ON GENERALIZATION OF A THEOREM OF FENYŐ

By

G. S. PANDEY

Department of Electrical Engineering Mathematics, Technical University Budapest

Received April 1, 1977

Presented by Prof. Dr. I. FENYŐ

1. It is well-known that, in general, the value of a no distribution value exists at a point does not exist. Łojasiewicz [2], however, has given a definition and criteria for the existence of the local value of a distribution at a point. According to his definition, if x_0 is a fixed point and $\tau_{\varkappa_0}$ is the operator adjoint to the shifting operator

$$\tau_{-\varkappa_0}\varphi(\varkappa) = \varphi(\varkappa - \varkappa_0),$$

where $\varphi(\varkappa)$ is an arbitrary testing function of the space $D(R)$; and $\varkappa_\gamma$ is the operator adjoint to the affinity operator

$$\frac{1}{|\gamma|}\varkappa_\gamma\,\varphi(\varkappa) = \frac{1}{|\gamma|}\varphi\left(\frac{\varkappa}{\gamma}\right), \qquad \gamma \neq 0\,;$$

then a distribution T has a local value at x_0 given by

$$\lim_{\gamma\to 0}\varkappa_\gamma\,\tau_{\varkappa_0}\,T = t_0, \tag{1}$$

provided the limit exists.

From the above definition it is clear that in general, the value of a distribution T at a point x_0 does not necessarily exist. But, if T is a distribution defined by a continuous function $f(\varkappa)$ and x_0 is in the domain of the definition of f, then the local value of this distribution exists (in the sense of Łojasiewicz) at x_0, and it is equal to $f(\varkappa_0)$.

Fenyő [1], on the other hand, has discussed in detail the problem of existence of the local value of a class of distribution of $D'(R^2)$. In fact, his investigations to assign a local value to a distribution at a point gave rise to a number of interesting problems in the field of applications of the distribution theory.

In the present paper our aim is to study the problem of existence of the local value of a certain class of distributions in the n-dimensional space.

2. Let $D(R^n)$ be the space of testing functions φ of n variables and let $D'(R^n)$ be its dual space with respect to the sequence topology of the space $D(R^n)$.

Let us define a mapping

$$l(a_1, a_2, \ldots, a_n) : D(R^n) \to D(R)$$

in the following way:

$$l(a_1, a_2, \ldots, a_n)\varphi = \frac{1}{|a_n|} \int\limits_{R^n} \varphi\left(\varkappa_1, \varkappa_2, \ldots, \frac{t - a_1\varkappa_1 - \ldots - a_{n-1}\varkappa_{n-1}}{a_n}\right) d\varkappa_1 \ldots d\varkappa_{n-1} =$$

$$= \frac{1}{|a_{n-1}|} \int\limits_{R^n} \varphi\left(\varkappa_1, \varkappa_2, \ldots, \frac{t - a_1\varkappa_1 - \ldots - a_{n-2}\varkappa_{n-2} - a_n\varkappa_n}{a_{n-1}}, \varkappa_n\right) d\varkappa_1 \ldots d\varkappa_{n-2} d\varkappa_n$$

— —

— —

$$= \frac{1}{|a_1|} \int\limits_{R^n} \varphi\left(\frac{t - a_2\varkappa_2 - \ldots - a_n\varkappa_n}{a_1}, \varkappa_2, \ldots, \varkappa_n\right) d\varkappa_2 \ldots d\varkappa_n, \tag{2}$$

where $a_1, a_2, \ldots, a_n$ are real numbers such that at least one $a_k \neq 0$; $k = 1, 2, \ldots, n$.

From our definition it follows that $l(a_1, a_2, \ldots, a_n)\varphi$ is a function of the class $D(R)$ and, with respect to the sequential topology of $D(R)$, it is also linear and continuous from the space $D(R^n)$ into $D(R)$.

