functional analysis)

$$T(t)=\exp(Bt)=I+\sum_{k=1}^{\infty}\frac{(tB)^k}{k!}\qquad(0\leqq t<\infty)\tag{2.15}$$

defines a uniformly continuous semi-group of operators ([8]). By formula (2.12) we might guess the same situation whenever B is a (weakly) continuous operator transformation. This is *not true generally;* (2.12) in our case could also be divergent. Consider, for example, the operator transformation $F_s: x\to sx$, which is obviously continuous, then there exists a continuous semi-group of operator transformations, namely $\gamma=\{e^{-\lambda s}/\lambda\geqq 0\}$, the semi-group of translations with $F_s=s$ as an infinitesimal generator; nevertheless the series

$$1+\sum_{n=1}^{\infty}\frac{\lambda^n s^n}{n!}(-1)^n$$

does not converge in the operator sense.

Otherwise, in the case of $B=k$, where k is a locally integrable function, the series

$$e^{-\lambda k}=1+\sum_{n=1}^{\infty}\frac{\lambda^n k^n}{n!}(-1)^n$$

converges in the operator sense, and $D(e^{-\lambda k})=M$.

When $B=D$, D is the operation of the algebraic derivation, and then

$$T_\lambda=I+\sum_{n=1}^{\infty}\frac{(\lambda D)^n}{n!} \qquad (0\leqq\lambda<\infty)$$

converges on C^0, as can be seen from

$$T_\lambda(f)=f+\sum_{n=1}^{\infty}\frac{\{(-1)^n\lambda^n t^n f(t)\}}{n!}=\{e^{-\lambda t}f(t)\},$$

therefore $T_\lambda(.)$ is a continuous semi-group of transformations on C^0 althought the transformations are continuous endomorphisms of M. We prove ([3]) that $T_\lambda(.)$ is a continuous semi-group on the set of distributions with half line support, D'_+. It is an open question whether $T_\lambda(.)$ is or is not a continuous semi-group on M.

It would be a confusion if an operator transformation generated more than one semi-group. The following theorem shows that we have no such case in certain circumstances.

2.9. *Theorem.* If A_0 is a linear operator transformation with domain $D(A_0)$, and $D(A_0)$ is dense in a linear subset M_1 of M, moreover A_0 is closed in M_1 then A_0 is the infinitesimal generator at most one continuous semi-group γ containing continuous operator transformations with domain $D(\gamma)=M_1$.

Proof. Let us assume that A_0 is the infinitesimal generator of two semi-groups $T_\tau(.)$ and $\hat{T}_\tau(.)$. If $f\in D(A_0)$ then $\hat{T}_\tau(f)\in D(A_0)$ and the operator function $g(\tau)=T_{t-\tau}\hat{T}(f)$ is continuously differentiable with respect to τ, and $g'(\tau)=0$. Indeed, $F(\tau)=\hat{T}_\tau(f)$ is a differentiable operator function, and by (2.7) and the continuity of T_μ we have

$$\frac{dT_{t-\tau}(f)}{d\tau}=-A_0T_{t-\tau}(F(\tau))+T_{t-\tau}\left(\frac{dF(\tau)}{d\tau}\right)=$$
$$=-A_0T_{t-\tau}\hat{T}_\tau(f)+T_{t-\tau}A_0\hat{T}_\tau(f)=0$$

since A_0 and $T_{t-\tau}$ commute. Therefore $g'(\tau)=0$ and $g(\tau)\in C_1([0,t])M$ imply that $g(\tau)\equiv c$, that is, $g(\tau)=g(t)=g(0)=T_t(f)=\hat{T}_t(f)$. Since $D(A_0)$ is dense in $D(\Gamma)$, $T_t(f=$ $=\hat{T}_t(f)$ if $f\in D(\Gamma)$.

2.10. *Corollary.* If A_0 satisfies the conditions of theorem 2.9, then A_0 is not the infinitesimal generator of any continuous semi-group being different from that generated by A_0 and contains weakly (pointwise) continuous operator transformations.

It is a simple fact, but we should remark that, if there is $\varepsilon>0$ such that T_μ is continuous for every $\mu\in[0,\varepsilon]$, then T_μ is continuous for each $\mu\geqq 0$. (The proof follows from property (c).)

3. Resolvent, strongly continuous semi-groups

Let U be a linear operator transformation with domain $D(U)$ and range $R(U)$ in M. The transformation $U_\lambda = \lambda I - U$ is defined also on $D(U)$ for all complex numbers λ. Let the range of U_λ be denoted by $R(U_\lambda)$. The resolvent set $\varrho(U)$ of U is the set of all complex numbers λ for which the inverse of U_λ exists and is unique. The inverse transformation U_λ^{-1} is called *resolvent* and denoted by $R(\lambda, U)$.

The continuous semi-group Γ is said to be *strongly continuous* if for each $f \in D(\Gamma)$ there is $q_f \neq 0$, $q_f \in C^0$ such that $q_f T_\tau(f)$ is a continuous parametric function, $q_f T_\tau(f) \in C[(0,\infty)]\,C^\circ$ and $\|q_f T_\tau(f)\|_\Omega \leqq \|g_f\|_\Omega$ $(0 \leqq \tau < \infty)$ for each $\Omega > 0$ with some fixed $f_f \in C^0$. (g_f is independent of Ω and depends only on f.)

