ON SOME PROBLEMS OF THE EQUILIBRIUM CONDITIONS OF THE FEEDBACK SYSTEMS OF SECOND ORDER

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1. Introduction

The study deals with the characteristic properties of such a feedback system of second order by its signal-transfer properties in which there is, in principle, a possibility for several equilibrium points to develop. The second-order system is considered to consist of a 'controller' and of a 'controlled process', and it is assumed that in the signal transfer of both the controller and the process a signal delay is caused by some energy storage element. This approach largely simplifies the discussion and offers a possibility of using the concepts and terms of control engineering.

2. Notations and Terms

Symbols usual in mathematics will be used in the notation system. The physical variables denoted by small letters of the alphabet mean the generally arbitrary, single-valued functions of time t. $x(t)$ stands for the time function of the physical variable and $x(t_0)$ for the instantaneous value of the variable at the time $t = t_0$. The physical variable constant in time will be noted without any notation of the time: x is the value of the variable, constant (steady state) in time. The symbol Δ will be used to denote the change with respect to the steady-state value: e.g. $\Delta x(t) = x(t) - x_0$ is the time function of the change with respect to the steady-state value x_0. For the denotation of vector variables a simple dash (—), and for the denotation of matrices the double dash (=) over the symbol will be used.

t	= time
$x(t)$	= modified signal
$y(t)$	= controlled signal
$a(t)$	= reference signal
$z(t)$	= disturbance signal
$\bar{q}(t) = [x(t), y(t)]T$	= state vector
$\bar{u}(t) = [a(t), z(t)]T$	= excitation vector
$g(x,y,a_0) = 0$	= static curve of the controller
$f(x,y,z_0) = 0$	= static curve of the process
x_b	= input signal of the opened loop
x_k	= output signal of the opened loop
$\mathrm{tg}\,\beta$	= slope of the curve $g(x,y,a_0) = 0$
$\mathrm{tg}\,\alpha$	= slope of the curve $f(x,y,z_0) = 0$
$x_k = k(x_b)_{\bar{0}}$	= curve of the loop factor
$K_h = \mathrm{tg}\,\alpha \cdot \mathrm{tg}\,\beta$	= loop factor

$\operatorname{tg}\gamma = \operatorname{tg}\alpha \cdot \operatorname{tg}\beta$	= slope of the curve $x_k = k(x_b)$
$\bar{\bar{A}}$	= state matrix (Jacobi matrix)
$\bar{\bar{B}}$	= coefficient matrix (Jacobi matrix)
$\bar{\bar{\Phi}}(t) = e^{\bar{\bar{A}}t}$	= transfer matrix
$D(\lambda) = \det(\lambda \bar{\bar{I}} - \bar{\bar{A}})$	= characteristic polynomial of $\bar{\bar{A}}$
λ_i	= root of the characteristic polynomial (eigenvalue of $\bar{\bar{A}}$)

3. Structure of the System Examined

The block diagram of the second-order feedback system examined is shown in Figure 1.

The state equations describing the dynamic properties of the system are

$$\begin{aligned} \dot{x}(t) &= g[x(t), y(t), a(t)] \\ \dot{y}(t) &= f[x(t), y(t), z(t)] \end{aligned} \tag{1}$$

In a shorter form:

$$\dot{\bar{q}}(t) = \bar{F}[\bar{q}(t), \bar{u}(t)] \tag{2}$$

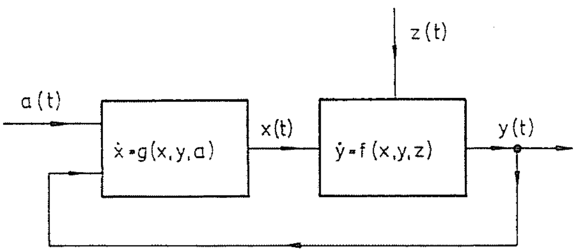


Fig. 1. Block diagram of the single input-single output nonlinear second-order system

For the function g and f in state equation (1) the following assumptions are made:

— the function g and f describe the signal transfer of the elements realizable also constructionally,

— the controller and the process have self-regulation. In the present case this means that the process gives a constant response y to constant input signals x and z, and the controller gives a constant output response x to constant input signals y and a,

— the function $g(x,y,a) = 0$ and $f(x,y,z) = 0$ describing the steady-state conditions of controller and process exhibit "usual behaviour" (they are single-valued, can be differentiated, etc.).

