CONTRIBUTION TO THE VARIATIONAL METHODS FOR SOLVING STATIC AND STATIONARY ELECTROMAGNETIC FIELD PROBLEMS

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1. Introduction

Static and stationary electromagnetic field problems lead to one and the same mathematical formalism. In linear cases, this means the solution of the Laplace equation satisfying some given Dirichlet, Neumann or mixed boundary conditions. There are several numerical approximate methods for the solution. Each method has its advantages as compared to others. The aim of the present paper is to discuss two methods based on variational principles and combine them in order to obtain a method suitable for the solution of certain practical problems.

One of the two methods mentioned above is the finite element method [1], [2], [3], [4]. According to this method the plane region studied (only two-dimensional problems are discussed) is divided into small triangles. The potential is approximated by different polynomials in each domain, applying Ritz's method. The degree of the polynomials is the same in each domain. The coefficients of the approximate functions are expressed with the aid of the node potentials of the elementary triangles, thus the potential function is continuous. At the boundary lines of the triangles the derivatives of the potential function are broken. The advantage of this method is its flexibility and the fact that, mainly in the critical places, the boundary conditions can be satisfied. However, the method needs a large computer storage.

The other method [5], [6], [7], [8] approximates the potential function by applying a continuous function set in the whole region. According to this method both the Laplace equation and the boundary conditions are satisfied by seeking the stationary point of a suitable functional. Ritz's method is applied to numerical solutions. In certain cases this method can yield quite a good solution and its need for computer storage is smaller. Since the boundary conditions are not exactly satisfied owing to the functional approximation, the solution will not be so good in the critical places.

This paper combines the two methods keeping their advantageous properties. For demonstrational purposes an example is presented as well.

2. Field equations

As it is known, static electrical, static magnetic and stationary flow current fields lead to one and the same problem using the scalar potential for the solution. Thereby we discuss the case of static magnetic field problems only. Static magnetic fields can be considered to be stationary fields excited by stationary currents in the region where currents do not flow.

The Maxwell equations to be solved are:

$$\text{rot } \mathbf{H}=\mathbf{0} \tag{1}$$

$$\text{div } \mathbf{B}=\mathbf{0} \tag{2}$$

$$\mathbf{B}=\mu\mathbf{H}, \tag{3}$$

where μ is the tensor of the inhomogeneous anisotrop substance, its elements depend on coordinates, but are independent of field intensities. The scalar potential is applied in the solution

$$\mathbf{H}=-\text{grad}\,\varphi, \tag{4}$$

and we get the Laplace equation

$$\text{div}(\mu\ \text{grad}\,\varphi)=0 \tag{5}$$

to be solved. The boundary conditions are of mixed type. This means that

$$\varphi=\Phi \text{ on the section } S_D, \tag{6}$$

$$\mathbf{n}\mu\text{grad}\varphi=B_n \text{ on the section } S_N, \tag{7}$$

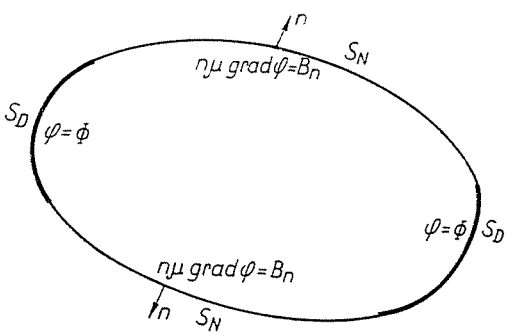

Fig. 1

where S is the bounding surface of the region studied (Fig. 1), Φ is the known value of the potential (it can vary on S_D), and B_n is the normal component of the magnetic induction vector on S_N.

Our object is to solve the Laplace equation (5) satisfying boundary conditions (6) and (7).

3. Variational formulas

The methods based on variational principles apply a suitable functional to the potential function. At zero variation of this functional on φ, both the Maxwell equations and the boundary conditions are satisfied. There are two main types of variational methods. One of them applies an approximate potential function with continuous derivatives which is valid for the whole region studied. Let this method be called the "whole region" method. According to the other method the region is divided into elementary triangle domains. In each domain, the potential function is approximated by different polynomials applying Ritz's method. The degree of the polynomials is the same in each domain, and thus the potential function is continuous, but its derivatives are broken at the bounding lines of the domains. This is the finite element method.

The two methods differ in the approximation of the potential function only. It appears reasonable to combine the two methods. This means that we use the finite element approximation in the subregions where the potential function and the boundary conditions are expected to vary strongly. In other subregions we use the so-called "whole region" approximation.