For the future we need:

3.1. *Proposition.* If $f(u) \in D(A)$ for all $u \in [0, \infty)$, $\int\limits_0^\infty \varphi(u) f(u)\, du$ does exist and belongs to $D(A)$, where $\varphi(u)$ is a numerical function, and if A is a closed operator transformation in $D(A)$, then

$$A\left(\int\limits_0^\infty \varphi(u) f(u)\, du\right) = \int\limits_0^\infty \varphi(u) A(f(u))\, du.$$

whenever the right side exists. (See [1].)

3.2. *Theorem.* If $\Gamma = \{T_\tau / \tau \geqq 0\}$ is a strongly continuous semi-group of weakly continuous operator transformations, and the infinitesimal generator of the semi-group is A_0, then for each λ, for which $\mathrm{Re}(\lambda) > 0$, we have

$$R(\lambda, A_0)(f) = \int\limits_0^\infty e^{-\lambda u}\, T_u(f)\, du \tag{3.1}$$

where $f \in D(A_0)$; moreover

$$M - \lim \lambda_n R(\lambda_n, A_0)(f) = f \tag{3.2}$$

for any sequences λ_n for which $|\arg \lambda_n| \leqq \alpha_0 < \dfrac{\pi}{2}$.

Proof. Let us consider the following transformation:

$$R_\lambda(f) = \int\limits_0^\infty e^{-\lambda u}\, T_u(f)\, du, \tag{3.3}$$

where $f\in D(\gamma)$ and $\mathrm{Re}(\lambda)>0$. First it will be shown that (3.3) is well defined for all $f\in D(\gamma)$. By the definition of the strong continuity we have $q_f\in C^0, q_f\neq 0$ and $g_f\in C^0$ such that

$$\|q_f T_u(f)\|_\Omega \leqq \|g_f\|_\Omega; \quad (\Omega>0, \quad u\geqq 0)$$

and therefore

$$q_f R_\lambda^{\Omega_n}(f)=\int_0^{\Omega_n} e^{-\lambda u} q T u_f(f)\,\mathrm{d}u$$

converges, as $\Omega_n\to\infty$. $R(R_\lambda)\subset D(A_0)$ and

3.4) $$(\lambda I-A_0)R_\lambda(f)=f$$

or each $f\in D(\gamma)$. Indeed,

$$A_v R_\lambda(f)=\frac{1}{v}\int_v^\infty e^{-\lambda u}(T_{v+u}-T_u)(f)\,\mathrm{d}u=$$

$$=\frac{e^{\lambda v}-1}{v}\int_v^\infty e^{-\lambda u}T_u(f)\,\mathrm{d}u-\frac{1}{v}\int_0^v e^{-\lambda u}T_u(f)\,\mathrm{d}u\to\lambda R_\lambda(f)-f=A_0R_\lambda(f).$$

Hence $R(R_\lambda)\subset D(A_0)$ and (3.4) is fulfilled. Let $f\in D(A_0)$; then $T_u(f)\in D(A_0)$ and $R_\lambda(f)\varepsilon D(A_0)$. Since A_0 is closed in $D(\Gamma)$,

$$R_\lambda A_0(f)=\int_0^\infty e^{-\lambda u}T_u(A_0(f))\,\mathrm{d}u=A_0\left(\int_0^\infty e^{-\lambda u}T_u(f)\,\mathrm{d}u\right)=A_0R_\lambda(f)$$

holds by 3.1. Comparing this result with (3.4), we obtain (3.1). To prove (3.2) let us consider the relation

$$\lambda R(\lambda, A_0)(f)-f=\lambda\int_0^\infty e^{-\lambda u}(T_u(f)-f)\,\mathrm{d}u,$$

for $f\in D(A_0)$ and $\mathrm{Re}(\lambda)>0$.

Since $\{T_u/u\geqq 0\}$ is strongly continuous, for any $\varepsilon\geqq 0$ there exists $\delta(\varepsilon,\Omega)>0$ such that $\|q_fT_u(f)-q_ff\|_\Omega<\varepsilon$ when $0\leqq u\leqq\delta$. In this case

$$|\lambda|\int_0^\delta e^{-\sigma u}\|q_fT_u(f)-q_ff\|_\Omega\,\mathrm{d}u<\frac{|\lambda|}{\sigma}\varepsilon=\varepsilon\sqrt{1+(\mathrm{tg}\,\alpha_0)^2},$$

when Re $(\lambda)=\sigma$. Let Re $(\lambda)=\sigma>\omega>0$, then

$$|\lambda|\int_{\delta}^{\infty} e^{-\lambda u}\|q_f T_u(f)-q_f f\|_{\Omega}\,du\leqq$$

$$\leqq|\lambda|\int_{\delta}^{\infty} e^{-\sigma u}(\|g_f\|_{\Omega}+\|q_f f\|_{\Omega})\,du=(\|g_f\|_{\Omega}+\|q_f f\|_{\Omega})e^{-\delta\sigma}\sqrt{1+(\operatorname{tg}\alpha_0)^2}<\varepsilon$$

if σ is large enough.

The proof of theorem 3.2 is complete.

The next theorem shows some advantages of the representation (3.1).

