

# A NEW METHOD FOR GENERATING ROOT LOCUS DIAGRAMS

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## Introduction

The root locus diagram is an efficient tool in the synthesis and analysis of linear systems with constant coefficients. Several methods and programs are available for determining the diagram [1], [2], [3], [4], [9].

The methods determining the root locus diagram by solving the characteristic equation are unsuitable for determining the root locus plot in equidistant points or for a system with dead time.

KRALL and FORNARO described for the first time an algorithm and a program bypassing the problem described above [6], [7].

A program for generating the generalized root locus diagram of multiloop linear systems has also been worked out [5], [8]. By the KRALL—FORNARO equations it can be decided whether a point of the complex plane is on the root locus diagram of a given system with dead time or not. In this paper a new method is given permitting fast, simple determination of the root locus diagram of a system with dead time at the desired accuracy in equidistant points of a given domain. The new method can be well used with minicomputers.

## Theoretical bases

The presented method is suitable for calculating both ordinary and time lag root locus diagrams. Previously we showed that generating the generalized root locus diagram of a multiloop linear system might be reduced to that of a root locus diagram of an ordinary system [5].

The characteristic equation of the system may be solved in the form

$$K(s) = G(s) + pe^{-s\tau}H(s) = 0 \quad (1)$$

where

$$G(s) = \sum_{i=0}^n g_i s^{n-i}$$

$$H(s) = \sum_{i=0}^m h_i s^{m-i}$$

and  $p$  is a variable parameter.

**Definition:** The root locus diagram is the geometrical locus of all the points  $s$  for which  $K(s) = 0$  is valid at some real value of the parameter  $p$ , or  $s$  is the zero of  $H(s)$ .

It may be observed that the definition assigns the zeros of  $H(s)$  separately. Since for the zeros of  $H(s)$  the  $p$  parameter is not finite, this distinction is useful. Also the points for which the parameter  $p$  is a negative real value are seen to be assigned by definition to the root locus diagram.

In this sense positive and negative branches of the root locus diagram can be spoken of.

By definition, the points for which  $p \geq 0$  ( $p < 0$ ) belong to the positive (negative) branches of the root locus diagram. In control engineering usually only the positive branches of the root locus diagram are considered.

**Theorem 1:** The point  $s_i = x_i + jy_i$  is on the root locus of  $K(s)$  if and only if  $z(x_i, y_i) = 0$ , where

$$z(x_i, y_i) = \cos(y_i\tau) \{ \text{Im}[G(s_i)] \text{Re}[H(s_i)] - \text{Re}[G(s_i)] \text{Im}[H(s_i)] \} + \\ + \sin(y_i\tau) \{ \text{Re}[G(s_i)] \text{Re}[H(s_i)] - \text{Im}[G(s_i)] \text{Im}[H(s_i)] \} = 0. \quad (2)$$

**Proof:** The proof is related to the case where  $G(s_i) \neq 0$  and  $H(s_i) \neq 0$ . First assume the point  $s_i = x_i + jy_i$  to be on the root locus diagram, i.e.  $K(s_i) = 0$ . By introducing the nominations

$$H(s_i) = H_1 + jH_2$$

$$G(s_i) = G_1 + jG_2$$

and

$$e^{-j\tau y_i} = T_1 - jT_2$$

$$pe^{-\tau x_i} = C,$$

$z(x_i, y_i)$ , and  $K(s_i)$  may be written, in correspondence with Eqs (1) and (2) as

$$z(x_i, y_i) = T_1(G_2H_1 - G_1H_2) + T_2(G_1H_1 - G_2H_2) = 0 \quad (3)$$

and

$$K(s_i) = G_1 + jG_2 + C(T_1 - jT_2)(H_1 + jH_2) = 0. \quad (4)$$

From Eq. (4) the variable  $C$  can be eliminated and with this, the theorem is proved.

Inversely, if Eqs (2) or (3) are valid for some point  $s_i = x_i + jy_i$  then Eq. (3) divided by  $G_1H_1 \neq 0$  yields the relationship

$$\frac{\text{tg } \gamma_i - \text{tg } \delta_i}{1 + \text{tg } \delta_i \text{tg } \gamma_i} = \text{tg}(y_i\tau) \quad (5)$$

where  $\text{tg } \delta_i = G_2/G_1$ ,  $\text{tg } \gamma_i = H_2/H_1$  and  $\text{tg } (y_i\tau) = \sin (y_i\tau)/\cos (y_i\tau)$ . From Eq. (5) the angular condition

$$\gamma_i - \delta_i - (y_i\tau) = \pm N180^\circ \quad (N = 1, 2, 3, \dots)$$

may be written up directly.