We seek the solution of the Laplace equation (5) considering boundary conditions (6) and (7). It can be proved that the solution of Eq. (5) considering boundary conditions (6) and (7) is the stationary function of the following functional (the stationary function of a functional is the function where the first variation of the functional equals zero):

$$W(\varphi)=\frac{1}{2}\iiint_V \operatorname{grad}\varphi\mu\operatorname{grad}\varphi\,\mathrm{d}V+$$

$$+\iint_{S_N}\varphi B_n\,\mathrm{d}s+\iint_{S_D}(\varphi-\Phi)\mathbf{n}\operatorname{grad}\varphi\,\mathrm{d}S. \tag{8}$$

We discuss two-dimensional problems only, this is why the functional (8) can be written in the form

$$W(\varphi)=\frac{1}{2}\iint_A \operatorname{grad}\varphi\mu\operatorname{grad}\varphi\,\mathrm{d}A+$$

$$+\int_{C_N}\varphi B_n\,\mathrm{d}C+\int_{C_D}(\varphi-\Phi)\mathbf{n}\mu\operatorname{grad}\varphi\,\mathrm{d}C. \tag{9}$$

Let us divide the studied region into two subregions and one of these subregions into several elementary triangle domains (Fig. 2). In region *I* we seek the solution applying the "whole region" method, in region *II* we seek the solution applying the "finite element" method.

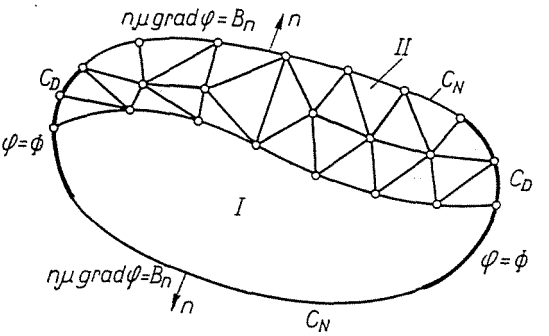

Fig. 2

In region I the potential function is approximated by Ritz's method:

$$\varphi \approx \varphi_I = \sum_{k=1}^{n} a_k f_k, \tag{10}$$

where f_k is the k-th element of an entire set of functions and a_k are the coefficients to be determined. It is assumed that there are n points inside region I including those on the curve separating the two regions (n is the number of the elements of the approximate function set in region I). The potential values of the points in region I and on its bounding curve can be written in the form

$$\begin{aligned}
\varphi_{Ip1} &= a_1 f_1(p_1) + a_2 f_2(p_1) + \ldots + a_n f_n(p_1) \\
\varphi_{Ip2} &= a_1 f_1(p_2) + a_2 f_2(p_2) + \ldots + a_n f_n(p_2) \\
&\cdot \\
&\cdot \\
&\cdot \\
\varphi_{Ipn} &= a_1 f_1(p_n) + a_2 f_2(p_n) + \ldots + a_n f_n(p_n)
\end{aligned} \tag{11}$$

where $f_k(P_i)$ is the value of the k-th element of the function set in the i-th point and $\varphi_{Ip\,i}$ means the value of the potential of the i-th point in region I. Since the number of equations is equal to the number of coefficients, the coefficients can be expressed by the values of potentials of the points in region I:

$$\mathbf{a} = A^{-1}\mathbf{\Phi}_I, \tag{12}$$

where

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix}, \tag{13}$$

$$\mathbf{\Phi}_I=\begin{bmatrix}\varphi_{Ip1}\\ \varphi_{Ip2}\\ .\\ .\\ .\\ \varphi_{Ipn}\end{bmatrix}, \tag{14}$$

and

$$A=\begin{bmatrix}f_1(P_1)\ f_2(P_1)\ \ldots\ldots\ f_n(P_1)\\ f_2(P_2)\ f_2(P_2)\ \ldots\ldots\ f_n(P_2)\\ .\\ .\\ .\\ f_n(P_n)\ f_2(P_n)\ \ldots\ldots f_n(P_n)\end{bmatrix} \tag{15}$$

Substituting into Eq. (10), the potential can be expressed with the potential values of points in region I:

$$\varphi_1=\mathbf{F}^+A^{-1}\mathbf{\Phi}_I, \tag{16}$$

where

$$\mathbf{F}=\begin{bmatrix}f_1\\ f_2\\ .\\ .\\ .\\ f_n\end{bmatrix} \tag{17}$$

and $\mathbf{F}^+$ means the transpose of $\mathbf{F}$.