3.3. *Theorem.* $R(\lambda, A_0)(f)=g(\lambda)$ is an analytical operator function for any $f\in D(A_0)$ on $\operatorname{Re}(\lambda)>0$; i.e.: $g(\lambda)\in A\big(\operatorname{Re}(\lambda)>0\big)M$.

Proof. Let $0<v<\infty$ and

$$q_f F_v(\lambda)(f)=\int_0^v q_f e^{-\lambda u}T_u(f)\,du.$$

$q_f F_v(\lambda)(f)\in A\big(\operatorname{Re}(\lambda)>0\big)C^0$ by the strong continuity of $\{T_u/u\geqq 0\}$. If $\operatorname{Re}(\lambda)>0$ and $\operatorname{Re}(\lambda+h)>0$, then

$$\triangle(q_f F_v(\lambda)(f))=h^{-1}(q_f F_v(\lambda+h)(f)-q_f F_v(\lambda)(f))+\int_0^v q_f e^{-\lambda u}T_u(f)u\,du=$$

$$=\int_0^v q_f T_u(f)\left(\frac{1}{h}(e^{-uh}-1)+u\right)e^{-\lambda u}\,du.$$

Since $\left|\frac{1}{h}(e^{-uh}-1)+u\right|\leqq|h|v^2e^{|h|v}$ when $0\leqq u\leqq v$, hence

$$\|\triangle(q_f F_v(\lambda)(f)\|_{\Omega}\leqq e^{|\operatorname{Re}(\lambda)v|}|h|v^2e^{|h|v}\cdot\int_0^v\|q_f T_u(f)\|_{\Omega}\,du\to 0,$$

as $h\to 0$. Therefore

$$\frac{dR(\lambda, A_0)(f)}{d\lambda}=\int_0^{\infty}(-u)e^{-\lambda u}T_u(f)\,du,$$

for Re $(\lambda)>0$. Repeating the previous argument

$$\frac{d^nR(\lambda, A_0)(f)}{d\lambda^n}=\int_0^{\infty}(-u)^n e^{-\lambda u}T_u(f)\,du.$$

Since $q_f F_v(\lambda)(f)\in A(\mathrm{Re}(\lambda)>0)C^0$ for any $v>0$ and its derivatives converge uniformly to the derivatives of $q_f R(\lambda, A_0)(f)$ in $\mathrm{Re}(\lambda)>0$, we obtain that $R(\lambda, A_0)(f)\in A(\mathrm{Re}(\lambda)>0)M$.

It is easy to see the following:

3.4. *Lemma.* If $\lambda_1, \lambda_2\in\varrho(A_0)$, $f\in D(R(\lambda, A_0))\cap D(A_0)$, then

$$(3.5)\qquad \big(R(\lambda_1, A_0)-R(\lambda_2, A_0)\big)(f)=(\lambda_2-\lambda_1)R(\lambda_1, A_0)R(\lambda_2, A_0)(f);\quad R(\lambda_1, A_0) \text{ and } R(\lambda_2, A_0) \text{ commute};$$

$$(3.6)\qquad \frac{d^n R(\lambda, A_0)(f)}{d\lambda^n}=(-1)^n\, n!\,(R(\lambda, A_0))^{n+1}(f),$$

for $n=0, 1, 2, \ldots, f\in D(R(\lambda, A_0))\cap D(A_0)$.

3.5. *Theorem.* Let A_0 be a linear closed operator transformation, with domain $D(A_0)$ being dense in the linear subspace $D(\Gamma)\subset M$. The following are necessary for A_0 to be the infinitesimal generator of a strongly continuous semi-group with domain $D(\Gamma)$: for each $f\in D(A_0)$ there exist $g_f\in C^0$, $q_f\in C^0$, $q_f\neq 0$ such that

$$(3.7)\qquad q_f\big(R(\lambda, A_0)\big)(f)\in C^0 \quad (n=1, 2, \ldots)$$

if $\mathrm{Re}(\lambda)>0$;

$$(3.8)\qquad \|q_f\big(R(\lambda, A_0)\big)^n(f)\|_\Omega\leqq\|g_f\|\cdot\sigma^{-n}$$

for each $\Omega>0$ where $\sigma=\mathrm{Re}(\lambda)$.

Proof. (3.7) follows from 3.2 and 3.3. By (3.1) and (3.6)

$$\frac{d^{r-1}R(\lambda, A_0)(f)}{d\lambda^{r-1}}=\int_0^\infty (-u)^{r-1}e^{-\lambda u}T_u(f)\,du,$$

therefore, by the strong continuity,

$$\left\|q_f\frac{d^{r-1}R(\lambda, A_0)(f)}{d\lambda^{r-1}}\right\|_\Omega\leqq\|g_f\|_\Omega\int_0^\infty u^{r-1}e^{-\sigma u}\,du=\|g_f\|_\Omega(r-1)!\,\sigma^{-r};$$

and now a comparison with (3.6) gives (3.8).

