A NEW FUNCTION IN THE TIME DOMAIN CHARACTERIZING THE DYNAMICS OF A LINEAR SYSTEM

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(Received Januar 15, 1976)

I. Symbols

$W(s)$ transfer function of a linear single input-single output system

$w(t) = L^{-1}[W(s)]$ weight function of the system. This is the response to a unit impulse input as well as the inverse Laplace transform of the transfer function.

$v(t) = \int_0^t w(t) \mathrm{d}t$ step response of the system. This is the response to a unit step input, i.e. the integral of the weight function

$V(t) = \int_0^t v(t) \mathrm{d}t$ response to a unit ramp function used as input.

$V_n(t) = \underbrace{\int_0^t \ldots \int_0^t}_{n} w(t) \underbrace{\mathrm{d}t \ldots \mathrm{d}t}_{n}$ response to the input $x(t)$ given by the following formula:

$$x(t) = \begin{cases} 0 & t < 0 \\ \dfrac{t^{n-1}}{(n-1)!} & t \geq 0 \end{cases} \qquad n = 1, 2, \ldots$$

and

$$x(t) = \delta(t) \qquad n = 0$$

Note that

$$V_0(t) = w(t)$$
$$V_1(t) = v(t)$$
$$V_2(t) = V(t)$$

and thus $V_n(t)$ may be termed as the n-th order generalized step function

$W(j\omega) = A(\omega) e^{j\Phi(\omega)}$ frequency function of the system. Its amplitude and phase function are $A(\omega)$ and $\Phi(\omega)$, respectively.

II. Definition of the rise function

We define the n-th order rise function $\alpha_n(t)$ for such linear single input-single output systems whose

1° transfer function has zeros only in the left half plane

2° transfer function has poles only in the left half plane and in the origin $(s = 0)$

3° n-th order step response $V_n(t)$ is positive for $t > 0$ and zero for $t = 0$.

The definition of the rise function is as follows:

$$\alpha_n(t) = \frac{t\, V_{n-1}(t)}{V_n(t)} - (n-1) \qquad n \geq 1\,. \tag{1}$$

Of course some k-th order step responses $V_k(t)$ may not meet condition 3°, while $V_{k+1}(t)$ does. In this case only rise functions of the order $n \geq k + 1$ exist.

A special but very important case is the first order rise function simply denoted as

$$\alpha_1(t) = \alpha(t)$$

$$\alpha(t) = \frac{t \cdot V_0(t)}{V_1(t)} = \frac{t \cdot w(t)}{v(t)} \tag{2}$$

Let us introduce new variables viz.

$$U_n = \ln \frac{V_n}{t^{n-1}} \qquad t > 0, \qquad n \geq 1 \tag{3}$$

$$\tau = \ln t \qquad t > 0 \tag{4}$$

It is easy to show (see App. I) that

$$\alpha_n = \frac{\mathrm{d}U_n}{\mathrm{d}\tau} \tag{5}$$

In particular, for $n = 1$,

$$U_1 = u = \ln v\,,$$

the first order rise function can be expressed as

$$\alpha = \frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\mathrm{d}\ln v}{\mathrm{d}\ln t}\,. \tag{6}$$

III. Some features of the rise functions

3.1 The rise function is a dimensionless quantity and does not depend on the steady state gain of the system. This statement follows directly from the definition (1) of the rise function.

3.2 If the system is a pure k-fold integrating element, i.e. its transfer function is

$$W(s) = K/s^k ,$$

then the rise function of any n-th order is

$$\alpha_n (t) = k = \text{const} . \tag{7}$$

Namely the n-th order step response of such a system is

$$V_n(t) = \frac{Kt^{n+k-1}}{(n+k-1)\,!} \tag{8}$$

and

$$V_{n-1}(t) = \frac{K \cdot t^{n+k-2}}{(n+k-2)\,!} . \tag{9}$$

Substituting (8) and (9) into the definition formula (1) of the rise function we get Eq. (7).

3.3 Let the transfer function of the system be

$$W(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + 1}{a_j s^j + a_{j-1} s^{j-1} + \ldots + 1} \frac{1}{s^i} \qquad \begin{array}{l} i \geq 0 \\ j > 0 \\ 0 \leq m < i + j \end{array} \tag{10}$$

then the initial value of the rise function is

$$\alpha_n (0) = j + i - m \tag{11}$$

and the final value

$$\alpha_n (\infty) = i . \tag{12}$$

The proof of these statements is given in App. II.

IV. Similarity between the rise function and the phase function of the frequency response

There are some remarkable similarities between the rise function and phase function $\Phi(\omega)$ of the frequency response

$$W(j\omega) = A(\omega)\, e^{j\Phi(\omega)}$$

4.1 Both the phase function and the rise function are dimensionless quantities and do not depend on the steady state gain of the system.

4.2 If the system is a pure k-fold integrating element, then the phase function is

$$\Phi(\omega) = -\frac{\pi}{2} k = -\frac{\pi}{2} \alpha_n(t) = \text{const} . \tag{13}$$

(Compare with Eq. (7))

4.3 If the transfer function of the system has the form of Eq. (10), then

$$\Phi(\infty) = -\frac{\pi}{2}(j + i - m) = -\frac{\pi}{2} \alpha_n(0) \tag{14}$$

and

$$\Phi(0) = -\frac{\pi}{2} i = -\frac{\pi}{2} \alpha_n(\infty) \tag{15}$$

(Compare with Eqs (11) and (12))

4.4 According to the Bode theorem, the phase function of a minimal phase system can be evaluated from its amplitude function $A(\omega)$ by the following relation:

$$\Phi(\omega_0) = \frac{\pi}{2} \left| \frac{\mathrm{d}a}{\mathrm{d}\Omega} \right|_{\Omega=0} + \frac{1}{\pi} \int_{-\infty}^{+\infty} \left\{ \left| \frac{\mathrm{d}a}{\mathrm{d}\Omega} \right| - \left| \frac{\mathrm{d}a}{\mathrm{d}\Omega} \right|_{\Omega=0} \right\} \ln \operatorname{cth} \left| \frac{\Omega}{2} \right| \mathrm{d}\Omega \tag{16}$$

where

$$a = \ln |W(j\omega)| = \ln A(\omega)$$

$$\Omega = \ln \frac{\omega}{\omega_0}$$

The Bode theorem expresses the fact that, using the logarithmic scale for the frequency as well as for the amplitude $A(\omega)$, the phase is approximately proportional to the slope of the amplitude function (see the first term of Eq. (16)). This approximation has to be corrected by the second term of Eq. (16) being an infinite limit integral. The integrand of the correcting term is the difference between the slope of the amplitude function at the actual frequency ω and its slope at the frequency ω_0, weighted by the function $\ln \operatorname{cth} |\Omega/2|$. Due to this weighting function, the slope changes affect the correcting term about frequency ω_0 more then far from it.