In region II the potential function is approximated by polynomials of order m over every elementary triangles. The potential function of the j-th domain can be expressed in the same way:

$$\varphi_{IIj}=\mathbf{F}^+A_j^{-1}H_j\mathbf{\Phi}_{II}, \tag{18}$$

where $\mathbf{\Phi}_{II}$ contains the potential values of the points in region II, and matrix H_j is constructed as follows:

$$H_j=\begin{matrix}\text{by local numbers}\\ \text{of the nodes of}\\ \text{the } j\text{-th domain}\end{matrix}\begin{bmatrix}0&1&0&\ldots\ldots&1\\ 1&0&.&\ldots\ldots&0\\ .&&&&\\ &&&&\\ 0&1&.&\ldots\ldots&0\end{bmatrix} \tag{19}$$

by general numbers of the nodes in region II

and

$$\boldsymbol{H}_j(k, l)=\begin{cases}1 \text{ if the } k\text{-th node of local number is the} \\ \quad \text{same as the } l\text{-th node of general number} \\ 0 \text{ else.}\end{cases} \tag{20}$$

The meanings of $\mathbf{F}$ and $\boldsymbol{A}_j$ are the same as before, but related to the j-th domain.

Since the region is divided into subregions and elementary domains in which the potential function is approximated by different analytical functions, integral (9) has to be partitioned in these regions:

$$\begin{aligned} W(\varphi)=&\frac{1}{2}\iint_{A_I} \operatorname{grad} \varphi_I \mu \operatorname{grad} \varphi_I \, \mathrm{d}A_I + \\ &+ \int_{C_{NI}} \varphi_I B_n \, \mathrm{d}C_I + \int_{C_{DI}} (\varphi_I - \Phi)\mathbf{n}\mu \operatorname{grad} \varphi_I \, \mathrm{d}C_I + \\ &+ \sum_{j=1}^{N} \left[\frac{1}{2} \iint_{A_{IIj}} \operatorname{grad} \varphi_{IIj} \mu \operatorname{grad} \varphi_{IIj} \, \mathrm{d}A_{IIj} + \right. \\ &\left. + \int_{C_{NIIj}} \varphi_{IIj} B_n \, \mathrm{d}C_{IIj} + \int_{C_{DIIj}} (\varphi_{IIj} - \Phi)\mathbf{n}\mu \operatorname{grad} \varphi_{Ij} \, \mathrm{d}C_{IIj} \right] \end{aligned} \tag{21}$$

where A_I and A_{IIj} are the surfaces of region I and the j-th domain in region II, respectively, C_{NI}, C_{NIIj}, C_{DI}, C_{DIIj} are sections of the bounding curve and N is the number of elementary domains in region II. (We note that curve integrals have to be performed only if j denotes a bounding domain.) Substituting the approximate potential functions (16) and (18) into Eq. (21), we get a function dependent on real coefficients. These coefficients are the potential values of the points.

$$\begin{aligned} W(\boldsymbol{\Phi}_I, \boldsymbol{\Phi}_{II})=&\frac{1}{2}\iint_{A_I} \operatorname{grad} (\mathbf{F}^{+}\boldsymbol{A}^{-1}\boldsymbol{\Phi}_I)\mu \operatorname{grad} (\mathbf{F}^{+}\boldsymbol{A}^{-1}\boldsymbol{\Phi}_I) \, \mathrm{d}A_I + \\ &+ \int_{C_{NI}} B_n \mathbf{F}^{+}\boldsymbol{A}^{-1}\boldsymbol{\Phi}_I \, \mathrm{d}C_I + \int_{C_{DI}} (\mathbf{F}^{+}\boldsymbol{A}^{-1}\boldsymbol{\Phi}_I - \Phi)\mu \operatorname{grad} \mathbf{F}^{+}\boldsymbol{A}^{-1}\boldsymbol{\Phi}_I)\mathbf{n} \, \mathrm{d}C_I + \\ &+ \sum_{j+1}^{N} \left[\frac{1}{2} \iint_{A_{IIj}} \operatorname{grad} (\boldsymbol{F}^{+}\boldsymbol{A}_j^{-1}\boldsymbol{H}_j\boldsymbol{\Phi}_{II})\mu \operatorname{grad} (\mathbf{F}^{+}\boldsymbol{A}_j^{-1}\boldsymbol{H}_j\boldsymbol{\Phi}_{II}) \, \mathrm{d}A_{IIj} + \right. \\ &\left. + \int_{C_{NIIj}} B_n \mathbf{F}^{+}\boldsymbol{A}_j^{-1}\boldsymbol{H}_j\boldsymbol{\Phi}_{II} \, \mathrm{d}C_{II} + \right. \\ &\left. + \int_{C_{DIIj}} (\mathbf{F}^{+}\boldsymbol{A}_j^{-1}\boldsymbol{H}_j\boldsymbol{\Phi}_{II} - \Phi)\mu \operatorname{grad} (\mathbf{F}^{+}\boldsymbol{A}_j^{-1}\boldsymbol{H}_j\boldsymbol{\Phi}_{II}) \, \mathrm{d}C_{II}. \right. \end{aligned} \tag{22}$$