4. Stability of the Equilibrium Point

Let us plot the characteristics related to the steady-state condition

$$g(x, y, a) = 0$$
$$f(x, y, z) = 0 \tag{3}$$

of the controller and of the process in the state plane $x \sim y$ with constant input signals $\bar{u} = [a_0, z_0]^T$ (Fig. 2). The state vector $\bar{q}_0 = [x_0, y_0]^T$ developing in the intersection point 0 of the characteristic curves will be the equilibrium position of the nonlinear system if, with $t \to \infty$, the solution $\bar{q}(t)$ of differential equation (2) satisfying the initial condition $\bar{q}(O)$ converges to $\bar{q}(\infty) = \bar{q}_0$, as it is shown in the state trajectory of Fig. 3.

In the general case it is complicated to determine the stability conditions, since stability, due to the nonlinearity of the function g and f, depends also on the excitation signal $\bar{u}(t)$ and on the initial condition $\bar{q}(0)$. With restriction to small changes, one can work, after transformation and linearization of the coordinate placed into point $\bar{0}$, also with the help of the linear approximation

$$\dot{\overline{\Delta q}}(t) = \bar{\bar{A}} \cdot \overline{\Delta q}(t) + \bar{\bar{B}} \cdot \overline{\Delta u}(t) \tag{4}$$

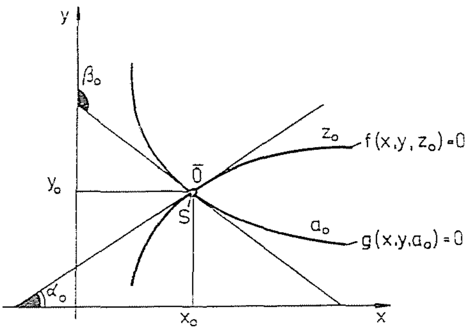


Fig. 2. Equilibrium operating point of the feedback system in intersection $\bar{o}$ of the steady-state curves

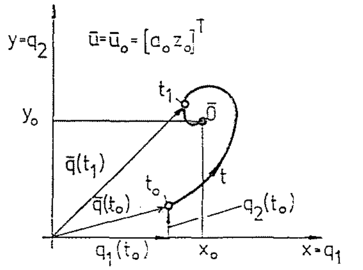


Fig. 3. Motion of the state vector $\bar{q}(t)$ in the state plane toward equilibrium pont $\bar{o}$. The initial condition is $\bar{q}(t_0)$ and the excitation $\bar{u}_0$ is constant

corresponding to (1). This equation describes the motion conditions of the motion in the environment of the operating point 0 according to Eq. (1), and in this environment

$$\bar{\bar{A}} = \begin{bmatrix} \left.\frac{\partial g}{\partial x}\right|_{\bar{0}} & \left.\frac{\partial g}{\partial y}\right|_{\bar{0}} \\ \left.\frac{\partial f}{\partial x}\right|_{\bar{0}} & \left.\frac{\partial f}{\partial y}\right|_{\bar{0}} \end{bmatrix} \text{ is the state matrix of the system,}$$

$$\bar{\bar{B}} = \begin{bmatrix} \left.\frac{\partial g}{\partial a}\right|_{\bar{0}} & 0 \\ 0 & \left.\frac{\partial f}{\partial z}\right|_{\bar{0}} \end{bmatrix} \text{ is the coefficient matrix of the excitations.}$$