Compare Eq. (16) with the expression of the first order rise function given by Eq. (6). The first term of the Bode theorem is similar to Eq. (6). $a(\omega) = \ln A(\omega)$ corresponds to $\ln v(t)$, and $\Omega = \ln \omega/\omega_0$ to $\ln t$. Remark that the factor $\pi/2$ appears in Eqs (13), (14) and (15) as well. Up to this time we could find no equivalent to the second (correcting) term of Eq. (16). (Perhaps it does not exist at all.)

To find other similarities, the rise functions of various structures have been plotted in Figs 1a—k. To visualize the relations, they are plotted in the same co-ordinate system. The scales of the ordinate axis are chosen so as to make value Φ coincide with the value $-\frac{\pi}{2}\alpha$.

On the other hand, the different arguments of the phase and rise functions are related by

$$t = \frac{\pi}{\omega} . \tag{17}$$

The reciprocal relation between t and ω is obvious. The constant π in Eq. (17) has been found intuitively, although in a time to angular frequency relationship, π is natural to occur. In order to compare the rise function with the phase function in Figs 1a—k, the logarithmic scale has been used for both arguments (ω and t). This results that the phase function is exactly the commonly used phase plot of the Bode diagram, but with co-ordinate axes directed oppositely than usual. By virtue of this — and other examples not demonstrated here — some further properties of the rise functions can be recognized.

1° If a closed loop control system is structurally (by any value of the loop gain) stable, then the value of the rise function of the open loop does not exceed 2. (Remember that in this case $\Phi(\omega) \geq -\pi$.)

2° If a closed loop control system is structurally (for every value of the loop gain) unstable, then the value of the rise functions of the open loop is everywhere greater than 2. (Remember that in this case $\Phi(\omega) \leq -\pi$.)

Note: Properties 1° and 2° are not exclusive, but contradictions have only been found in few cases, with no practical significance for the process control.

To demonstrate one case of contradiction, in Figs 2 the first-order rise function and the phase function of a system having the transfer function

$$W(s) = \frac{1}{s}\left(\frac{1 + s\tau}{1 + sT}\right)^2 \tag{18}$$

have been plotted with various values of τ/T. The phase plots show that the system is structurally stable, if $\tau/t > 0{,}172$, else it is conditionally stable (A system is said to be conditionally stable if the closed loop is unstable only when the loop gain lies between two finite values, $K_{\text{krit } 1}$ and $K_{\text{krit } 2}$ i.e.: $K_{\text{krit } 1} < K < K_{\text{krit } 2}$. The phase function of a conditionally stable system intersects the line $\Phi = -\pi$ twice.) On the other hand, the first order rise function intersects the line $\alpha = 2$ for $\tau/t > 0{,}137$. Thus, concluding to the stability from the rise function may cause a mistake: i.e. for $0{,}137 <$ $< \tau/t > 0{,}172$, the system is regarded as structurally stable, although it may be unstable even near the stability limit.

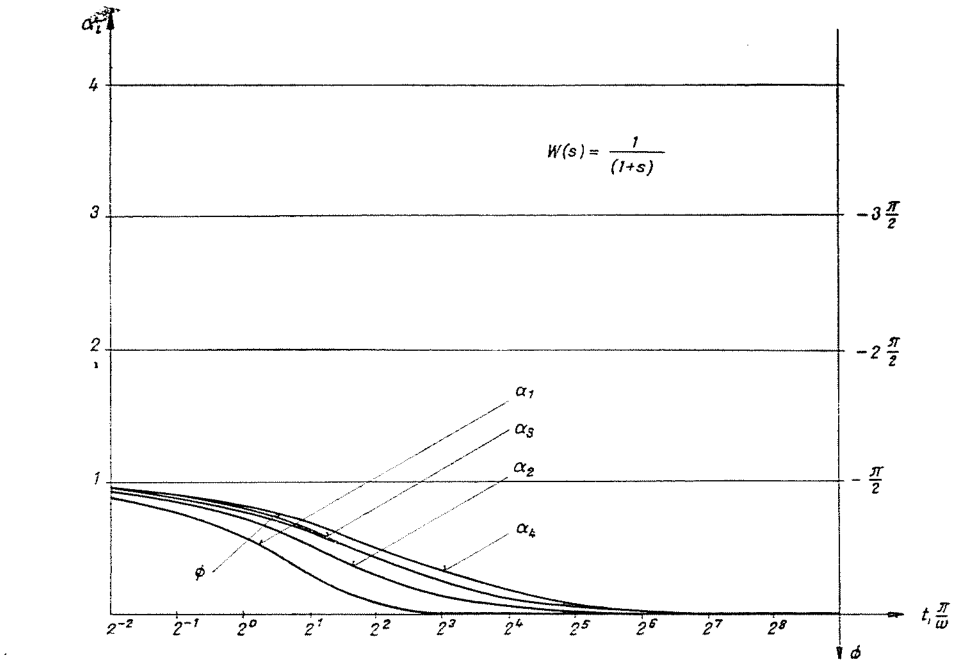

α_i
4
3
2
1
$W(s) = \frac{1}{(1+s)}$
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
α_1
α_3
α_2
α_4
ϕ
2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/a