When seeking the stationary point of function (22), $\mathbf{\Phi}_I$ and $\mathbf{\Phi}_{II}$ can be calculated. In the knowledge of $\mathbf{\Phi}_I$ and $\mathbf{\Phi}_{II}$ the approximate potential function can be determined.

Since the degree of the approximation is different in region I and in the elementary domains, the potential function is broken on the curve separating region *I* and *II*. Increasing the number of points in region *II* and the degree of approximation in region *I*, the potential function must approach to becoming continuous on the separating curve. The potential function is of course continuous at the points of the separating curve, owing to the fact that the parameters are the values of the potentials at the points.

4. Numerical methods

The necessary condition for the existence of a stationary point of function (22) is:

$$\frac{\partial W(\mathbf{\Phi}_I, \mathbf{\Phi}_{II})}{\partial \mathbf{\Phi}_I} = \mathbf{0}, \tag{23}$$

$$\frac{\partial W(\mathbf{\Phi}_I, \mathbf{\Phi}_{II})}{\partial \mathbf{\Phi}_{II}} = \mathbf{0}. \tag{24}$$

Performing the derivations a set of linear equations can be obtained for the potential values $\mathbf{\Phi}_I$ and $\mathbf{\Phi}_{II}$. These equations are:

$$\boldsymbol{B}_I \mathbf{\Phi}_I = \mathbf{b}_I \tag{25}$$

$$\boldsymbol{B}_{II} \mathbf{\Phi}_{II} = \mathbf{b}_{II}, \tag{26}$$

where

$$\boldsymbol{B}_I = \boldsymbol{A}^{-1+} \left[\iint_{A_I} \operatorname{grad} \mathbf{F} \mu \operatorname{grad} \mathbf{F}^+ \, \mathrm{d}A_I + \right.$$

$$\left. + \int_{C_{DI}} \mu \operatorname{grad} (\mathbf{F}\mathbf{F}^+)\mathbf{n} \, \mathrm{d}C_I \right] \boldsymbol{A}^{-1}, \tag{27}$$

$$\mathbf{b}_I = \boldsymbol{A}^{-1+} \left[- \int_{C_{NI}} B_n \mathbf{F} \, \mathrm{d}C_I + \int_{C_{DI}} \Phi \mu \operatorname{grad} \mathbf{F}\mathbf{n} \, \mathrm{d}C_I \right] \tag{28}$$

$$\boldsymbol{B}_{II} = \sum_{j=1}^{N} \boldsymbol{H}_j^+ \boldsymbol{A}_j^{-1+} \left[\iint_{A_{IIj}} \operatorname{grad} \mathbf{F} \mu \operatorname{grad} \mathbf{F}^+ \, \mathrm{d}A_{IIj} + \right.$$

$$\left. + \int_{C_{DIIj}} \mu \operatorname{grad} (\mathbf{F}\mathbf{F}^+)\mathbf{n} \, \mathrm{d}C_{IIj} \right] \boldsymbol{A}_j^{-1} \boldsymbol{H}_j, \tag{29}$$

$$\mathbf{b}_{II} = \sum_{j=1}^{N} \boldsymbol{H}_j^+ \boldsymbol{A}_j^{-1+} \left[- \int_{C_{NIIj}} B_n \mathbf{F} \, \mathrm{d}C_{IIj} + \int_{C_{DIIj}} \Phi \mu \operatorname{grad} \mathbf{F}\mathbf{n} \, \mathrm{d}C_{IIj} \right] \tag{30}$$

The integrals and the grad operations have to be performed on the elements of arrays. The evaluation of integrals over elementary domains can be performed by means of numerical methods. As $\mathbf{\Phi}_I$ means the potential values of region I and $\mathbf{\Phi}_{II}$ those of region II, equations (25) and (26) are not independent, because $\mathbf{\Phi}_I$ and $\mathbf{\Phi}_{II}$ have joint elements (the potential values of the points on the separating curve.) Therefore these equations can be summed. Thus the equations

$$\boldsymbol{B\Phi}=\mathbf{b}$$

can be written.