The conditions (3.7) and (3.8) offer the possibility of defining a new semi-group on the set $D(R(\lambda, A_0))\cap D(A_0)$, which is very closely connected with the original semi-group. Let us assume that (3.7) and (3.8) are fulfilled for some linear closed operator transformation A_0. Then in the case of $U_\lambda=\lambda^2R(\lambda, A_0)-\lambda I$ we obtain

$$(3.9)\qquad F_v(\lambda,f)=\sum_{k=0}^{\infty}\frac{v^k}{k!}U_\lambda^k(f)\qquad (0\leqq v<\infty),$$

which forms a strongly continuous semi-group on the set $D\big(R(\lambda, A_0)\big)\cap D(A_0)$. Indeed, using (3.7) and (3.8) we could show that

$$q_f F_v(\lambda, f)=q_f\left(e^{-\lambda v}\sum_{k=0}^{\infty}\frac{(\lambda^2 v)^k}{k!}(R(\lambda, A_0))^k(f)\right)$$

converges uniformly with respect to v in any finite interval [0, a]. From (3.9) it follows that for $\mathrm{Re}(\lambda)>0$, $Re(\mu)>0$

$$U_\mu F_v(\lambda, f)=F_v(\lambda,\cdot)U_\mu(f).$$

Assuming that (3.7) and (3.8) hold, we have

$$\|q_f\big(\lambda R(\lambda, A_0)(f)-f\big)\|_\Omega=\|q_f R(\lambda, A_0)A_0(f)\|_\Omega=0(|\lambda|^{-1})$$

when $|\lambda|\to\infty$, therefore $U_\lambda(f)\to A_0(f)$ as $|\lambda|\to\infty$.

3.6. *Theorem.* If $\{T_u/u\geqq 0\}$ is a strongly continuous semi-group of continuous linear operator transformations, A_0 is its infinitesimal generator and $U_\lambda=\lambda^2R(\lambda, A_0)-\lambda I$, then

$$F_v^\lambda(f)=\sum_{k=0}^{\infty}\frac{v^k}{k!}U_\lambda^k(f)=e^{-\lambda v}\sum_{k=0}^{\infty}\frac{(\lambda^2 v)^k}{k!}(R(\lambda, A_0))^k(f) \tag{3.10}$$

is a strongly continuous semi-group on the set $D(R(\lambda, A_0))\cap D(A_0)=D(F_v^\lambda)\cdot$ $D(F_v^\lambda)$ is dense in $D(\{T_u\})$ and for any $f\in D(F_v^\lambda)$

$$F_v^\lambda(f)\to T_v(f),$$

as $|\lambda|\to\infty$, uniformly with respect to v in any compact subset of $[0, \infty]$.

Proof. By 2.5, $D(F_v^\lambda)$ is dense in $D(\{T_u\})$. If $F_v^\lambda(f)$ converges, as $|\lambda|\to\infty$, then $F_v^\lambda(f)\to T_v(f)$; indeed,

$$F_v^\lambda(f)-f=\int_0^v F_u U_\lambda(f)\,\mathrm{d}u$$

and by the previous argument

$$\hat{T}_v(f)-f=\int_0^v \hat{T}_u A_0(f)\,\mathrm{d}u$$

holds. $\hat{T}_v(.)$ is a semi-group with the infinitesimal generator A_0, hence $\hat{T}_v=T_v$ by the theorem of unicity.

Now we are going to show that $F_v^\lambda(f)$ converges for each

$$f\in D(R(\lambda, A_0))\cap D(A_0).$$

By (3.1) we obtain

$$(R(\lambda, A_0))^n(f)=\frac{1}{(n-1)!}\int_0^\infty u^n e^{-\lambda u}T_u(f)\,\mathrm{d}u$$

for Re $(\lambda)>0$, $n=1, 2, \ldots$. Hence

$$q_f F_v^\lambda(f)=q_f\left(e^{-\lambda v}\sum_{k=1}^{\infty}\frac{(\lambda^2 v)^k}{k!}\cdot\frac{1}{(k-1)!}\int_0^\infty e^{-\lambda u}u^k T_u(f)\,\mathrm{d}u+e^{-\lambda v}f\right)=$$

$$=\int_0^\infty\left(\sum_{k=0}^{\infty}e^{-\lambda(u+v)}\frac{(\lambda^2 v)^{k+1}u^k}{k!(k+1)!}\right)q_f T_u(f)\,\mathrm{d}u+e^{-\lambda v}q_f f.$$

Thus

$$q_f F_v^\lambda(f)=\int_0^\infty K(\lambda, v, u)q_f T_u(f)\,\mathrm{d}u+e^{-\lambda v}fq_f,$$

where

$$K(\lambda, v, u)=e^{-\lambda(u+v)}\sum_{k=0}^{\infty}\frac{(\lambda^2 v)^{k+1}}{k!(k+1)!}u^k=\lambda\left(\frac{v}{u}\right)^{\frac{1}{2}}e^{-\lambda(u+v)}I_1(2\lambda(uv)^{\frac{1}{2}}).$$

Since $e^{-\lambda v}fq_f\to 0$ as $|\lambda|\to\infty$, it is sufficient to show that

$$\int_0^\infty K(\lambda, v, u)\,q_f T_u(f)\,\mathrm{d}u\to q_f T_v(f)\qquad \text{as } |\lambda|\to\infty.$$

The kernel function $K(\lambda, v, u)$ has the following properties:

(I) $K(\lambda, v, u)\geqq 0$ when $\lambda, v, u\geqq 0$;

(II) $\displaystyle\int_0^\infty K(\lambda, v, u)\,\mathrm{d}u=1-e^{-\lambda v}.$

Because $q_f T_v(f)\in C([0,\infty))C^0$, it follows that

$$G(\beta, v)=\int_v^\beta \|q_f T_v(f)-q\ T_u(f)\|_\Omega\,\mathrm{d}u=\mathrm{o}(|\beta-v|).$$

Hence we should show, using (II), that

$$\int_0^\infty K(\lambda, v, u)\|q_f T_v(f)-q_f T_u(f)\|_\Omega\,\mathrm{d}u\to 0$$

when $|\lambda|\to\infty$.