Solution of the state differential equations (4) is

$$\overline{\Delta q}(t) = \bar{\bar{\Phi}}(t) \cdot \overline{\Delta q}(0) + \int_0^t \bar{\bar{\Phi}}(t-\tau) \cdot \bar{\bar{B}}\overline{\Delta u}(\tau)\, \mathrm{d}\tau \tag{5}$$

If the motion is generated only by the initial conditions $\bar{q}(0)$ (i.e. $\overline{\Delta u}(t) = 0$). then the systems come into equilibrium in the point $\bar{q}_0$ if

$$\lim_{t\to\infty} \bar{\bar{\Phi}}(t)\, \overline{\Delta q}(0) = \bar{0} \tag{6}$$

The equilibrium point marked by the index 0 is then defined as a stable equilibrium point. To satisfy condition (6), the state matrix $\bar{\bar{A}}$ must have eigenvalues with negative real parts ($Re\lambda_i < 0$), since in this case — with $t \to \infty$ — $\exp(\lambda_i \cdot t) \to 0$, and thus all the components of $\bar{\bar{\Phi}}(t)$ disappear.

The eigenvalues of the state matrix are the solutions of the characteristics equation

$$\frac{1}{g_x f_y}\lambda^2 - \frac{g_x + f_y}{g_x f_y}\lambda + \left(1 - \frac{f_x g_y}{f_y g_x}\right) = 0 \tag{7}$$

whose both roots contain a negative real part, if

$$\begin{aligned} \frac{g_x + f_y}{g_x f_y} &< 0 \\ \frac{f_x g_y}{f_y g_x} &< 1 \end{aligned} \tag{8}$$

To the analysis of conditions (8), on the basis of expression (4), let us draw the block diagram (Fig. 4) giving a linear approximation of the system.

Since, according to our assumptions, both the controller and the process have separate self-regulation, this means that the feedback of the integrating elements can be only negative, for the case $g_x \geqq 0$ and $f_y \geqq 0$ would exclude the possibility of self-regulation (Fig. 5).

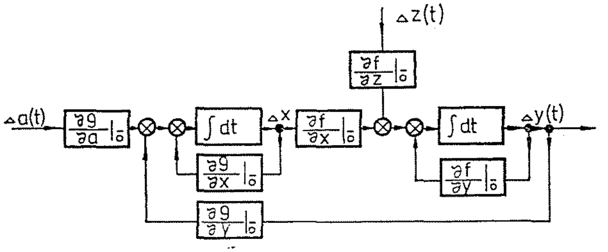


Fig. 4. Block diagram giving a linear approximation for the vicinity of the equilibrium point of a nonlinear system

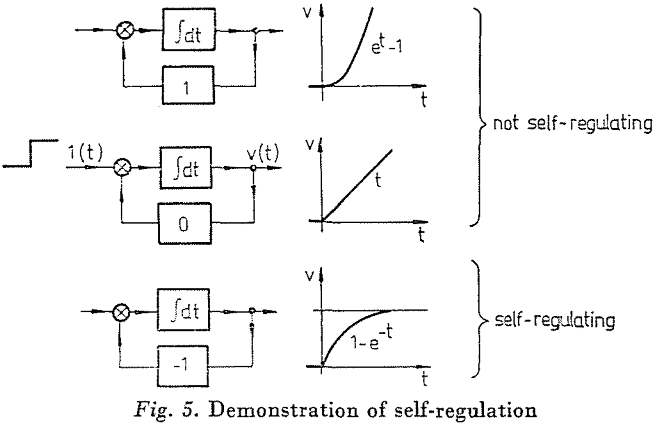


Fig. 5. Demonstration of self-regulation

Hence follows that $g_x < 0$ and $f_y < 0$, and thus the first inequality condition of (8) is fulfilled automatically. The second condition of inequality imposes a restriction on the loop factor of the feedback system, since the quotient $f_x g_y / f_y g_x$ in it is the loop factor in compliance with the expression