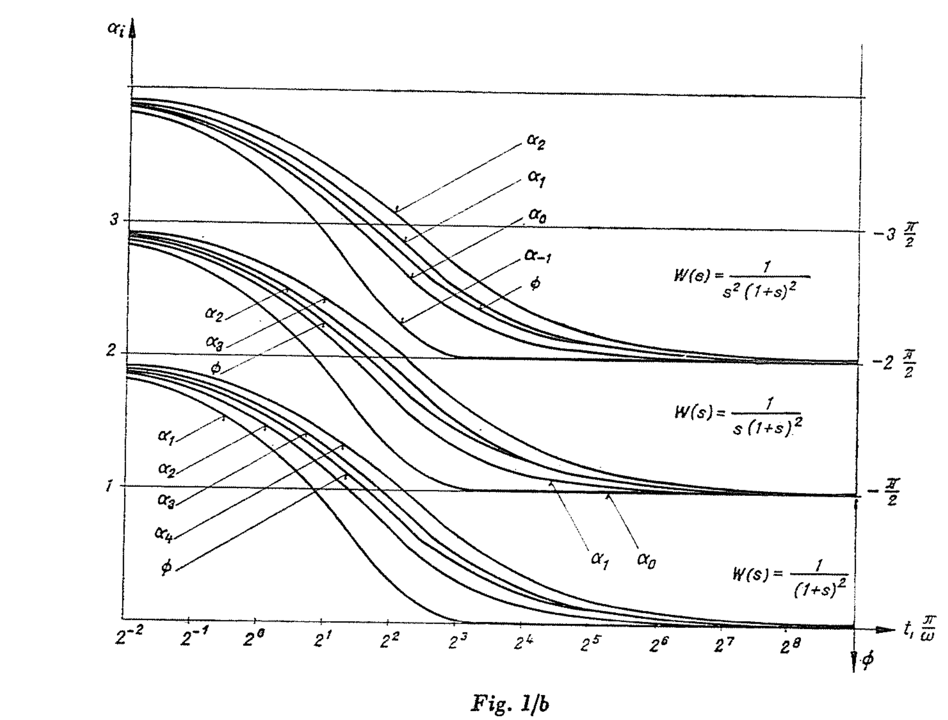
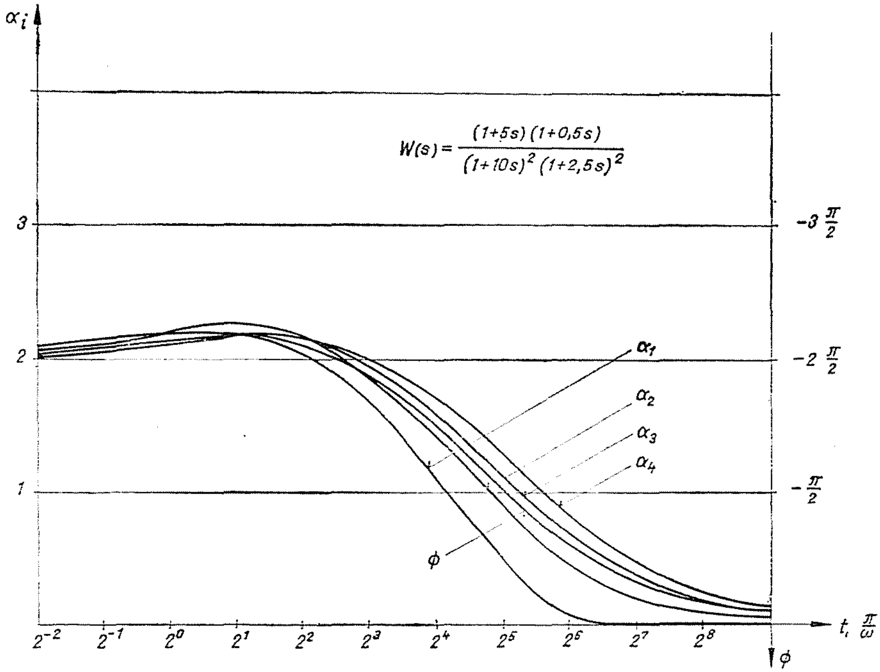
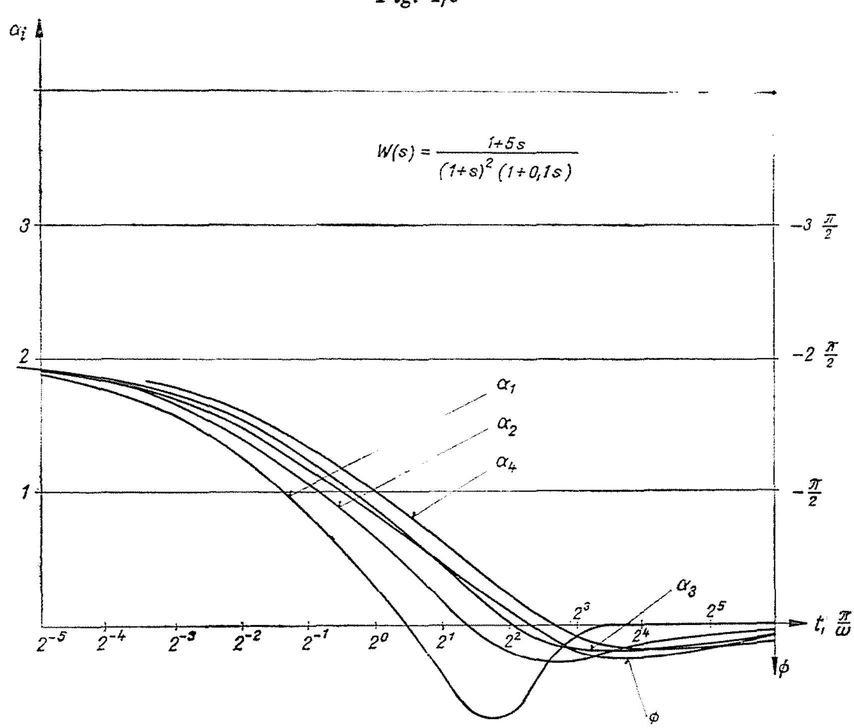

α_i
3
2
1
α_2
α_1
α_0
α_{-1}
ϕ
$W(s) = \frac{1}{s^2(1+s)^2}$
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
$W(s) = \frac{1}{s(1+s)^2}$
α_2
α_3
ϕ
α_1
α_2
α_3
α_4
ϕ
α_1 α_0
$W(s) = \frac{1}{(1+s)^2}$
2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/b