The proof of this assertion might be taken from [8]. We should recognize that he uniform convergence with respect to v follows from the strong continuity.

3.7. *Corollary*. In theorem 3.6 we should have assumed only the weak continuity of the operator transformations.

3.8. *Corollary.* (Theorem of unicity) If A_0 is the infinitesimal generator of a strongly continuous semi-group which contains weakly continuous operator transformations, then there exists no other strongly continuous semi-group with the infinitesimal generator A_0.

Comments. The converse of theorem 3.5 and 3.6 would be more interesting. In the classical theory of semi-groups this problem leads to the necessary and sufficient conditions due to *Hille* and *Yosida.* In the theory of operator transformations it cannot be realized generally. This arries from the fact that, if F is a continuous mapping from a Banach space X into X, then $\|Ff\| < C\|f\|$, with some constant C, and in the operator field there is no norm with respect to the usual convergence.

In some cases this bar can be left out; for instance, when $A_0 = F_c : x \to cx$. Now we get the bounded logarithm defined by *L. Máté* ([9]). In this case the strong continuity of the semi-group implies the boundedness of the logarithm in Máté's sense, and the *Hille–Yosida* theorem holds: (3.7) and (3.8) form necessary and sufficient conditions to the boundedness of a logarithm, and also to the strong continuity of the semi-group. The logarithm w is bounded in *Máté's* sense when for each $\lambda > 0$ there exists $f \in C^0$, $f \neq 0$, such that $\|f \exp(-\lambda w)\|_\Omega \leqq \|f\|_\Omega$; in our case w is bounded logarithm if w generates a strongly continuous semi-group, that is, for $\mathrm{Re}(\lambda) > 0$ there exist $f, g \in C^0$, $f \neq 0$ such that $\|f \exp(-\lambda w)\|_\Omega \leqq \|g\|_\Omega$. Obviously our bounded logarithm is more general than Máté's but not too much. We might apply the argument of *Máté* without any modification ([9]) and obtain the following: $A_0 = F_w : x \to wx$, generates a strongly continuous semi-group if and only if there exist $f \in C^0$, $g \in C^0$, $(f \neq 0)$ such that

$$\frac{1}{x} f(R(\lambda, A_0))^n(x) \in C^0 \qquad (\text{for } \mathrm{Re}(\lambda) > 0)$$

and

$$\left\| \frac{1}{x} f(R(\lambda, A_0))^n(x) \right\|_\Omega \leqq \|g\|_\Omega \qquad (\text{for } \mathrm{Re}(\lambda) > 0).$$

Describing by term w, we obtain Máté's conditions:

$$\text{(a)} \left(\frac{\lambda}{\lambda + w} \right)^k f \in C^0;$$

$$\text{(b)} \left\| \left(\frac{\lambda}{\lambda + w} \right)^k f \right\|_\Omega \leqq \|g\|_\Omega \qquad (\text{for } \mathrm{Re}(\lambda) > 0,\ \Omega > 0).$$

We should remark finally that formula (3.10) gives an assymptotic procedure to construct bounded logarithm, since in this case U_λ is an operator transformation of type F_c.

4. Cauchy-problem; application of the transformation semi-group

Every exponential function represents a continuous semi-group of transformations which could easily be verified by the definition. They can be applied to solve partial differential equations ([4], [5], [11]). Here we deal with some similar kinds of questions studying "transformation-differential" equations. (It might be a partial differential equation.) Let us formulate a so-called "abstract Cauchy-problem". U is a given linear operator transformation with domain $D(U)$ and range $R(U)$ in M. Moreover, given $x_0 \in D(U)$, a fixed operator; and we should find an operator function $y(\lambda)=y(\lambda, x_0)$ satisfying the following:

(4.1)
(i) $y(\lambda)\in C_1((0, \infty))M$;
(ii) for each $\lambda>0$ $y(\lambda)\in D(U)$ and
$y'(\lambda)=U(y(\lambda))$;
(iii) $y(\lambda)\to x_0$ if $\lambda\to 0$.

The third condition could determine the CP's describing what kind of convergence is taken into consideration. We have CP_1 if $y(\lambda)\in C_1([0, \infty])M$ and $y(0)=x_0$; CP_2 if $y(\lambda)\in C_1((0, \infty))M$ and $y(\lambda)=a\{y(\lambda, t)\}$ with $\{y(\lambda, t)\}\in C_1((0, T))C^\circ$, and $\|y(\lambda, t) - a^{-1}x_0\|_\Omega\to 0$ for all $\Omega>0$ as $\lambda\to 0$, and finally CP_3: when $y(\lambda)_n\to x_0$ in M whenever $\lambda_n\to 0$. *Remark.* If $U=F_c$: $x\to cx$, then CP_1 can be solved and the solution is the exponential function $y(\lambda)=x_0e^{\lambda c}$, provided c is a logarithm in *Mikusinski*'s sense. If c is a bounded logarithm, then the solution $y(\lambda)=x_0e^{\lambda c}$ is a bounded operator function (see for more detail later on). If $U=D$,D is the operation of algebraic derivation, then for all $x_0\in D_+$ (the set of distributions with leftsided bounded supports) the solution is $T^\alpha(x_0)$ ([3]). If $x_0\in M$ is an arbitrary operator, then $T^\alpha(x_0)$ is a solution of the problem in a weaker sense, and works as a sequentially continuous (in II.-type convergence — see [3]) operator function.