Denote by $L(a_1, a_2, \ldots, a_n)$ the mapping adjoint to $l(a_1, a_2, \ldots, a_n)$ given by

$$L(a_1, a_2, \ldots, a_n)(T) \cdot \varphi = T \cdot l(a_1, a_2, \ldots, a_n)(\varphi), \tag{3}$$

where $T \in D'(R)$ and $\varphi \in D(R^n)$.

In case $T = f$, where f is a locally integrable function, we have

$$L(a_1, a_2, \ldots, a_n)(f)(\varkappa_1, \varkappa_2, \ldots, \varkappa_n) = f(a_1\varkappa_1 + a_2\varkappa_2 + \ldots + a_n\varkappa_n).$$

Also, it is easy to see that

$$\frac{\partial}{\partial \varkappa_1} L(a_1, a_2, \ldots, a_n)(T) = a_1 L(a_1, a_2, \ldots, a_n)\left(\frac{dT}{dt}\right),$$

$$\frac{\partial}{\partial \varkappa_2} L(a_1, a_2, \ldots, a_n)(T) = a_2 L(x_1, a_2, \ldots, a_n)\left(\frac{dT}{dt}\right),$$

— — — — — — — — — — — — — — — — —

— — — — — — — — — — — — — — — — —

$$\frac{\partial}{\partial \varkappa_n} L(a_1, a_2, \ldots, a_n)(T) = a_n L(a_1, a_2, \ldots, a_n)\left(\frac{dT}{dt}\right).$$

3. Let ψ be an arbitrary, but fixed, testing function of the class $D(R)$. On the lines of Fenyő (loc. cit.), let us define a linear and continuous mapping from $D'(R^n)$ into $D'(R^{n-1})$ denoted by $\odot$, as follows:

$$(S \odot \psi)\cdot\varphi := S\cdot\varphi\otimes\psi \qquad \forall\,\varphi \in D(R^{n-1}),$$
$$(\psi \odot S)\cdot\varphi := S\cdot\psi\otimes\varphi \qquad \forall\,\varphi \in D(R^{n-1}), \tag{4}$$

where $S \in D'(R^n)$ and $\varphi\otimes\psi$ denote the tensor products of the functions given by

$$(\varphi\otimes\psi)(\varkappa_1, \varkappa_2, \ldots, \varkappa_{n-1}, \varkappa) = \varphi(\varkappa_1, \varkappa_2, \ldots, \varkappa_{n-1})\cdot\psi(\varkappa),$$

that is, $S \odot \psi$ and $\psi \odot S$ are distributions in $D'(R^{n-1})$.

We shall prove the following:

Theorem. If $T \in D'(R)$ and $\psi \in D(R)$, then the distribution

$$L(a_1, a_2, \ldots, a_n)(T) \odot \psi$$

has a local value (in the sense of Łojasiewicz) at every point $(\mathring{\varkappa}_1, \mathring{\varkappa}_2, \ldots, \mathring{\varkappa}_{n-1})$ given by

$$\big(L(a_1, a_2, \ldots, a_n)\, T \odot \psi\big)_{(\mathring{\varkappa}_1, \mathring{\varkappa}_2, \ldots, \mathring{\varkappa}_{n-1})} = (\varkappa_{a_n}\, \tau_{a_1\mathring{\varkappa}_1 + \ldots + a_{n-1}\mathring{\varkappa}_{n-1}} \cdot T \cdot \psi).$$

4. *Proof of the theorem.* Let us consider the function

$$F(\varkappa_1, \varkappa_2, \ldots, \varkappa_n) = \varkappa_{a_n}\tau_{a_1\varkappa_2 + \ldots + a_{n-1}\varkappa_{n-1}} \cdot T\psi =$$
$$= \frac{1}{a_n} T\cdot\psi\left(\frac{t - a_1\varkappa_1 - \ldots - a_{n-1}\varkappa_{n-1}}{a_n}\right). \tag{5}$$

Now, in order to prove the theorem, it will be enough to show that F is a continuous function at every point. Let $\mathring{X} = (\mathring{\varkappa}_1, \mathring{\varkappa}_2, \ldots, \mathring{\varkappa}_{n-1})$ be a fixed point in the space R^{n-1}.