α_i
$W(s) = \frac{(1+5s)(1+0,5s)}{(1+10s)^2(1+2,5s)^2}$
3
2
1
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
α_1
α_2
α_3
α_4
ϕ
2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/c



α_i
$W(s) = \frac{1+5s}{(1+s)^2(1+0,1s)}$
3
2
1
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
α_1
α_2
α_4
α_3
2^{-5} 2^{-4} 2^{-3} 2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5
$t, \frac{\pi}{\omega}$
ϕ
ϕ

Fig. 1/d

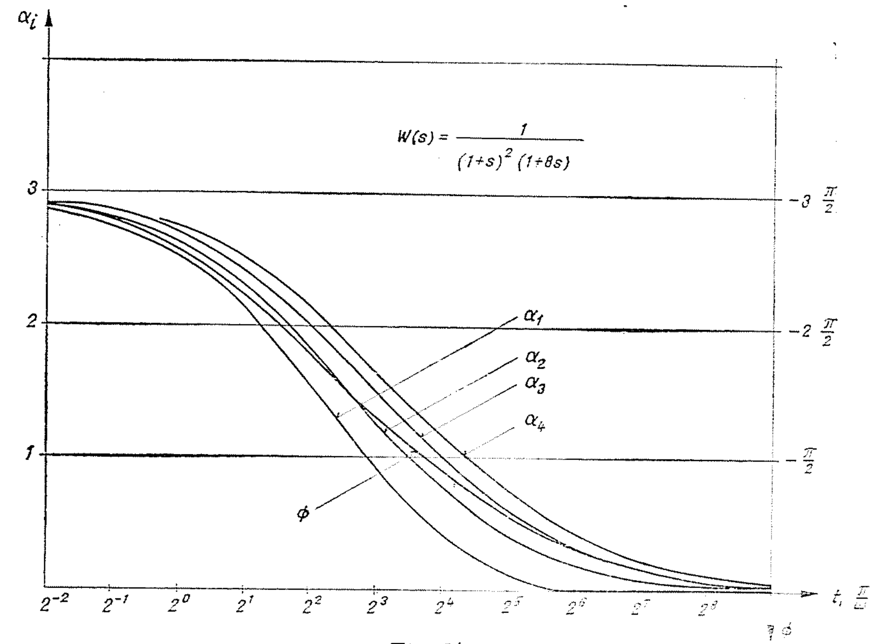

α_i
$W(s) = \frac{1}{(1+s)^2(1+8s)}$
3
2
1
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
α_1
α_2
α_3
α_4
ϕ
2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/e

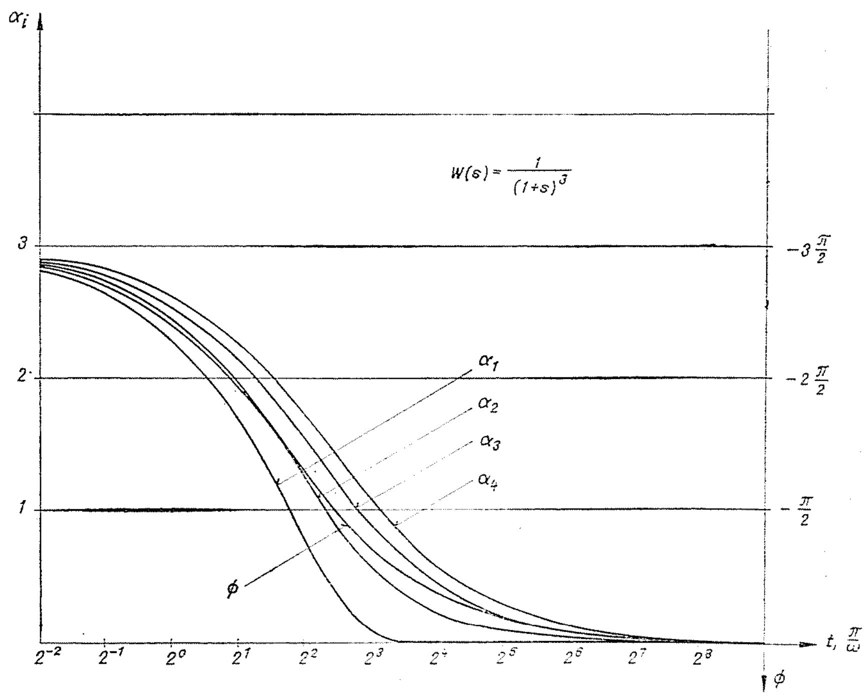

α_i
$W(s) = \frac{1}{(1+s)^3}$
3
2
1
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
α_1
α_2
α_3
α_4
ϕ
2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/f

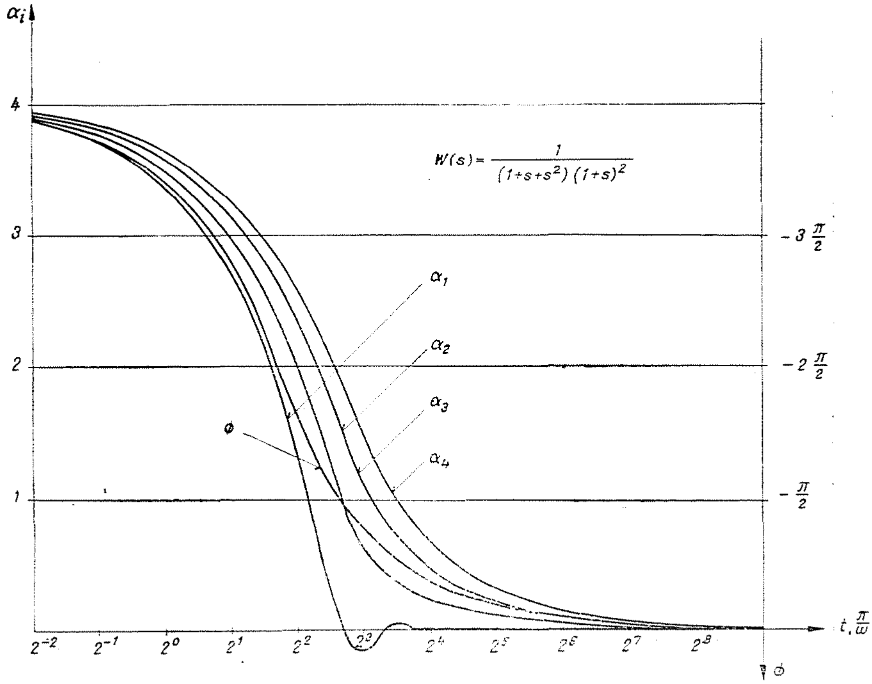

α_i
$W(s) = \frac{1}{(1+s+s^2)(1+s)^2}$
α_1
α_2
α_3
α_4
ϕ
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/g