We are going to find the solution of CP in a special class of the operator functions.

4.1. *Definition.* $y(\lambda)$ belongs to the class $B^{(n)}(M)$ if $y(\lambda)\in C_n([0, \infty])M$ and there exist $f\in C^\circ, f\neq 0$ and $g\in C^\circ$ such that

$$\| fy^{(i)}(\lambda) \|_\Omega \leqq \| g \|_\Omega \quad (\lambda\geqq 0,\ \Omega>0)$$

for $i=0, 1, \dots n$. (Here f and g may depend on $y(\lambda)$, but are independent of Ω).

Obviously, if $k<n$, then $B^{(n)}(M)\subset B^{(k)}(M)$. We have defined the class $B^{(n)}(M)$, since it is desired to use the method of the Laplace transformation.

4.2. *Lemma.* If $y(u)\in B^{(0)}(M)$, then

$$L(y, v)=\int_0^\infty e^{-vu}y(u)du$$

exists, for $\mathrm{Re}(v)\geqq\varrho_0>0$ and $L(y, v)\equiv 0$ implies $y(u)\equiv 0$.

Proof. By the definition of $B^{(0)}(M)$ we have

$$\left\| g \int_0^\infty e^{-uv} y(u)\, du \right\|_\Omega \leqq \int_0^\infty e^{-Re(uv)} \|gy(u)\|_\Omega\, du \leqq \frac{\|f\|_\Omega}{\mathrm{Re}(v)},$$

therefore the above integral exists. If $L(y, v)\equiv 0$, then

$$\int_0^\infty gy(u) e^{-uv}\, du = \left\{ \int_0^\infty y(u, t) e^{-uv}\, du \right\} = 0,$$

hence $y(u, t)=0$ for almost all u with fixed t, which implies $y(u, t)\equiv 0$ for fixed t, and therefore $y(u, t)\equiv 0$ for all u, $t\geqq 0$.

For further investigation we need the following assertion:

4.3. *Lemma*. Let U be a linear operator transformation with domain $D(U)$, closed in A, $D(U)$ is dense in the linear subspace A of M. Moreover U can be extended from $D(U)$ to A, i.e.; for any $x\in A$, if we take $x_n\in D(U)$, $x_n\to x$ and then $U(x_n)\to \overline{U}(x)$ independently of the choice of the sequence x_n. Let us assume that $y(u)\in D(U)$ (for $\alpha\leqq u\leqq\beta$ and $\alpha=-\infty$, $\beta=\infty$ are possible) and

$$\int_\alpha^\beta y(u)\, du \in A$$

then

$$\int_\alpha^\beta y(u)\, du \in D(U)$$

and

$$\int_\alpha^\beta U(y(u))\, du = U\left(\int_\alpha^\beta y(u)\, du \right). \tag{4.1}$$

The proof immediately follows from the definition above and the properties of the Stieltjes integral of operator functions ([1], [7]).

Now CP_1 will be examined. $IB^1D(U)$ stands for the set of all operators which are in the form

$$\int_u^v y(z)\, dz$$

($v=\infty$ is also possible), where $u\geqq 0$, $y(z)\in D(U)\cap B^{(1)}(M)$ for $z\geqq 0$.

4.4. *Theorem*. Let U be a linear operator transformation on $D(U)$, extendable from $D(U)$ onto $IB^1D(U)$ and closed in $IB^1D(U)$. Moreover, assume that $\varrho(U)$ contains a right halfplane, i.e. there is $\sigma_0>0$ such that $\{\mathrm{Re}(z)\geqq\sigma_0>0\}\subset\varrho(U)$. Then in $B^{(1)}(M)$ there is at most one solution of CP_1 related to the equation

$$y'(z) = U(y(z)) \tag{4.2}$$

with the initial condition $y_0\in D(U)$.

Proof. It is sufficient to show that the theorem holds in the case of $y_0=0$, and then $y(z)\equiv 0$. Assume that $y(z)\in B^{(1)}(M)$, $y_0=0$, and satisfies equation (4.2). The integrals

$$\int_u^v e^{-pz}y(z)\,dz$$

and

$$\int_u^v {}^{-pz}U(y(z))\,dz$$

exist. By virtue of (4.2) and lemma 4.3

$$\int_u^v e^{-pz}y'(z)\,dz=\int_u^v e^{-pz}U(y(z))\,dz=U\left(\int_u^v e^{-pz}y(z)\,dz\right).$$

Integrating on the left hand side by part,

$$e^{-vp}y(v)-e^{-up}y(u)+p\int_u^v e^{-pz}y(z)\,dz$$

follows. Hence, in limit relation, by 4.3 and $y(z)\in B^{(1)}(M)$:

$$U(L(y,p))=pL(y,p)$$

for all $\mathrm{Re}(p)\geqq\sigma_0>0$. Since $\{\mathrm{Re}(p)\geqq\sigma_0>0\}\subset\varrho(U)$ it follows $L(y,p)=0$. Lemma 4.2 concludes the proof.