$$\left(\frac{1}{-\left.\dfrac{\partial g}{\partial x}\right|_{\bar{0}}} \right) \left(\left.\frac{\partial f}{\partial x}\right|_{\bar{0}} \right) \left(\frac{1}{-\left.\dfrac{\partial f}{\partial y}\right|_{\bar{0}}} \right) \left(\left.\frac{\partial g}{\partial y}\right|_{\bar{0}} \right) = K_{ho} \tag{9}$$

written on the basis of Figure 4. In addition, considering that

$$-\frac{\left.\frac{\partial f}{\partial x}\right|_{\bar{0}}}{\left.\frac{\partial f}{\partial y}\right|_{\bar{0}}} = \left.\frac{dy}{dx}\right|_{\bar{0}} = \operatorname{tg}\alpha_0\,; \qquad -\frac{\left.\frac{\partial g}{\partial y}\right|_{\bar{0}}}{\left.\frac{\partial g}{\partial x}\right|_{\bar{0}}} = \left.\frac{dx}{dy}\right| = \operatorname{tg}\beta_0 \tag{10}$$

an expression will be obtained for the stability condition of the working point 0 in accordance with

$$K_{h0} = \operatorname{tg}\alpha_0 \cdot \operatorname{tg}\beta_0 < 1 \tag{11}$$

This expression is also geometrically easily interpretable [1]. If, based on the characteristic curve of the process, the working point data of the feedback system were normalized to the value $\alpha = \pi/4$, the condition of stability could described also by the angle of the tangent drawn to the characteristic curve of the controller. Accordingly, the stability requires that the angle β lies in one of the intervals.

$$0 < \beta < \pi/2; \; \pi/2 < \beta < \pi \tag{12}$$

It is expedient to distinguish between the cases shown in Fig. 6.

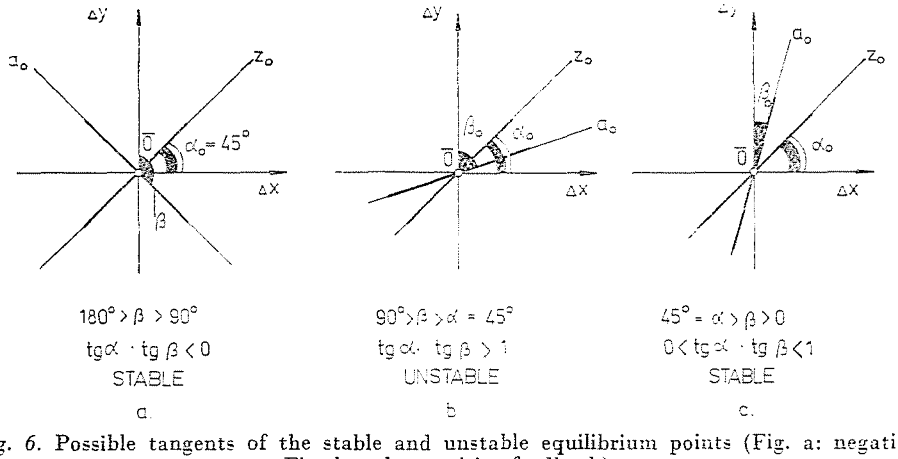


Fig. 6. Possible tangents of the stable and unstable equilibrium points (Fig. a: negative; Figs b and c: positive feedback)

Working point stability, feedback types and loop factor values in various cases

β	K_h	Stability of equilibrium point	Feedback type	Figure
$\pi/2 < \beta < \pi$	$K_h < 0$	stable	negative	6/a
$\pi/4 < \beta < \pi/2$	$K_h > 1$	unstable	positive	6/b
$0 < \beta < \pi/4$	$0 < K_h < 1$	stable	positive	6/c

The cases $\beta = \pi/2$ ($K_h = \infty$); $\beta = \pi/4$ ($K_h = 1$); $\beta = \pi(K_h = 0)$; $\beta = 0$ ($K_h = 0$) define special modes of operation, which will not be examined in the present paper. The root distribution of the characteristic equation for changes $-\infty < K_h < \infty$ of the loop factor are given in Fig. 7 with the assumption of $g_x = f_y = -1$.