Since $\psi \in D(R)$, there exists a positive number c such that supp $\psi \in \in (-c, c)$.

Let N_ε be a neighbourhood of the point $\mathring{X}$, $\varepsilon > 0$, then for any point $X = (\varkappa_1, \varkappa_2, \ldots, \varkappa_{n-1}) \in N_\varepsilon$, we have

$$\operatorname{supp}_t \psi\left(\frac{t - a_1\varkappa_1 - \ldots - a_{n-1}\varkappa_{n-1}}{a_n}\right) \subset$$
$$\subset \{|t| \le |a_1(\mathring{\varkappa}_2 + \varepsilon) + \ldots + a_{n-1}(\mathring{\varkappa}_{n-1} + \varepsilon)| + |a_n c|\}. \tag{6}$$

Thus, we get

$$\lim_{X\to\mathring{X}} \frac{d^k}{dt^k}\psi\left(\frac{t - a_1\varkappa_2 - \ldots - a_{n-1}\varkappa_{n-1}}{a_n}\right) = \frac{d^k}{dt^k}\psi\left(\frac{t - a_1\mathring{\varkappa}_2 - \ldots - a_{n-1}\mathring{\varkappa}_{n-1}}{a_n}\right), \tag{7}$$

$$k = 0, 1, 2, \ldots,$$

uniformly for all real t.

Hence, it follows that

$$\left|\frac{d^k}{dt^k}\psi\left(\frac{t-a_1\varkappa_2-\ldots-a_{n-1}\varkappa_{n-1}}{a_n}\right)-\frac{d^k}{dt^k}\psi\left(\frac{t-a_1\mathring{\varkappa}_2-\ldots-a_{n-1}\mathring{\varkappa}_{n-1}}{a_n}\right)\right|\leq$$

$$\leq\frac{AM_k}{a_n^2}\left|(\varkappa_1-\mathring{\varkappa}_2)+\ldots+(\varkappa_{n-1}-\mathring{\varkappa}_{n-1})\right|,$$

where A is a positive constant depending on $a_1, a_2, \ldots, a_n$ and

$$M_k=\max_{t\in R}\left|\psi^{(k+1)}(t)\right|.$$

Now, from (6) and (7), we have

$$\psi\left(\frac{t-a_1\varkappa_1-\ldots-a_{n-1}\varkappa_{n-1}}{a_n}\right)\to\psi\left(\frac{t-a_1\mathring{\varkappa}_1-\ldots-a_{n-1}\mathring{\varkappa}_{n-1}}{a_n}\right)$$

as $X\to\mathring{X}$ in the pseudo-topology of the space $D(R)$.

Thus, finally, we get

$$\lim_{X\to\mathring{X}}T\cdot\psi\left(\frac{t-a_1\varkappa_1-\ldots-a_{n-1}\varkappa_{n-1}}{a_n}\right)=T\cdot\psi\left(\frac{t-a_1\mathring{\varkappa}_1-\ldots-a_{n-1}\mathring{\varkappa}_{n-1}}{a_n}\right).$$

This proves that $F(\varkappa_1, \varkappa_2, \ldots, \varkappa_{n-1})$ is continuous.

Let φ be an arbitrary function of class $D(R^{n-1})$ and N_c a neighbourhood of the origin such that $\operatorname{supp}\varphi\subset N_c$.

Hence, we find that:

$$\int_{N_c}\varphi(\varkappa_1,\varkappa_2,\ldots,\varkappa_{n-1})\,\psi^{(k)}\left(\frac{t-a_1\varkappa_1-\ldots-a_{n-1}\varkappa_{n-1}}{a_n}\right)d\varkappa_1\ldots d\varkappa_{n-1}=$$

$$=\int_{R^n}\varphi(\varkappa)\,\psi^{(k)}\left(\frac{t-a_1\varkappa_1-\ldots-a_{n-1}\varkappa_{n-1}}{a_n}\right)d\varkappa_1\ldots d\varkappa_{n-1}.$$