One can easily see that the semi-group theory of transformations and CP are closely related. It is obvious that, when $y_0\in D(A_0)$ and A_0 is the infinitesimal generator of the semi-group, then $T_z(y_0)=y(z,y_0)=y(z)$ is a solution of CP_1. Comparing this with 4.4, we obtain:

4.5. *Theorem.* $T_z(y_0)$ is the unique solution of CP_1 related to (4.2) in $B^{(1)}(M)$, if U is the infinitesimal generator of the strongly continuous semi-group $\{T_z/z\geqq 0\}$ and satisfies the conditions of theorem 4.4.

There is a natural way to generalize $CP=CP^1$ in a higher order n. Given a U linear operator transformation with $D(U)\subset M$, $R(U)\subset M$ and $y_0, y_1, y_2, \dots y_{n-1}\in D(U)$. Find the operator function $y(z)=y(z, y_0, \dots, y_{n-1})$ which satisfies:

(i) $y(z)\in C_n((0,\infty))M$;
(ii) $y^{(k)}(z)\in D(U^{n-k})$, $U^{n-k}(y^{(k)}(z))\in C_1((0,\infty))M$ for $k=0, 1, \dots (n-1)$ and

(4.3.) $\quad y^{(n)}(z)=U^n(y(z))$ for $z>0$;

(iii) $y^{(k)}(z)\to y^k$ if $z\to 0$.

The third condition gives the possibility of defining, as already mentioned, different kinds of CP^{n}'s : CP_1^n, CP_2^n and CP_3^n.

We could define $IB^nD(U)$ similarly to $IB^1D(U)$ by replacing $B^{(1)}(M)$ by $B^{(n)}(M)$, and using the same Laplace method as in theorem 4.4 we obtain:

4.6. *Theorem.* If U is a linear operator transformation on $D(U)$, extendable from $D(U)$ onto $IB^nD(U)$, closed in $IB^nD(U)$, and $\varrho(U)$ contains a right half plane, then the CP_1^n related to (4.3) with initial conditions $y_0, \ldots, y_{n-1} \in D(U^n)$ has at most one solution in $B^{(n)}(M)$.

The CP^n also has relation with the semi-group theory in the case of $n>1$. While in the case of $n=1$ U should be an infinitesimal generator if $n>1$,

$$U, \eta U, \ldots, \eta^{n-1}U$$

should be infinitesimal generators, where $\eta = \exp(2\pi i/n)$.

The operator transformation U is said to be bounded on $B(U) \subset D(U)$ when for each $g \in B(U)$ there exist $p \in C^\circ$, $q \in C^\circ$ $(q \neq 0)$ such that $\| qU^n(g) \|_\Omega \leqq \| p \|_\Omega$, $(n=1, 2, \ldots)$.

4.7. *Theorem.* If U satisfies the conditions of theorem 4.6 and is bounded on $B(U) \subset D(U)$, then for any initial system $y_0, \ldots, y_{n-1} \in B(U)$ the CP_1^n related to (4.3) has only one solution in $B(U)$ and can be represented by

$$y(z, y_0, \ldots, y_{n-1}) = \sum_{k=0}^{n-1} \sum_{m=0}^{\infty} \frac{z^{mn+k}}{(mn+k)!} U^{mn}(y_k). \tag{4.4}$$

The proof immediately follows from 4.6, and $y(z) \in B^{(n)}(M)$ implies the convergence of (4.4).

We remark if $U = F_k : x \to kx$ and k is a locally integrable function, then $B(U) = M$. Now we present a theorem which gives an other type sufficient condition for solving CP_1^n.

4.8. *Theorem.* If $\eta^k U$ is the infinitesimal generator of some strongly continuous semi-group for each $k (k=0, 1, \ldots (n-1))$ where $\eta = \exp(2\pi i/n)$ and

$$y_0 \in D(U^n), \; y_1 \in D(U^n) \cap R(U), \; \ldots, \; y^k \in D(U^n) \cap R(U^k), \; \ldots$$

for $k = 1, 2, \ldots, (n-1)$, then CP_1^n can be uniquely solved in $B^{(n)}(M)$ and

$$y(z) = \sum_{i=0}^{n-1} S(z, \eta^i U)(a_i), \tag{4.5}$$

where $S(z, \eta^i U)$ is the semi-group generated by $\eta^i U$ and a_i is the solution of the system

$$\sum_{i=0}^{n-1} \eta^{ik} a_i = c_k \qquad (k=0, 1, \ldots (n-1)), \tag{4.6}$$

where $c_k = U(y_k)$.