This condition shows if point  $s_i$  is on the positive or on the negative branch of the root locus diagram. So theorem 1 is proved fully.

*Theorem 2:* If point  $s_i = x_i + jy_i$  is on the root locus diagram and  $H(s_i) \neq 0$ , then the real parameter  $p$  can be computed from the formula

$$p = -\text{Re}[G(s_i)] e^{\tau x_i} / (\cos (y_i\tau) \text{Re}[H(s_i)] + \sin (y_i\tau) \text{Im}[H(s_i)]) . \quad (6)$$

Note: The statement of the theorem is directly evident from Eq. (4).

If  $\tau = 0$ , the points of root locus diagram and the parameter  $p$  can be computed on the basis of theorems 1 and 2 by the formula

$$z(x, y) = G_2H_1 - G_1H_2 = 0 \quad (7)$$

and

$$p = -G_1/H_1 . \quad (8)$$

We note that Eq. (6) of theorem 2 may also be given in the form

$$p = -G_1 e^{\tau x_i} / (T_1H_1 + T_2H_2) , \quad (9)$$

or in the form

$$p = -G_2 e^{\tau x_i} / (T_1H_2 - T_2H_1) . \quad (9a)$$

### Computation method

Denote the pair  $(x_n, y_m)$  a point in the rectangular region  $i$  of the complex plane. Let be given this region by the corner points  $xb_i, ya_i, xb_i + N_i dx_i, ya_i + M_i dy_i$ , where  $dx_i, dy_i$  are increments in  $x$  and in  $y$ , respectively, and  $N_i, M_i$  are the number of dividing points along the co-ordinate axes  $x$  and  $y$ .

Decision on the points  $(x, y)$  can be done by Eqs (3), (7), (8), (9) in the following way:

1. Let us fix  $x$ :

$$x_n = xb_i + ndx_i \quad (n = 0, 1, \dots, N_i) .$$

2. At a given  $x_n$  mark out the intervals

$$I_k = (y_k, y_{k+1}) \quad (k = 0, 1, \dots, M_i - 1)$$

by the points:

$$y_m = ya_i + mdy_i \quad (m = 0, 1, \dots, M_i).$$

3. Find those intervals  $I_k$  where  $\varphi(y) = z(x_n, y)$  changes its sign. This is controlled by the validity of the condition:

$$\text{sign}(\varphi(y_k)) \neq \text{sign}(\varphi(y_{k+1}))$$

(i.e. only a single sign change is assumed to occur in the interval  $I_k$ ).

4. In the intervals  $I_k$  where the characteristic equation has a root, the series  $\{c_i\}$  ( $i = 1, 2, \dots$ ) approximating this root is computed by the following steps:

$$\begin{aligned} b &\leftarrow y_k \\ j &\leftarrow y_{k+1} \\ c &\leftarrow (y_k + y_{k+1})/2 \end{aligned} \quad (10)$$

*A*: if  $\text{sign}(\varphi(b)) = \text{sign}(\varphi(c))$  then  $b \leftarrow c$  else

$$j \leftarrow c; \quad c \leftarrow (j + b)/2$$

For  $j - b > \varepsilon$  the computation is to be continued from the label *A*.

It may be observed that this mode of computation is rather advantageous if  $z(x_n, y)$  can be written as

$$z(x_n, y) = \varphi_{x_n}(y) = \varphi(y). \quad (11)$$

The KRALL-FORNARO equations are more complicated than these presented here, and they give the separation corresponding to formula (11) in a closed form, analytically.

The computation will still simplify if the separation is not made analytically but numerically by using the HORNER scheme.

The steps of this are:

1. Polynomials  $G(s)$  and  $H(s)$  are expanded into a series around  $x_n$ , i.e. the coefficients  $gr_i, hr_j$  ( $i = 1, 2, \dots, n; j = 1, 2, \dots, m$ ) in the following power series

$$\begin{aligned} G(s) &= \sum_{i=0}^n g_i s^{n-i} = \sum_{i=0}^n gr_i (s - x_n)^i \\ H(s) &= \sum_{j=0}^m h_j s^{n-j} = \sum_{j=0}^m hr_j (s - x_n)^j \end{aligned} \quad (12)$$

are determined.

