A NEW METHOD FOR TRANSFORMATION OF DISCRETE TRANSFER FUNCTIONS TO CONTINUOUS ONE

By

L. KEVICZKY

Department of Automation, Technical University, Budapest

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1. Introduction

Nowadays, most computerized identification methods apply discrete technique. These methods determine the continuous system parameters in two steps. In the first step the parameters of a discrete-time model are estimated which ensure a good fit to the input and output signals of the process at the sampling instants. In the second step the equivalent continuous system has to be computed from the discrete model obtained. In this paper this second transformation step is considered for linear time-invariant systems. This problem has already been treated by several authors with different approaches: SMITH [1], HAYKIN [2], HSIA [3], SINHA [4]. Most of the proposed methods involve decomposition into subsystems and the case of multiple poles is out of consideration. JEŽEK [5] pointed out that the relationships of the equivalent transformation are simple to obtain by direct integration of the state equations. Following this course, well computerizable algorithms of the unit step and ramp response transformations are given here by using state description forms. These methods can also be applied in case of multiple poles with no extra difficulty and there is no need to decompose the discrete-transfer function to be transformed into partial-fraction subsystems.

2. State space approach

Let us consider a state space description of a single–input single–output linear continuous system of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \tag{1}$$

$$y(t) = \mathbf{c}^T\mathbf{x}(t) + \beta_0 u(t) \tag{2}$$

(Here T means the transposition). The solution of the continuous state space equations on the sampling interval $kh \leq t < (k+1)h$ is

$$\mathbf{x}((k+1)h) = e^{\mathbf{A}h}\mathbf{x}(kh) + \int\limits_{kh}^{(k+1)h} e^{\mathbf{A}((k+1)h-\tau)}\mathbf{b}u(\tau)\,d\tau \tag{3}$$

hence, the integro-difference equation of an equivalent discrete system is

$$\mathbf{x}_{k+1} = e^{\mathbf{A}h}\mathbf{x}_k + \int\limits_{kh}^{(k+1)h} e^{\mathbf{A}((k+1)h-\tau)}u(\tau)\,d\tau\mathbf{b} = e^{\mathbf{A}h}\mathbf{x}_k + \mathbf{Q}[u(t), \mathbf{A}, h]\,\mathbf{b}. \tag{4}$$

Comparing this latter equation with the state space equations of a discrete-time linear system

$$\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k + \mathbf{g}u_k \tag{5}$$

$$y_k = \mathbf{c}^T\mathbf{x}_k + b_0u_k \tag{6}$$

and evaluating the input integral

$$\mathbf{Q}[u(\tau), \mathbf{A}, h] = \int\limits_{kh}^{(k+1)h} e^{\mathbf{A}((k+1)h-\tau)}u(\tau)\,d\tau = \int\limits_{0}^{h} e^{\mathbf{A}(h-\vartheta)}u(kh+\vartheta)\,d\vartheta \tag{7}$$

for a given approximation of the input signal $u(t)$ in the interval $kh = t < < (k+1)h$, unambiguous relationships are obtained between the continuous and the discrete state space equations by comparing the coefficient matrices, assuming the applied approximation to be time-invariant.

If suitable canonic equations are chosen for comparison then the application of common notations in the output equations is legitimate. (In this case $\mathbf{c}$ usually contains no system parameter and the coefficients of $u(t)$ and u_k are the same.) Otherwise the change to such a form can be performed by simple transformations [6], [7].

Comparing (4) and (5) the transformation rule of $\mathbf{F}$ is seen to be

$$\mathbf{A} = \frac{1}{h}\ln(\mathbf{F}) \tag{8}$$

independent of the approximation of the input signal. ln ($\mathbf{F}$) means a matrix function [8] which — among others — can be defined by its matrix power-series and there are relevant computer routines available. The necessary condition of the existence of $\mathbf{A}$ is that $\mathbf{F}$ has all its eigenvalues inside the unit circle (provided negative real roots $\mathbf{A}$ cannot be computed). Let us investigate the input integral for two kinds of approximations of the input $u(t)$. First let $u(t)$ be constant during the whole sampling interval $u(kh+\vartheta) \cong u(kh) = u_k$, this assumption being required by the step response equivalent transformation [2]. Then

$$\mathbf{Q}[u(\tau), \mathbf{A}, h] = \int\limits_{0}^{h} e^{\mathbf{A}(h-\vartheta)}u_k d\vartheta = u_k \int\limits_{0}^{h} e^{\mathbf{A}(h-\vartheta)}d\vartheta = u_k[-\mathbf{A}^{-1}e^{\mathbf{A}(h-\vartheta)}]_0^h =$$

$$= \mathbf{A}^{-1}(e^{\mathbf{A}h} - \mathbf{I})\,u_k \tag{9}$$

considering $e^{0} = \mathbf{I}$ [9].