The Neumann boundary conditions are natural for functional (21), thus in that case no integrals are calculated on curves of C_N type. If the Dirichlet boundary conditions are satisfied in such a way that a satisfactory number of points are chosen also on the bounding curve of Dirichlet type in region I, no integration is performed even on curves of C_D type. In this case the coefficients of the linear set of equations are simpler. The right side of the equations can be formulated by substituting the known potential values into the equations.

Example

For demonstration, let us calculate the approximate potential function over the region which can be seen in Fig. 3.

The boundary conditions are of the Dirichlet type:

$$\Phi=\begin{cases}\text{varies linearly } 0-1 & \text{on } 1\\ 1 & \text{on } 2\\ \text{varies linearly } 1-0 & \text{on } 3\\ 0 & \text{on } 4\\ 0 & \text{on } 5\\ 0 & \text{on } 6\end{cases}$$

It is to be seen that the approximation would be not good in the narrow gap if

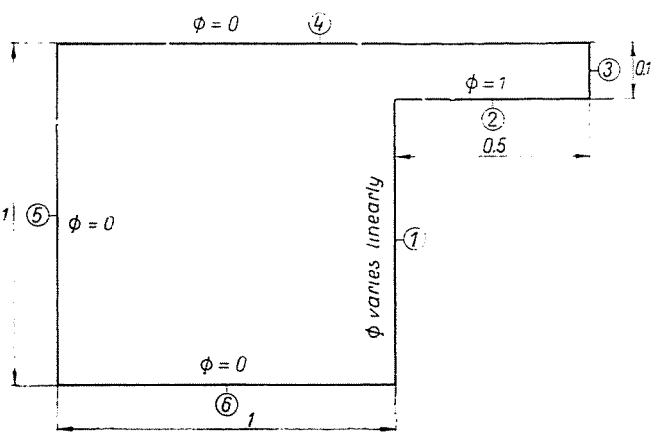

Fig. 3

the "whole region" method were applied. The reason of this fact is that the value of the integral over the narrow gap is negligible compared with the integral performed over the larger part of the region. However, the application of the finite element method would lead to a large number of unknown quantities if the figure of the larger part were more complicate. We apply the method for the solution presented above.

Let us divide the region into subregions as can be seen in Fig. 4.

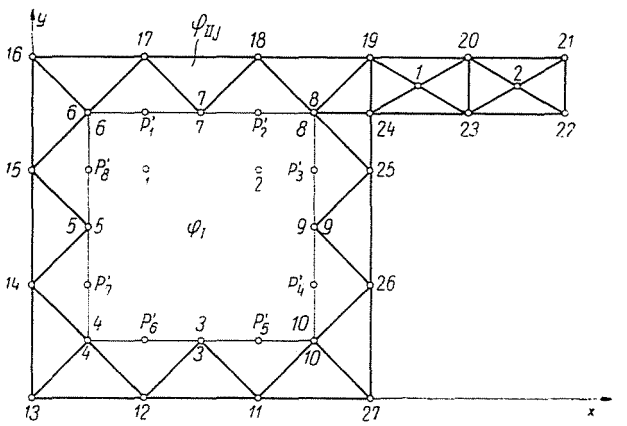

Fig. 4

Performing the calculations, the potential of the points in region *I* are:

$$\Phi_I = \begin{bmatrix} 0.0665 \\ 0.0908 \\ -0.3315 \\ -0.0150 \\ 0.0150 \\ 0.0026 \\ 0.1478 \\ 0.4266 \\ 0.4794 \\ 0.2109 \end{bmatrix}$$

As was mentioned, the potential function is not continuous on the curve separating regions *I* and *II*.

The greatest deviation is in points $P_1', P_2', \ldots, P_8'$ (Fig. 4). Table 1 shows a comparison of these values.

Table 1

Points	Potential values	
	Region I	Region II
P_1'	0.0364	0.0752
P_2'	0.3062	0.2872
P_3'	0.4906	0.4530
P_4'	0.3597	0.3452
P_5'	−0.1165	−0.1583
P_6'	−0.0054	0.0
P_7'	0.0361	0.0752
P_8'	0.0083	0.0088

Summary

The paper elaborates a method for solving static and stationary electromagnetic fields based on variational techniques. The potential function is approximated in different ways in the studied region. The finite element approximation is applied to the subregions where the potential function is expected to vary strongly. The potential function is approximated by means of a set of continuous functions in other, not elementary subregions. The potential function is broken along the curve separating different subregions, except in discrete points. The method needs not so large computer storage and is flexible considering boundary conditions.

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