2. For determining  $\varphi(y)$ , the substitute values

$$G(s) |_{s=x_n+jy} = \sum_{i=0}^n gr_i(jy)^i = G_1(y) + jG_2(y)$$

and

$$H(s) |_{s=x_n+jy} = \sum_{k=0}^m hr_k(jy)^k = H_1(y) + jH_2(y) \tag{13}$$

are computed by the HORNER scheme.

In Eqs (13),  $G_1, G_2, H_1, H_2$  are mere functions of  $y$  at a given  $x_n$ . The series expansion according to formula (12) for  $x_n$  is seen to have to be performed only once.

For the series expansion and the computation of the substitute value according to formula (13), different HORNER scheme must be used. This substitute value may be computed, e.g. by the following algorithm, introducing the notation:

$$A(s) |_{s=jy} = \sum_{i=0}^n a_i s^i |_{jy} = \sum_{i=0}^n a_i (jy)^i = R + jI . \tag{14}$$

The steps of the algorithm are:

$$\text{step 1: } R = a_n, I = 0$$

$$\text{step } k: (k = 2, 3, \dots, n + 1)$$

$$b = -I \cdot y + a_{n-k-1}$$

$$I = R \cdot y$$

$$R = b$$

And for the series expansion the algorithm of PANKIEWITZ [8] may be used. For the sake of completeness we note that it is useful to involve the following services into the computer program:

1. Generating polynomials from the pole-zero configuration and vice versa.
2. Placing the point of the pole-zero configuration into the given ranges.
3. In the case if  $n - m$  is even, placing the point computed by

$$x_a = (h_1/h_0 - g_1/g_0)/(n - m) \tag{15}$$

into the sequence of equidistant points for computing also the possible vertical branch of root locus diagram.

Here the coefficients  $g_0, g_1$  and  $h_0, h_1$ , and the degree numbers  $n, m$  are defined by Eq. (12).

In the following item the flow-chart and the program to be referred to as RL will be described.

### Description of the RL Program

We have prepared an R10 FORTRAN program according to the present-  
ed computation procedure. The simplified flow-chart of the program is given  
in Fig. 1 with the following notations:

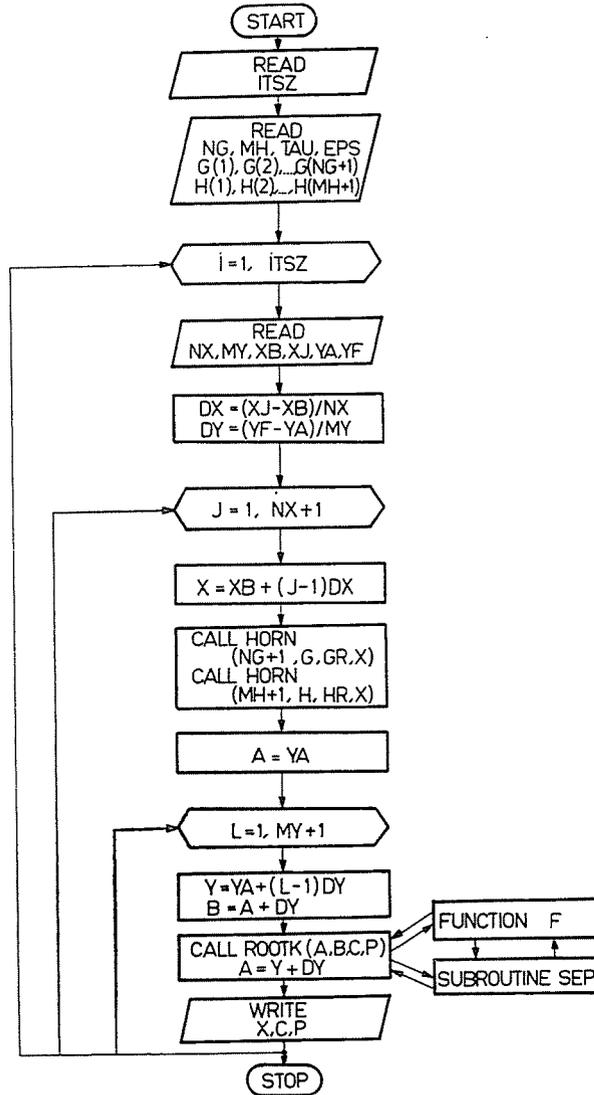


Fig. 1

Input variables:

ITSZ	number of ranges to be investigated
NG	degree of the polynomial $G(s)$
G(1)	} coefficients of polynomial $G(s)$ where $G(1)$ is the coefficient of the highest-degree term
G(2)	
⋮	
G(NG + 1)	
MH	degree of polynomial $H(s)$
H(1)	} coefficients of polynomial $H(s)$ where $H(1)$ is the coefficient of the highest-degree term
H(2)	
⋮	
H(MH + 1)	
TAU	dead time
EPS	accuracy of the root
NX	number of division points in direction $X$
MY	number of division points in direction $Y$
XB	abscissa of the left-side end point of a range
XJ	abscissa of the right-side end point of a range
YA	ordinate of the lower end point of a range
YF	ordinate of the upper end point of a range

Output and dummy variables:

J	} loop variables
L	
DX	equidistant increment of $X$
X	actual value of the abscissa
DY	equidistant increment of $Y$
Y	actual value of the ordinate
A	one end point of a range containing a single possible root
B	second end point of a range containing a single possible root
C	ordinate of the root
P	value of parameter $p$ belonging to the root locus point $(x, c)$
GR(1)	} power series expansion of polynomial $G(s)$ at a given point $\alpha$ . The coefficient of the highest-degree variable: GR(NG + 1)
⋮	
GR(NG + 1)	
HR(1)	
⋮	} power-series-expanded form at $H(s)$ . The coefficient of the highest-degree variable: HR(MH + 1)
HR(MH + 1)	

The subroutines of RL are:

HORN  
 ROOTK  
 SEP  
 F

At the first level HORN and ROOTK are called. At the second level ROOTK calls SEP and F, and F calls SEP.

The calling and the list of the parameters of subroutines HORN and ROOTK called at the first level are given by the flow chart in Fig. 1.

Their task are as follows:

SUBROUTINE HORN performs the power series expansion,

SUBROUTINE ROOTK determines the possible single root found in the interval (A, B) at an accuracy EPS and computes the value of the parameter  $p$  for the found root.

The task of SUBROUTINE F is to compute the left-hand side of Eq. (3).

SUBROUTINE SEP computes the substitute values  $G_1, G_2, H_1, H_2$  of Eqs (13).

### Examples, outputs

#### Example 1

This example shows the application of the elaborated new method. Be the characteristic equation of the studied system

$$K(s) = (s^2 + 6s + 25) + p(s + 6) = 0 .$$

For the function  $z(x, y)$  the substitute values corresponding to Eq. (7) are:

$$G_1(x, y) = x^2 - y^2 + 25 + 6x$$

$$G_2(x, y) = 2xy + 6y$$

$$H_1(x, y) = x + 6$$

$$H_2(x, y) = y.$$

So

$$\begin{aligned} z(x, y) &= y(2x + 6)(x + 6) - y(x^2 - y^2 + 25 + 6x) = \\ &= y[(x + 6)^2 + y^2 - 25] = 0 , \end{aligned}$$

i.e. equations of the root locus diagram branches are:

$$\begin{aligned} y &= 0 \\ (x + 6)^2 + y^2 - 25 &= 0 . \end{aligned}$$

Let us compare the obtained result with the positive branches of the root locus diagram shown in Fig. 2. Let us find now the points of the root locus diagram at  $x = 6$  by series expansion according to formula (12). The series obtained in this way are :