Eq. (4) becomes

$$\mathbf{x}_{k+1} = e^{\mathbf{A}h}\mathbf{x}_k + \mathbf{A}^{-1}(e^{\mathbf{A}h} - \mathbf{I})\,\mathbf{b}u_k \tag{10}$$

and comparing with (5) we get

$$\mathbf{b} = \frac{1}{h}\ln(\mathbf{F})(\mathbf{F} - \mathbf{I})^{-1}\,\mathbf{g}\ . \tag{11}$$

Now

$$\beta_0 = b_0\ . \tag{12}$$

Let us approximate the input signal according to a linear interpolation in the sampling interval with

$$u(kh + \vartheta) = u_k + \frac{u_{k+1} - u_k}{h}\,\vartheta \tag{13}$$

which corresponds to the ramp equivalent transformation [2]. Now the input integral is

$$\mathbf{Q}[u(\tau), \mathbf{A}, h] \cong \int_0^h e^{\mathbf{A}(h-\vartheta)}\left[u_k + \frac{u_{k+1} - u_k}{h}\,\vartheta\right] d\vartheta =$$

$$= u_k \int_0^h e^{\mathbf{A}(h-\vartheta)} d\vartheta + \frac{u_{k+1} - u_k}{h}\int_0^h e^{\mathbf{A}(h-\vartheta)} d\vartheta\ . \tag{14}$$

After not too complicated calculations we get

$$\mathbf{Q}[u(\tau), \mathbf{A}, h] \cong \mathbf{A}^{-1}(e^{\mathbf{A}h} - \mathbf{I}) - \frac{1}{h}[\mathbf{A}^{-2}(e^{\mathbf{A}h} - \mathbf{I}) - h\mathbf{A}^{-1}]u_k +$$

$$+ \frac{1}{h}[\mathbf{A}^{-2}(e^{\mathbf{A}h} - \mathbf{I}) - h\mathbf{A}^{-1}]\,u_{k+1} = \mathbf{Q}_1 u_k + \mathbf{Q}_2 u_{k+1}\ . \tag{15}$$

Thus, the integro-difference equation for (3) gives the state equation

$$\mathbf{x}_{k+1} = \mathbf{F}\mathbf{x}_k + \mathbf{Q}_1\mathbf{b}u_k + \mathbf{Q}_2\mathbf{b}u_{k+1}\,. \tag{16}$$

Introducing a new state vector

$$\tilde{\mathbf{x}}_{k+1} = \tilde{\mathbf{x}}_{k+1} - \mathbf{Q}_2\mathbf{b}u_{k+1} \tag{17}$$

the Eqs (3) and (4) become

$$\tilde{\mathbf{x}}_{k+1} = \mathbf{F}\tilde{\mathbf{x}}_k + (\mathbf{F}\mathbf{Q}_2 + \mathbf{Q}_1)\,\mathbf{b}u_k \tag{18}$$

$$y_k = \mathbf{c}^T\tilde{\mathbf{x}}_k + (\mathbf{c}^T\mathbf{Q}_2\mathbf{b} + \beta_0)\,u_k. \tag{19}$$

The transformation equation (8) is valid further on, but (11) becomes:

$$\mathbf{b} = (\mathbf{F}\mathbf{Q}_2 + \mathbf{Q}_1)^{-1}\mathbf{g} = \frac{1}{h}\,[\ln(\mathbf{F})]^2(\mathbf{F} - \mathbf{I})^{-2}\,\mathbf{g} \tag{20}$$

and

$$\beta_0 = b_0 - \mathbf{c}^T\mathbf{Q}_2\mathbf{b} = b_0 + \mathbf{c}^T(\mathbf{F} - \mathbf{I})^{-1}\ln(\mathbf{F})[(\mathbf{F} - \mathbf{I})^{-1} - \mathbf{I}]\,\mathbf{g}. \tag{21}$$

Both of the above state space transformation methods map the static gain of the system without error. This can easily be checked by the equation

$$K = \beta_0 - \mathbf{c}^T\mathbf{A}^{-1}\mathbf{b} = b_0 - \mathbf{c}^T(\mathbf{F} - \mathbf{I})^{-1}\mathbf{g}\ . \tag{22}$$

3. Computational algorithm

On the basis of the above state space transformation methods the discrete transfer function

$$G(z^{-1}) = \frac{B(z^{-1})}{A(z^{-1})} = \frac{b_0 + b_1 z^{-1} + \ldots + b_n z^{-n}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} \tag{23}$$

can be transformed into the equivalent continuous system

$$H(s) = \frac{\beta(s)}{\alpha(s)} = \frac{\beta_0 s^n + \beta_1 s^{n-1} + \ldots + \beta_n}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n} \tag{24}$$

according to the following algorithm:

1. Let us construct the coefficient matrices of the discrete state equations (5), (6), on the basis of coefficients in (23) in the form

$$\mathbf{F} = \begin{bmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ -a_{n-1} & 0 & 0 & \ldots & 1 \\ -a_n & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{25}$$

$$\mathbf{g} = [b_1 - b_0 a_1, \ldots, b_n - b_0 a_n]^T \tag{26}$$

and

$$\mathbf{c} = [1, 0, \ldots, 0]^T\ . \tag{27}$$

2. The change to the continuous state equations is made according to the following formulae

$$\mathbf{A} = \frac{1}{h} \ln (\mathbf{F}) , \tag{28}$$

and

$$\mathbf{b} = \begin{cases} \frac{1}{h} \ln (\mathbf{F}) (\mathbf{F} - \mathbf{I})^{-1} \mathbf{g} & \text{(step equivalent)} \\ \frac{1}{h} [\ln (\mathbf{F})]^2 (\mathbf{F} - \mathbf{I})^{-2} \mathbf{g} & \text{(ramp equivalent)} \end{cases} \tag{29}$$

and

$$\beta_0 = \begin{cases} b_0 & \text{(step equivalent)} \\ b_0 - \mathbf{c}^T (\mathbf{F} - \mathbf{I})^{-1} \mathbf{g} & \text{(ramp equivalent)} \end{cases} \tag{30}$$

3. Then the continuous system given by **A** and **b** is transformed into phase variable canonic form $\mathbf{A}^*$ and $\mathbf{b}^*$ by any standard procedure [6], [7], e.g. by the transformation matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{c}^T \\ \mathbf{c}^T \mathbf{A} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{c}^T \mathbf{A}^{n-1} \end{bmatrix} . \tag{31}$$

In the canonic form

$$\mathbf{A}^* = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{I} \\ \hline \multicolumn{2}{c}{-\mathbf{k}^T} \end{array} \right] \tag{32}$$

where

$$\mathbf{k} = [\alpha_n, \alpha_{n-1}, \ldots, \alpha_1]^T \tag{33}$$

is the canonic vector. The vector **k** is given by

$$\mathbf{k} = -[\mathbf{c}^T \mathbf{A}^n] \mathbf{T}^{-1} \tag{34}$$

[6], [7], furthermore

$$\mathbf{b}^* = \mathbf{T}\mathbf{b}. \tag{35}$$

The denominator of $H(s)$ being already available, its numerator is obtained by

$$\mathbf{q} = [\gamma_1, \gamma_2, \ldots, \gamma_n]^T = \mathbf{P}\mathbf{b}^* = \mathbf{P}\mathbf{T}\mathbf{b} \tag{36}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \alpha_1 & 1 & 000 & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{n-1} & \alpha_{n-2} & \ldots & 1 \end{bmatrix} . \tag{37}$$

The coefficients of the numerator of $H(s)$ are

$$\beta_i = \gamma_i + \beta_0 \alpha_i; \quad i = 1, \ldots, n \tag{38}$$

where β_0 corresponds to (30).

4. Conclusions

This study was conducted to determine a state space transformation algorithm for computing the step and ramp response equivalent continuous system models of identified discrete transfer functions. The suggested transformation equations need only the use of a matrix functions and simple matrix operations. This approach fits better the state space methods and the transformation equations do not depend on the multiplicity of poles. Subroutines computing the equivalent continuous system models by this way may be useful elements of identification program libraries.

Summary

A state space transformation method is given to determine both the step and ramp equivalent continuous plant models to a discrete transfer function. The method does not require decomposition of the system to partial-fraction subsystems and it suits cases of multiple poles. It needs only matrix operations simple to computerize and the method fits practical identification tasks.

References

1. SMITH, F. W.: System Laplace-Transform Estimation from Sampled Data. IEEE Trans. on Aut. Control, Vol. AC-13, No 1, 37—44 (1968).
2. HAYKIN, S. S.: A Unified Treatment of Recursive Digital Filtering. IEEE Trans. on Aut. Control, Vol. AC-17, No 1, 113—116 (1972).
3. HSIA, T. C.: On Sampled Data Approach to Parameter Identification of Continuous Linear Systems. IEEE Trans. on Aut. Control, Vol. AC-17, 247—249 (1972).
4. SINHA, N. K.: Estimation of Transfer Function of Continuous System from Sampled Data. Proc. IEE, Vol. 119, No 5, 612—614 (1972).
5. JEŽEK, J.: Algorithm for Computing the Transfer Function of a Linear System from the Measurement Values of the Unit Step Transient Response. IFAC Symp., Prague (1970).
6. CSÁKI, F.: Chapters from Control Theory. State Space Equations (in Hungarian). Műszaki Könyvkiadó, Budapest (1973).
7. LEE, R. C. K.: Optimal Estimation, Identification and Control. The MIT Press, Cambridge, Massachusetts (1964).
8. GANTMAHER, F. R.: The Theory of Matrices. Chelsea Publishing Co. New York (1959).
9. CSÁKI, F.: Modern Control Theories. Akadémiai Kiadó, Budapest (1972).

Dr. László KEVICZKY H-1521 Budapest