Proof. Since $\eta^i \neq \eta^j$ if $i \neq j$, $\det | \eta^{ik} | \neq 0$ and (4.6) can be solved. The CP_1^n can be solved with initial conditions $a_i^k = \eta^{ik} U^k(a_i)$ $(k=0, 1, \ldots (n-1))$, and (2.11) shows that the solution is $S(z, \eta^i U)(a_i)$. Therefore

$$\sum_{i=0}^{n-1} S(z, \eta^i U)(a_i) = y(z) \tag{4.7}$$

is a solution of CP_1^n whenever $\{a_i\}$ are choiced sufficiently. Taking the derivative of (4.7) and substituting $z=0$ we get

$$U^k\left(\sum_{i=0}^{n-1} \eta^{ik} a_i\right)=y_k \qquad (k=0, 1, \ldots (n-1)). \tag{4.8}$$

Since $y_k \in D(U^n) \cap R(U^k)$, there is a $c_k \in D(U^k)$ such that $c_k = U^k(y_k)$; hence the solution of (4.7) is a solution of (4.8). By the assumption, $S(z, \eta^i U)$ is strongly continuous, $y(z) \in B^{(n)}(M)$, which completes the proof.

Finally let us mention the case, when there are i's, that $\eta^i U$ does not generate a semi-group. p is said to be the *degree of freedom of* CP^n if there are exactly p transformations among $\eta^i U$ $(i=0, 1, \ldots, (n-1))$ which generate semi-groups. It is obvious that p the number of initial conditions which can be given "arbitrarily".

4.9. *Theorem.* Let p be the degree of freedom of CP_1^n. If

$$y_i \in D(U^n) \cap R(U^i) \quad (i=0, 1, \ldots, (p-1)), \quad y(z) \in B^{(p)}(M), \; y^{(k)}(0)=y_k$$

$(k=0, 1, \ldots, (p-1))$ and $y(z)$ is a solution of CP_1^n with initial values $y_0, \ldots, y_{p-1}, \ldots, y_{n-1}$ and U satisfies the conditions of theorem 4.6, then it is sufficient and necessary for $y(z) \in B^{(n)}(M)$ that

$$y_j = U^j\left(\sum_{k=1}^{p} \eta^{i_k j} a_k\right) \qquad (j=p, \ldots, (n-1)), \tag{4.9}$$

where $\{a_k\}$ is the solution of the system

$$y_r = U^r\left(\sum_{k=1}^{p} \eta^{r i_k} a_k\right) \qquad (r=0, 1, \ldots, (p-1)), \tag{4.10}$$

where $\eta = \exp(2\pi i/n)$ and $\eta^{i_k} U$ generates a semi-group for $k=1, \ldots p$. *Proof.* Since U satisfies the conditions of theorem 4.6, there is only one solution in $B^{(n)}(M)$. If η^{i_1} $\eta^{i_1} U$ generates a semi-group, then $y_{i_1}(z) = S(z, \eta^{i_1} U)(a_1)$ is the solution of CP_1^n with initial values $y_{i_1}^{(j)}(0) = \eta^{i_1 j} U^j(a_1)$. If

$$y(z) = \sum_{k=1}^{p} S(z, \eta^{i_k} U)(a_k)$$

is a solution of CP^p, then (4.9) and (4.10) follow from (4.6). Conversely, if (4.9) and (4.10) are fulfilled, then $y(z)$ is a solution of CP^n. Since the solution of CP^p is unique the solution of CP^n is, too.

We remark that in the case $n=2$, if U and $-U$ generate semi-groups, then U actually generates a group and the solution with initial values y_0, y_1 will be

$$y(z) = \frac{1}{2}\left(S(z, U)(y_0 + x) + S(-z, U)(y_0 - x)\right)$$

where $U(x) = y_1$. (The solution is independent of the choice of x.)

Summary

In a previous paper [in (2)] we gave a method to construct new type of operator transformations with domain M, the whole operator field. This paper deals with another version to deal with operator transformations that have domain possibly only a proper subset of the operator field. Here the main idea of the semi-group theory of transformations, due to *Hille, Yosida* and *Phillips*, will be adapted to the set of linear operator transformations.

References

1. BLEYER, A.: A note to the general theory of the operator transformations, Doklady (Sofia) (1975) Tom 28 No. 6
2. BLEYER, A.: A note to the construction of continuous operator transformations, Acta Math. Acad. Sci. Hung. (in print). (1973)
3. BLEYER, A.: Kandidátusi disszertáció (Hungarian) (1974) MTA (Hungarian Academy of Sciences)
4. BOEHME, T. K.: On Mikusinski operators, Studia Math. 33 (1969)
5. ERDÉLYI, A.: Operational Calculus and Generalized Functions, New York 1963
6. GESZTELYI, E.: Über lineare Operatoren Transformationen, Publ. Math. (Debrecen) 14 (1967)
7. GESZTELYI, E.: Über das Stieltjes-Integral von Operatorenfunktionen. II., Publ. Math. (Debrecen) 13 (1966)
8. HILLE, E.–PHILLIPS, R. S.: Functional analysis and semi-groups, Amer. Math. Soc. 1957
9. MÁTÉ, L.: On the problem of Mikusinski's logarithm, MTA Közl. 7 (1962)
10. MÁTÉ, L.: Note on Mikusinski's logarithm, Bull. Acad. Pol. XIII. tom. 9 (1965)
11. MIKUSIŃSKI, J.: Operational Calculus, New York, 1959
12. URBANIK, K.: Sur la structure non-topologique du corps des opérateurs, Studia. Math. 14 (1954)

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