$$GR(s) = 25 - 6(s + 6) + (s + 6)^2$$

and

$$HR(s) = s + 6 .$$

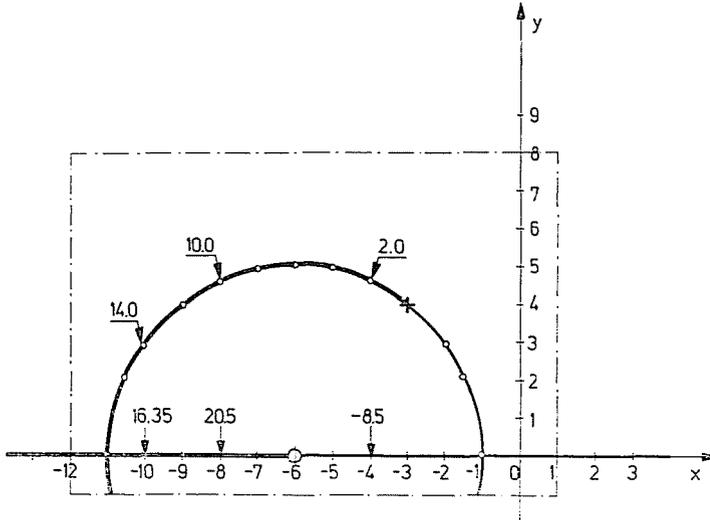


Fig. 2

Along the line  $x = -6$ :

$$\begin{aligned} GR(s) |_{s=-6+jy} &= 25 - 6(jy) + (jy)^2 \\ &= 25 - y^2 + j(-6y) \end{aligned}$$

and

$$HR(s) |_{s=-6+jy} = 0 + jy .$$

So

$$G_1(y) = 25 - y^2$$

$$G_2(y) = -6y$$

and

$$H_1(y) = 0$$

$$H_2(y) = y .$$

With these values Eq. (7) yields:

$$\varphi(y) = y(-25 + y^2) = 0 .$$

So the points of the diagram on the line  $x = -6$  are:

$$P_1(-6, 0)$$

$$P_2(-6, 5)$$

$$P_3(-6, -5)$$

the corresponding values of parameter  $p$  are:

$$p_1 = \text{indefinite}$$

$$p_2 = p_3 = 6$$

Values  $p_2$  and  $p_3$  have been determined by Eq. (9a). The example was solved also by the RL program, the input data were:

$$\text{ITSZ} = 1 \quad \text{NG} = 2 \quad \text{MH} = 1 \quad \text{TAU} = 0.0 \quad \text{EPS} = 5 \cdot 10^{-7}$$

$$\text{G}(1) = 1 \quad \text{G}(2) = 6 \quad \text{G}(3) = 25$$

$$\text{H}(1) = 1 \quad \text{H}(2) = 6$$

$$\text{NX} = 13 \quad \text{MY} = 9 \quad \text{XB} = -12.0 \quad \text{XJ} = 1.0 \quad \text{YA} = 1.0 \quad \text{YF} = 8.0$$

The following is a sample of the output results:

Co-ordinates $x$	Co-ordinates $y$	Parameters $p$
-10.0000	0.0000	16.2500
-10.0000	3.0000	14.0000
- 9.0000	0.0000	17.3333
- 9.0000	4.0000	12.0000
- 8.0000	0.0000	20.5000
- 8.0000	4.5826	10.0000
- 7.0000	0.0000	32.0000
- 7.0000	4.8990	8.0000

### Example 2

Be the characteristic equation of the studied system

$$K(s) = 153s^4 + 1836s^3 + 12\,393s^2 + 41\,310s + pe^{-s\tau}(270s^2 + 1620s + 41\,310) = 0.$$

Let us compute the points of the root locus diagram by the RL program for the case  $\tau = 0$ . The example is especially interesting, as the line  $x = -3$  belongs to the root locus diagram.

Input data of the RL program are:

$$ITSZ = 1$$

$$NG = 4 \quad MH = 2 \quad TAU = 0.0 \quad EPS = 10^{-4}$$

$$G(1) = 153.0 \quad G(2) = 1836.0 \quad G(3) = 12\,393.0 \quad G(4) = 41\,310.0$$

$$G(5) = 0.0$$

$$H(1) = 270.0 \quad H(2) = 1620.0 \quad H(3) = 41\,310$$

$$NX = 50 \quad MY = 50 \quad XB = -9.0 \quad XJ = 3.0 \quad YA = -1.0$$

$$YB = 21.0$$

A sample of the output results:

Co-ordinates $x$	Co-ordinates $y$	Parameters $p$
-4.4400	0.0000	1.0230
-4.4400	4.2560	2.8787
-3.9600	0.0000	1.1663
-3.9600	4.0761	2.4851
-3.4800	0.0000	1.2483
-3.4800	3.9675	2.2788
-3.0000	2.9600	2.0272
-3.0000	3.4000	2.1500
-2.7600	0.0000	1.2684
-2.7600	3.9403	2.2306

The root locus diagram is shown in Fig. 3.