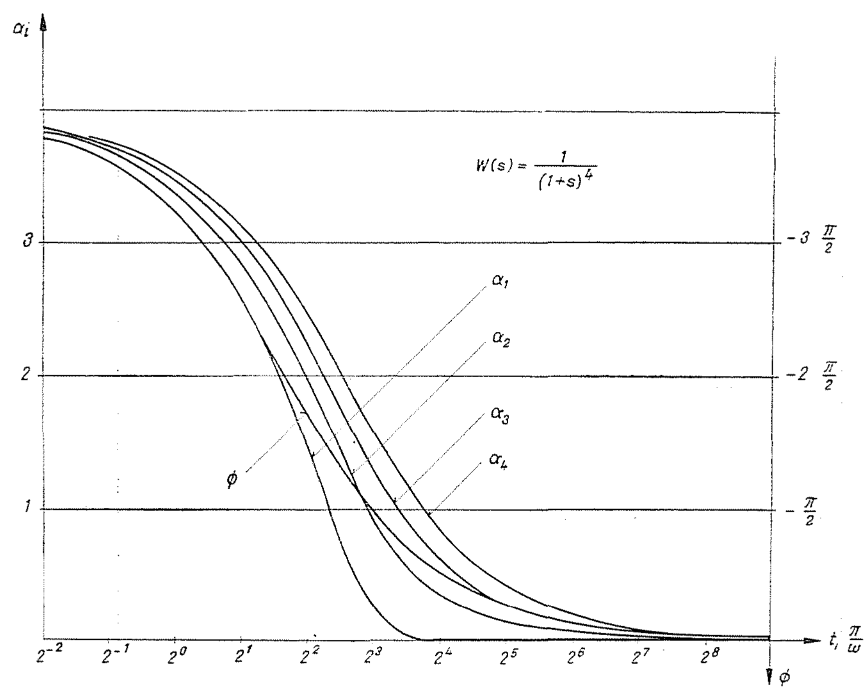
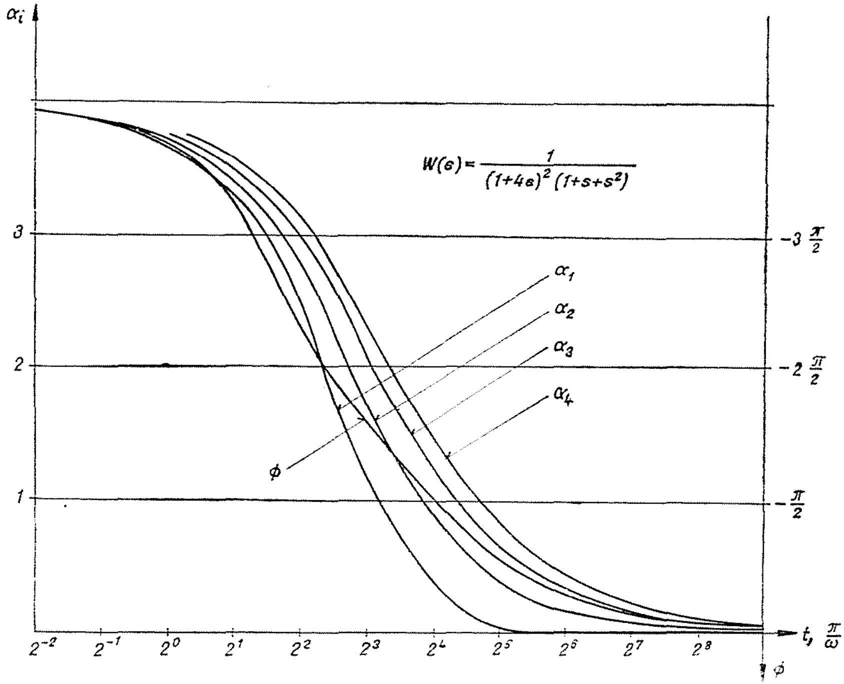
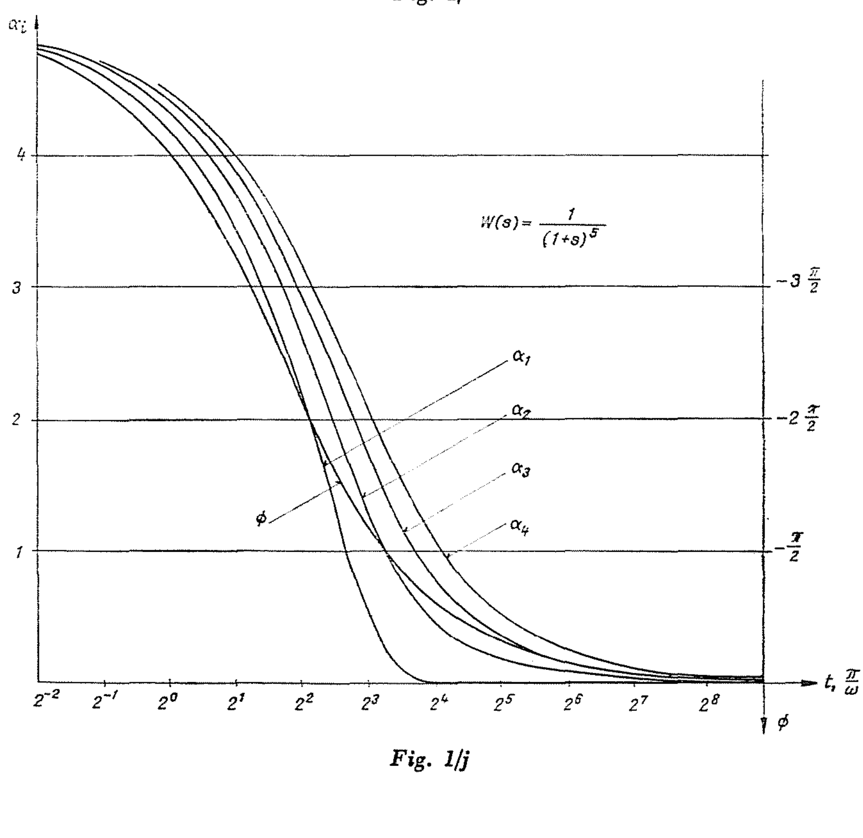

