RELATIONS BETWEEN THE MINIMUM VARIANCE CONTROL AND THE OPTIMUM TRANSFER FUNCTION IN THE WIENER SENSE

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Nowadays a huge progress of the stochastic control theory is seen in the field of control engineering. An enormous amount of books and papers deal with the problem to control a process influenced by stochastic environment. Not only theoretical results [1, 2], but a lot of successful industrial applications [3, 4, 5, 6] are known from the last few years.

One of the most applicable and powerful methods from the stochastic control strategies is the minimum variance control [3, 4, 6, 7, 8], where the purpose of the control is to minimize the variance of the controlled signal about its desired value. For the solution of that sort of problems, the optimum transfer function in the WIENER sense is available. The purpose of this paper is to show that the minimum variance control strategy can be derived by means of the optimum transfer function in the WIENER sense. While in the general case the determination of the optimum transfer function in the WIENER sense has difficulties in connection with the proper factorizations of the power spectra, in the case where the system is described by an ÅSTRÖM model, this factorization is easy to perform through a simple polynomial separation.

1. Statement of the problem

Consider a single input–single output discrete time (sampled) linear system with constant and known parameters, which can be described by the difference equation:

$$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) = b_0 u(t-d) + b_1 u(t-d-1) + \ldots + b_m u(t-d-m), \qquad (1)$$

$$(m \leq n;\ t = 0, \pm 1, \pm 2, \ldots)$$

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where y is the controlled signal, u is the control signal, and d is the time delay of the process. Introducing the polynomials

$$A(z^{-1}) = a_0 + a_1 z^{-1} + \ldots + a_n z^{-n}$$

and

$$B(z^{-1}) = b_0 + b_1 z^{-1} + \ldots + b_m z^{-m},$$

in accordance with the interpretation of the backward shift operator z^{-1} the system equation (1) can be written in the following form:

$$A(z^{-1})\, y(t) = B(z^{-1})\, u(t-d)\,. \tag{2}$$

By choosing

$$a_0 = 1$$

the minimal number of the necessary parameters is ensured.

The stochastic environment and the noisy measurement situation are taken into account by an additive stochastic process reduced to the output of the system. This stochastic process is driven by a white noise $e(t)$, characterized by zero mean value and variance 1. By means of polynomial

$$R(z^{-1}) = 1 + r_1 z^{-1} + \ldots + r_R z^{-R}$$

a moving average (x_m) or an autoregressive (x_a) stochastic process of $e(t)$ can be generated in the following way [3]:

$$x_a(t) = \frac{1}{R(z^{-1})}\, e(t) = e(t) - r_1 x_a(t-1) - \ldots - r_R x_a(t-R)\,,$$

$$x_m(t) = R(z^{-1})\, e(t) = e(t) + r_1 e(t-1) + \ldots + r_R e(t-R)\,.$$

In the general case the process can be disturbed by an autoregressive moving average stochastic process of the driving noise $e(t)$ with variance 1:

$$x_n(t) = \lambda \frac{C(z^{-1})}{A(z^{-1})}\, e(t)\,,$$

where

$$C(z^{-1}) = 1 + c_1 z^{-1} + \ldots + c_k z^{-k} \qquad (k \leq n).$$

Taking into consideration the additive disturbance x_n in Eq. (2) we get

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})}\, u(t-d) + \lambda \frac{C(z^{-1})}{A(z^{-1})}\, e(t)\,. \tag{3}$$

In order to have a stable system and to avoid numerical instability the polynomials $z^n A(z^{-1})$ and $z^k C(z^{-1})$ are assumed to have all their zeros inside the unit circle [2, 9]. Notice that the coincidence of the nominators of the process $B(z^{-1})/A(z^{-1})$ to be controlled and that of the $C(z^{-1})/A(z^{-1})$ model of disturbances mean no special condition, because a system

$$y(t) = \frac{B_1(z^{-1})}{A_1(z^{-1})} u(t-d) + \lambda \frac{C_1(z^{-1})}{A_2(z^{-1})} e(t)$$

can always be arranged to have the form of Eq. (3), where

$$\begin{aligned} A(z^{-1}) &= A_1(z^{-1})\, A_2(z^{-1}) \\ B(z^{-1}) &= B_1(z^{-1})\, A_2(z^{-1}) \\ C(z^{-1}) &= C_1(z^{-1})\, A_1(z^{-1}) \,. \end{aligned}$$

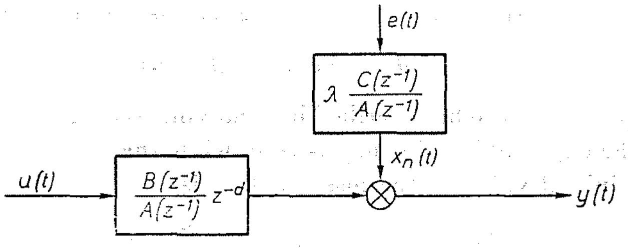

Fig. 1

The flow chart of the so-called ÅSTRÖM-model by Eq. (3) is shown in Fig. 1.