#### *Example 3*

Be the studied system identical with that in example 2, but now with  $\tau = 0.5$ . The results are shown in Fig. 4.

### Summary

This paper presents a new method for generating the root locus diagram of ordinary and delayed systems. The diagrams are computed much faster and more accurately than ever before. The new method can be well used with minicomputers. Also proof and detailed description of the suggested algorithm are given. The results are supported by numerical examples.

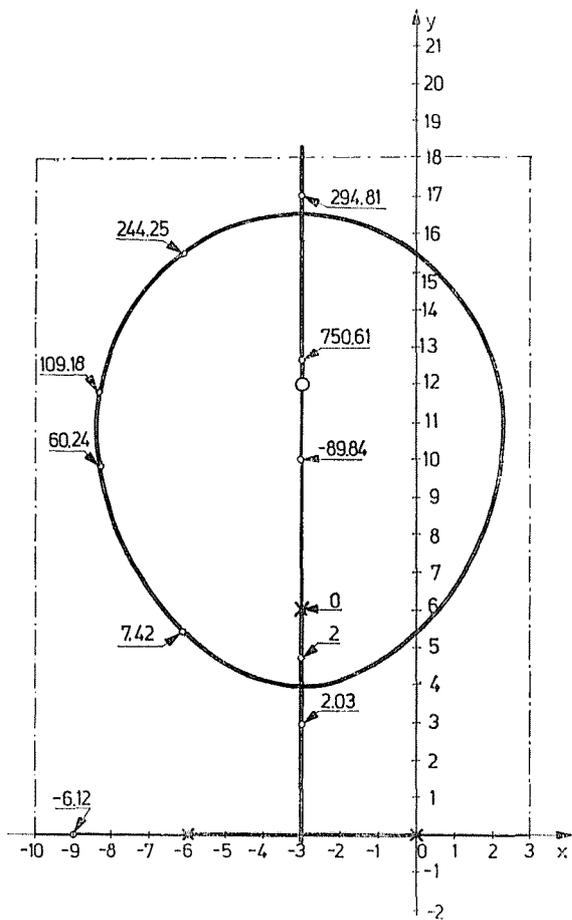


Fig. 3

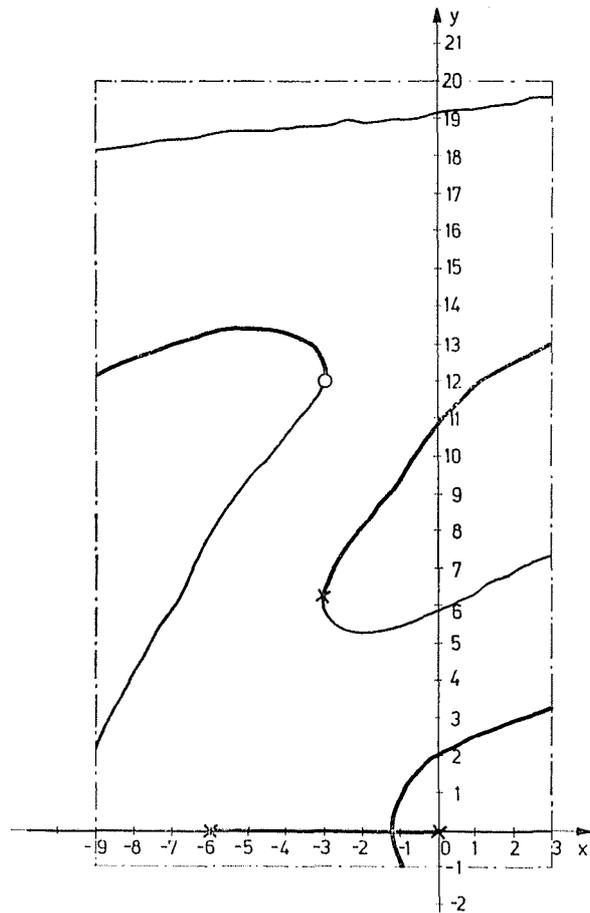


Fig. 4

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