α_i
$W(s) = \frac{1}{(1+s)^4}$
α_1
α_2
α_3
α_4
ϕ
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/h



α_i
$W(s) = \frac{1}{(1+4s)^2(1+s+s^2)}$
3
2
1
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
α_1
α_2
α_3
α_4
ϕ
2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/i



α_i
$W(s) = \frac{1}{(1+s)^5}$
4
3
2
1
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
α_1
α_2
α_3
α_4
ϕ
2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/j

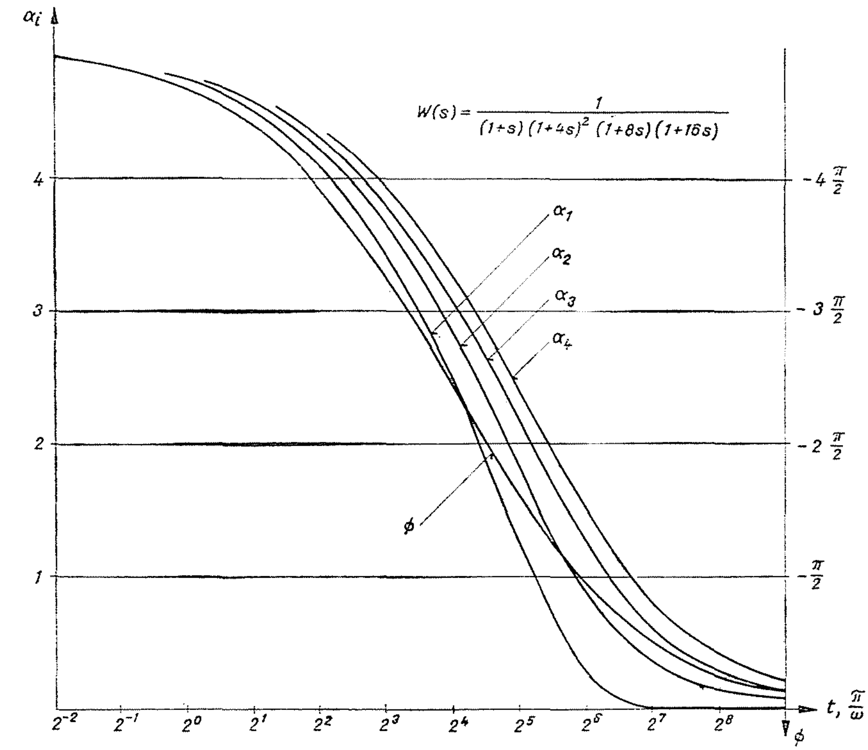

α_i
$W(s)=\frac{1}{(1+s)(1+4s)^2(1+8s)(1+16s)}$
α_1
α_2
α_3
α_4
ϕ
4
3
2
1
$-4\frac{\pi}{2}$
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/k

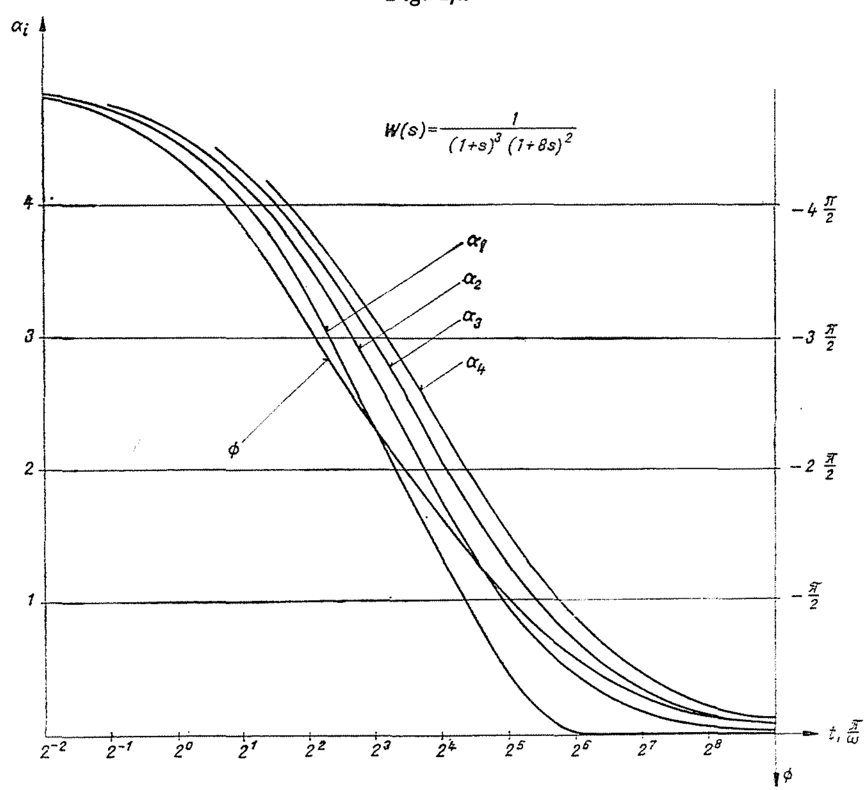

α_i
$W(s)=\frac{1}{(1+s)^3(1+8s)^2}$
α_1
α_2
α_3
α_4
ϕ
4
3
2
1
$-4\frac{\pi}{2}$
$-3\frac{\pi}{2}$
$-2\frac{\pi}{2}$
$-\frac{\pi}{2}$
2^{-2} 2^{-1} 2^0 2^1 2^2 2^3 2^4 2^5 2^6 2^7 2^8
$t, \frac{\pi}{\omega}$
ϕ