Therefore, it follows that:

$$T\cdot\int_{R^n}\varphi(\varkappa_1,\ldots,\varkappa_{n-1})\,\psi\left(\frac{t-a_1\varkappa_1-\ldots-a_{n-1}\varkappa_{r-1}}{a_n}\right)d\varkappa_1\ldots d\varkappa_{n-1}=$$

$$=\int_{R^n}\varphi(\varkappa_1,\ldots,\varkappa_{n-1})\,T\cdot\psi\left(\frac{t-a_1\varkappa_1-\ldots-a_{n-1}\varkappa_{n-1}}{a_n}\right)\cdot d\varkappa_1\ldots d\varkappa_{n-1}=$$

$$=|a_n|\int_{R^n}F(\varkappa_1,\ldots,\varkappa_{n-1})\,\varphi(\varkappa_1,\ldots,\varkappa_{n-1})\,d\varkappa_1\ldots d\varkappa_{n-1}. \tag{8}$$

Finally, using (8) and (4), we have:

$$\big(L(a_1, a_2, \ldots, a_n) \odot \psi\big)\cdot\varphi = L(a_1, \ldots, a_n)\, T\cdot \varphi(\varkappa_1, \ldots, \varkappa_{n-1}) \otimes \psi(\varkappa_n) =$$

$$= T\cdot l(a_1, \ldots, a_n)\, \varphi(\varkappa_1, \ldots, \varkappa_{n-1}) \otimes \psi(\varkappa_n) =$$

$$= T\frac{1}{|a_n|}\int\limits_{R^{n-1}} \varphi(\varkappa_1, \ldots, \varkappa_{n-1})\cdot\psi\left(\frac{t - a_1\varkappa_1 - \ldots - a_{n-1}\varkappa_{n-1}}{a_n}\right)\cdot d\varkappa_1 \ldots d\varkappa_{n-1} =$$

$$= \frac{1}{|a_n|}\int\limits_{R^{n-1}} \varphi(\varkappa_1, \ldots, \varkappa_{n-1})\, T\psi\left(\frac{t - a_1\varkappa_1 - \ldots - a_{n-1}\varkappa_{n-1}}{a_n}\right)\cdot d\varkappa_1 \ldots d\varkappa_{n-1} =$$

$$= \int\limits_{R^{n-1}} F(\varkappa_1, \ldots, \varkappa_{n-1})\, \varphi(\varkappa_1, \ldots, \varkappa_{n-1})\, d\varkappa_1 \ldots d\varkappa_{n-1}\,.$$

This implies that the distribution $L(a_1, \ldots, a_n)(T) \odot \psi$ is identical with the function $F(\varkappa_1, \ldots, \varkappa_{n-1})$, which is continuous at every point.

Now, using the theorem of Łojasiewicz (loc. cit., p. 7), we find that

$$\big(L(a_1, a_2, \ldots, a_n)(T) \odot \psi\big)_{(\mathring{\varkappa}_1, \ldots, \mathring{\varkappa}_{n-1})} = F(\mathring{\varkappa}_1, \ldots, \mathring{\varkappa}_{n-1}) =$$

$$= \varkappa_{a_n}\tau_{a_1\mathring{\varkappa}_1 + \ldots + a_{n-1}\mathring{\varkappa}_{n-1}}\, T\cdot\psi.$$

The author is highly thankful to Professor Dr. István FENYŐ for his kind help and valuable advice during the preparation of this paper.

Summary

Criteria for the existence of the local value of a certain class of distributions in the n-dimensional space are investigated.

References

1. FENYŐ, I.: Sur un théorème de M. Picone, *Accademia Nazionale dei Lincei, Serie* VIII, *Vol.* L, *fasc.* 5 (1971), 517–523.
2. ŁOJASIEWICZ, S.: Sur la valeur et la limite d'une distribution en un point, Studia Math., Vol. **16** (1958), 1–36.

Dr. G. S. PANDEY Department of Mathematics Vikram University
Ujjain, M. P. India.