The problem is to determine the control signal $u(t)$ from the knowledge of the system parameters and the observations $\{y(t), y(t-1), \ldots, u(t-1), u(t-2), \ldots\}$ in such a way that the loss function

$$V = E\{y^2(t)\}$$

will be as small as possible. $E\{.\}$ denotes mathematical expectation.

2. Determination of the optimal control by means of the system equations

To determine the optimal $u(t)$ let us write the earliest value of y influenced by $u(t)$ on the basis of Eq. (3):

$$y(t+d) = \frac{B(z^{-1})}{A(z^{-1})} u(t) + \lambda \frac{C(z^{-1})}{A(z^{-1})} e(t+d) \,.$$

The second term in the right side is a linear function of $e(t+d)$, $e(t+d-1),\ldots$ $\ldots, e(t+1)$, $e(t)$, $e(t-1),\ldots$ Since $e(t)$, $e(t-1),\ldots$ can be exactly computed from the observations $\{y(t), y(t-1),\ldots u(t-d), u(t-d-1),\ldots\}$ by Eq. (3), but $e(t+d)$, $e(t+d-1),\ldots e(t+2)$, $e(t+1)$ are independent of these observations, this separation is easy by means of the polynomial equation

$$C(z^{-1}) = A(z^{-1})\, F(z^{-1}) + z^{-d}G(z^{-1})\,, \tag{4}$$

where

$$F(z^{-1}) = 1 + f_1 z^{-1} + \ldots + f_{d-1} z^{1-d}$$

$$G(z^{-1}) = g_0 + g_1 z^{-1} + \ldots + g_{n-1} z^{1-n}.$$

Straightforward algebraic manipulations then give the optimal control law [3, 10, 11]:

$$u^{o}(t) = \frac{-G(z^{-1})}{B(z^{-1})\, F(z^{-1})}\, y(t)\,. \tag{5}$$

By such a control strategy the controlled signal has the following form:

$$y(t+d) = \lambda F(z^{-1})\, e(t+d)\,. \tag{6}$$

From Eq. (6) $y(t)$ is seen to be described by a moving average stochastic process of $e(t)$. Thus the expectable value of y is zero, while the variance of the output, that is, the minimal value of the loss function is

$$V_{\min} = \lambda^2(1 + f_1^2 + \ldots f_{d-1}^2)\,.$$

In connection with Eq. (6) the minimum variance output has to be evidently a moving average stochastic process of $e(t)$ at least of the order $(d-1)$. It is obvious, because the independent noises $e(t+d)$, $e(t+d-1),\ldots$ $\ldots, e(t+2)$, $e(t+1)$ appear after generating the control signal $u(t)$.

The flow chart of the optimally controlled system is shown in Fig. 2.

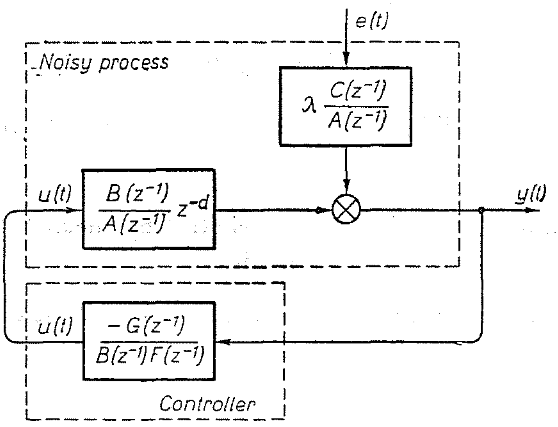

Fig. 2

3. Determination of the minimum variance control by means of the optimum transfer function in the WIENER sense

Let us transform the system equation (3), the polynomial separation equation (4), as well as the equation of the optimal control by Eq. (5) in such a way that the argument of the polynomials will be z instead of z^{-1}. Define

$$\begin{aligned} A^*(z) &= z^n A(z^{-1}) = z^n + a_1 z^{n-1} + \ldots + a_n \\ B^*(z) &= z^n B(z^{-1}) = b_0 z^n + b_1 z^{n-1} + \ldots + b_m z^{n-m} \\ C^*(z) &= z^n C(z^{-1}) = z^n + c_1 z^{n-1} + \ldots + c_k z^{n-k} \\ G^*(z) &= z^n G(z^{-1}) = g_0 z^n + g_1 z^{n-1} + \ldots + g_{n-1} z \\ F^*(z) &= z^d F(z^{-1}) = z^d + f_1 z^{n-1} + \ldots + f_{d-1} z \end{aligned}$$

hence, the system equation is

$$y(t+d) = \frac{B^*(z)}{A^*(z)} u(t) + \lambda \frac{C^*(z)}{A^*(z)} e(t+d), \tag{7}$$

the polynomial separation equation is

$$z^d C^*(z) = A^*(z)\, F^*(z) + G^*(z), \tag{8}$$

and the optimal control is

$$u^0(t) = \frac{-z^d G^*(z)}{B^*(z)\, F^*(z)} y(t) = Y_v^0(z) y(t). \tag{9}$$

The pulse transfer function $Y_v^0(z)$ will be shown to equal the optimum transfer function in the WIENER sense.