Fig. 1/l

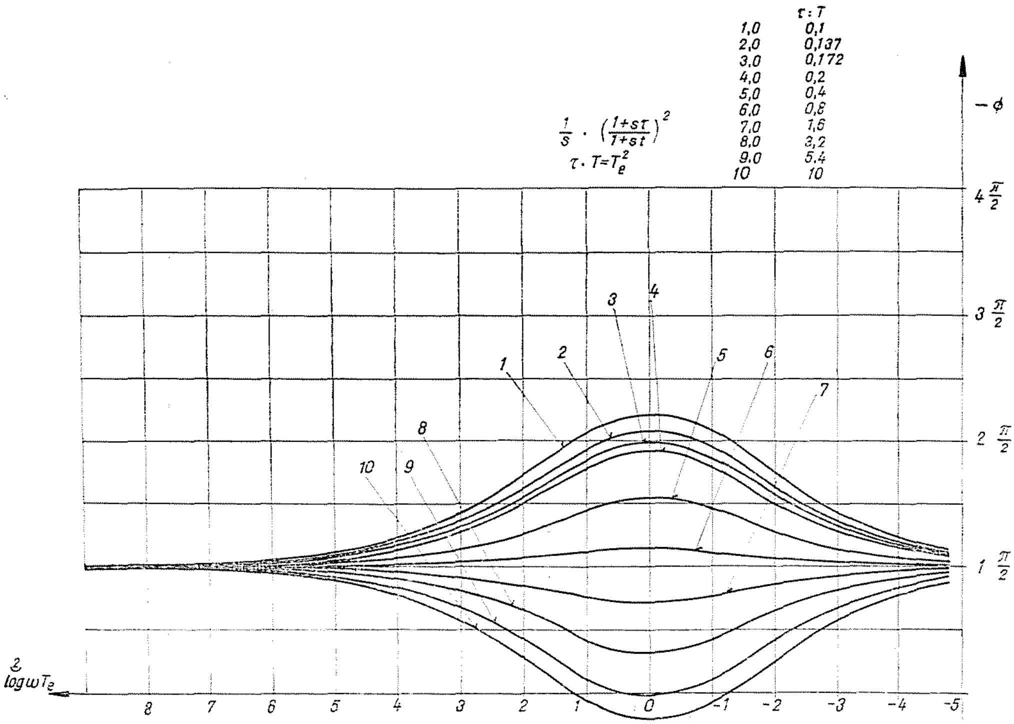

τ : T
1,0 0,1
2,0 0,137
3,0 0,172
4,0 0,2
5,0 0,4
6,0 0,8
7,0 1,6
8,0 3,2
9,0 5,4
10 10
$\frac{1}{s} \cdot \left(\frac{1+s\tau}{1+sT}\right)^2$
$\tau \cdot T = T_e^2$
$-\phi$
$4\frac{\pi}{2}$
$3\frac{\pi}{2}$
$2\frac{\pi}{2}$
$1\frac{\pi}{2}$
$^2\log \omega T_e$
8 7 6 5 4 3 2 1 0 -1 -2 -3 -4 -5

Fig. 2/a

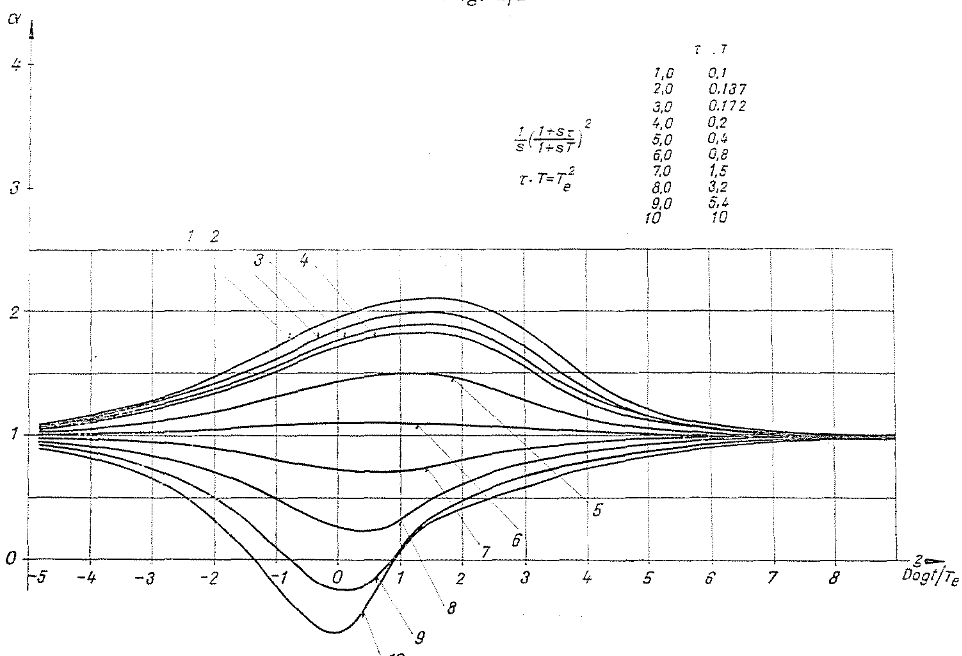

α
4
3
2
1
0
τ . T
1,0 0,1
2,0 0.137
3,0 0.172
4,0 0,2
5,0 0,4
6,0 0,8
7,0 1,5
8,0 3,2
9,0 5,4
10 10
$\frac{1}{s}\left(\frac{1+s\tau}{1+sT}\right)^2$
$\tau \cdot T = T_e^2$
-5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
$^2\log t/T_e$

Fig. 2/b

3° In the region of $\omega \to 0$ the phase function fits to the j-th order rise function a_j for which

$$j = 3 - i \tag{19}$$

(The meaning of i can be found in Eq. (10))

4° In the region of $\omega \to \infty$ the phase function fits to the p-th order rise function, α_p, for which

$$p = 5 - k = 5 - (n + i - m) \tag{20}$$

(The meaning of n, i and m can be found in Eq. (10)). Note: Properties 3° and 4° can be prove exactly. For the sake of conciseness, the proof will be, however, omitted here.

5° If the phase function is a monotonic one, then it lies mainly in the bundle limited by the rise functions of the j-th and p-th order. (The meanings of j and p are defined by Eqs (19) and (20)).

6° If the poles of the transfer function do not lie close to each other, then the slope of the phase plot is relatively small, and this is valid for the rise functions, too. In this case the phase plot runs uniformly through the bundle of the rise functions described in statement 5°.

7° If there are more poles close to each other which can be recognized by the steeper slope of the phase function, then the slope of the rise functions is also steeper. In this case the phase plot runs less uniformly thorugh the bundle of the rise functions. It removes faster from the rise function to which it fits on the right hand side ($t \to \infty$) (see Figs 4 or 9).

8° The monotonic phase function intersects the first-order rise function $\alpha_1(t)$ in the region $3 > \alpha_1 > 2$. If phase function fits to $\alpha_1(t)$ on the left-hand side, then $\Phi(t)$ leaves $\alpha_1(t)$ in the above region.

9° The phase function intersects the second-order rise function $\alpha_2(t)$ in the region $2 > \alpha_2(t) > 1$.

V. Conclusion

The rise function gives a full description of the dynamics of a linear system in the time domain. Except for the gain, the information content of rise functions is equivalent to that of step responses.

If a logarithmic scale is used for the argument t of the rise function then the shape of the rise function plot has several similarities to the phase curve of the Bode plot.

The rise functions may be of use for control system analysis and synthesis as well as process identification methods. Some examples will be presented in subsequent papers.

Appendix I

$$u_n = \ln \frac{V_n}{t^{n-1}}$$

$$\tau = \ln t$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}\tau} = \frac{\mathrm{d}u_n}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\tau}$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{V_n}{t^{n-1}} = \frac{t^{n-1}}{V_n} \; \frac{V_{n-1} \cdot t^{n-1} - (n-1)\, t^{n-2}\, V_n}{t^{2(n-1)}}$$

and

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = t$$

From the above three equations:

$$\frac{\mathrm{d}u_n}{\mathrm{d}\tau} = \frac{t\, V_{n-1}}{V_n} - (n-1)\,. \qquad \text{Q. E. D.}$$

Appendix II

The transfer function of the system is

$$W(s) = \frac{b_m s^m + \ldots + b_1 s + 1}{a_j s^j + \ldots a_1 s + 1} \cdot \frac{1}{s^i} \tag{21}$$

By long division we can expand this function to an infinite power series with respect to $1/s$:

$$W(s) = \frac{C_k}{s^k} + \frac{C_{k+1}}{s^{k+1}} + \ldots \tag{22}$$

$$k = j + i - m \tag{23}$$

(22) is the Laurent series of the transfer function which approximates the latter in the vicinity of $s \to \infty$. The corresponding step response is

$$v(t) = V_1(t) = \frac{C_k}{k!}\, t^k + \frac{C_{k+1}}{(k+1)!}\, t^{k+1} + \ldots \tag{24}$$

According to the final value theorem this is a Taylor series of the step response of the system, which approximates the latter in the vicinity of $t \to 0$.

The power series of the generalized *n-th* and $(n-1)$-th order step response are

$$V_n(t) = \frac{C_k}{(k+n-1)!}\, t^{k+n-1} + \frac{C_{k+1}}{(k+n)!}\, t^{k+n} + \ldots \tag{25}$$

and

$$V_{n-1}(t) = \frac{C_k}{(k+n-2)!}\, t^{k+n-2} + \frac{C_{k+1}}{(k+n-1)!}\, t^{k+n-1} + \ldots \tag{26}$$

Substituting (25) and (26) into the expression of the rise function

$$\alpha_n(t) = \frac{\dfrac{C_k}{(k+n-2)!}\, t^{k+n-1} + \dfrac{C_{k+1}}{(k+n-1)!}\, t^{k+n} + \ldots}{\dfrac{C_k}{(k+n-1)!}\, t^{k+n-1} + \dfrac{C_{k+1}}{(k+n)!}\, t^{k+n} + \ldots} - (n-1) \tag{27}$$

Since if $t \to 0$, the second and further terms of the numerator and denominator can be neglected:

$$\alpha_n(t)_{t\to 0} = \frac{\dfrac{C_k}{(k+n-2)!}\, t^{k+n-1}}{\dfrac{C_k}{(k+n-1)!}\, t^{k+n-1}} - (n-1) = (k+n-1) - (n-1) = k = j + i - m \tag{28}$$

Q. E. D.

Eq. (15) can be proven similarly: Now the transfer function (21) will be expanded in power series in the vicinity of $s = 0$. For this reason a long division is needed by writing both the numerator and the denominator according to increasing power:

$$W(s) = \frac{1 + b_1 s + b_2 s^2 + \dots}{s^i + a_1 s^{i+1} + a_2 s^{i+2} + \dots}$$

The result of the long division is a power series of the following form:

$$W(s)_{s\to 0} = \frac{1}{s^i} + \frac{d_{(-i+1)}}{s^{i-1}} + \dots + d_0 + d_1 s + \dots \tag{29}$$

The corresponding series of the step response approximates it at $t \to \infty$.

$$v(t) = \frac{t^i}{i!} + \frac{d_{(-i+1)}\, t^{i-1}}{(i-1)!} + \dots + d_0 + \Delta(t) \tag{30}$$

$\Delta(t)$ denotes the sum of the first and higher order impulse (Dirac) functions corresponding to the positive power members of Eq. (29).

Due to $t \to \infty$, on the one hand, Δt can be omitted, and on the other hand, the first $t^i/i!$ term maiorizes the other terms. So for sufficiently high t values, the system can be regarded as an i-fold integrating element and therefore, according to Eq. (7), also Eq. (12) holds.

Note that no series expansion can be done if the system has no poles in the right half plane, because in this case the Laplace transform does not converge at $s = 0$. This fact restricts the validity of Eq. (12) to system having no poles in the right half plane.

Summary

A new function named "rise function" is defined.

The rise function gives a full description of the dynamics of a linear system in the time domain. Except for the gain, the information content of rise functions is equivalent to that of step responses.

If a logaritmic scale is used for the argument t of the rise function, then the shape of the rise function plot has several similarities to the phase curve of the Bode plot.

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