On the basis of Fig. 3 the pulse transfer function between $e(t)$ and $y(t)$ is written as:

$$W(z) = \frac{\lambda C^*(z)\, z^d}{A^*(z)\, z^d - Y_v(z)\, B^*(z)}. \tag{10}$$

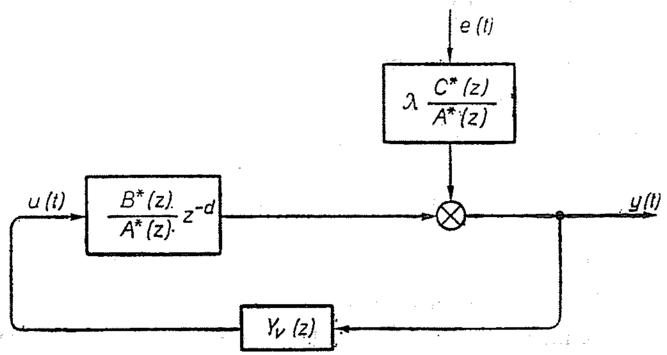

Fig. 3

The optimal $Y_v(z)$ is sought for, by which $y(t)$ follows

$$x_i(t) = \lambda F(z^{-1})\, e(t) = \lambda F^*(z)\, z^{-d} e(t)$$

as an ideal signal.

Fig. 4 shows the optimal transfer function

$$W_c(z) = \frac{1}{A^*(z)\, z^d - Y_v(z)\, B^*(z)} \tag{11}$$

to be determined in a semi-free configuration with

$x_b(t) = e(t)$	as input signal
$x_i(t) = \lambda F^*(z)\, z^{-d} e(t)$	as ideal signal
$Y_f(z) = \lambda C^*(z)\, z^d$	as the pulse transfer function of the fixed elements.

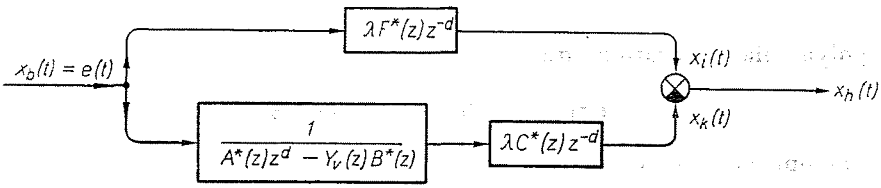

Fig. 4

According to the theory of the sampled stochastic processes [3, 9, 12], the physically realizable optimal pulse transfer function can be determined by

$$W_{cr}(z) = \frac{1}{Y_f(z)} \; \frac{\left[\dfrac{\Phi_{bi}(z)}{\Phi_{bb}^{-}(z)}\right]_+}{\Phi_{bb}^{+}(z)}\,, \tag{12}$$

assuming that $Y_f(z)$ has no poles or zeros outside the unit circle. This condition is met in our case. In Eq. (12)

$\Phi_{bb}(z)$ is the power spectrum of x_b,
$\Phi_{bi}(z)$ is the cross-power spectra of x_b and x_i,
$\Phi_{bb}^{+}(z)$ and $\Phi_{bb}^{-}(z)$ are obtained by spectrum factorization

$$\Phi_{bb}(z) = \Phi_{bb}^{+}(z)\Phi_{bb}^{-}(z),$$

where the factor $\Phi_{bb}^{+}(z)$ contains all the poles and zeros of $\Phi_{bb}(z)$ inside the unit circle, while the factor $\Phi_{bb}^{-}(z)$ contains all the poles and zeros of $\Phi_{bb}(z)$ outside the unit circle. Finally, the superscript $+$ refers to the positive time functions [9, 12].

In our case of interest the following relationships hold:

$$\Phi_{bb}(z) = 1 ,$$

$$\Phi_{bi}(z) = \Phi_{bb}(z)\, Y_i(z) = \lambda F^*(z)\, z^{-d}.$$

Taking into consideration these relationships in Eq. (12) we get

$$W_{cr}(z) = \frac{F^*(z)\, z^{-d}}{C^*(z)\, z^d} . \tag{13}$$

Substituting Eq. (13) into Eq. (11) and expressing $Y_v(z)$, the optimal pulse transfer function $Y_v^o(z)$ becomes:

$$Y_v^o(z) = \frac{A^*(z)\, F^*(z) - z^d C^*(z)}{F^*(z)\, B^*(z)\, z^{-d}} = - \frac{z^d G^*(z)}{F^*(z)\, B^*(z)} .$$

Comparing the above expression of $Y_v^o(z)$ with Eq. (9) the two ways of solution of the minimum variance control problem are seen to give the same result.

Thus it has been shown that the minimum variance control strategy can be derived by means of the optimum transfer function in the WIENER sense.

Summary

In this paper a derivation of the minimum variance control law is given by means of the optimum transfer function in the WIENER sense. It is shown that the optimum transfer function has to be determined in a semi-free configuration. The ideal signal is chosen after taking into consideration the independence of the disturbances related to the output of the system.

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