The condition $K_h < 1$ of the working point stability can easily be read off from here, too: if this condition is satisfied, then both roots of (7) will have a negative real point.

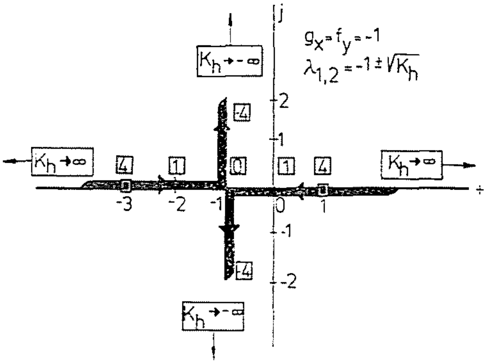


Fig. 7. Description of transient phenomena in the vicinity of the operating point with the help of the root locus plot

5. Characteristic Curve of the Loop Factor

Let the feedback system be opened in the signal path of x. Then x_b is the input signal of the open loop, influencing the process itself. The signal appearing at the output of the controller is the output signal x_k of the open loop (Figure 8).

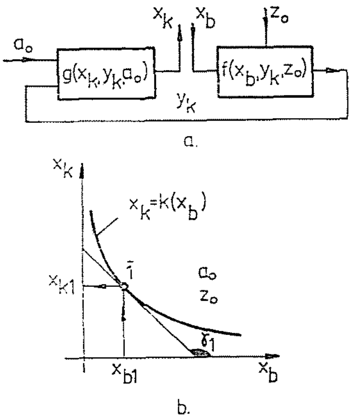


Fig. 8. Interpretation of the loop factor curve $x_k = k(x_b)$

In this case the function describing the steady-state conditions of the controller and the controlled process are

$$\begin{aligned} g(x_k, y_k, a_0) &= 0 \\ f(x_b, y_k, z_0) &= 0 \end{aligned} \tag{13}$$

Expressing $y_k = f^*(x_b, z_0)$ from the second equation and substituting it into the first equation describing the controller's properties, one obtains

$$g[x_k, f^*(x_b, z_0), a_0] = 0 \tag{14}$$

This equation defines the relation between the signals x_k and x_b. Supposing that x_k can be transformed into an explicit form, we get:

$$x_k = k(x_b, a_0, z_0) = k(x_b)_{\bar{0}} \tag{15}$$

A graphical plotting of relation (15) in the coordinate system $x_k \sim x_b$ will result in the characteristic curve of the loop factor (Fig. 8/b). This has the characteristic property that it intersects the straight $x_k = x_b$ at the input signal $x_b = x_0$, and thus the output signal, too, will be here $x_k = x_0$. The intersection point determines the coordinates corresponding to the equilibrium point marked by the index 0 of the feedback system, since — closing the system with such an input signal, one may obtain equilibrium state.

The slope of the tangent drawn to the intersection point of $x_k = k(x_b)_{\bar{0}}$ — and $x_k = x_b$ is the operating point value of the loop factor:

$$K_{ho} = \left.\frac{\partial k(x_b)}{\partial x_b}\right|_{\bar{0}} = \operatorname{tg} \gamma_0 \tag{16}$$

If $x = k(x_b)_{\bar{0}}$ decreases monotonically ($K_h < 0$), the feedback is negative and if it increases monotonically (i.e. $K_h > 0$), the feedback is positive.

The stability of the working point can be described also with the use of the angle γ_0: the equilibrium point is stable if γ_0 is not $\pi/4 < \gamma_0 < \pi/2$. If $\gamma_0 > \pi/2$, the feedback is negative ($K_{h0} < 0$), and if $0 < \gamma_0 < \pi/4$ the feedback is stable positive ($0 < K_h < 1$; See Fig. 9).

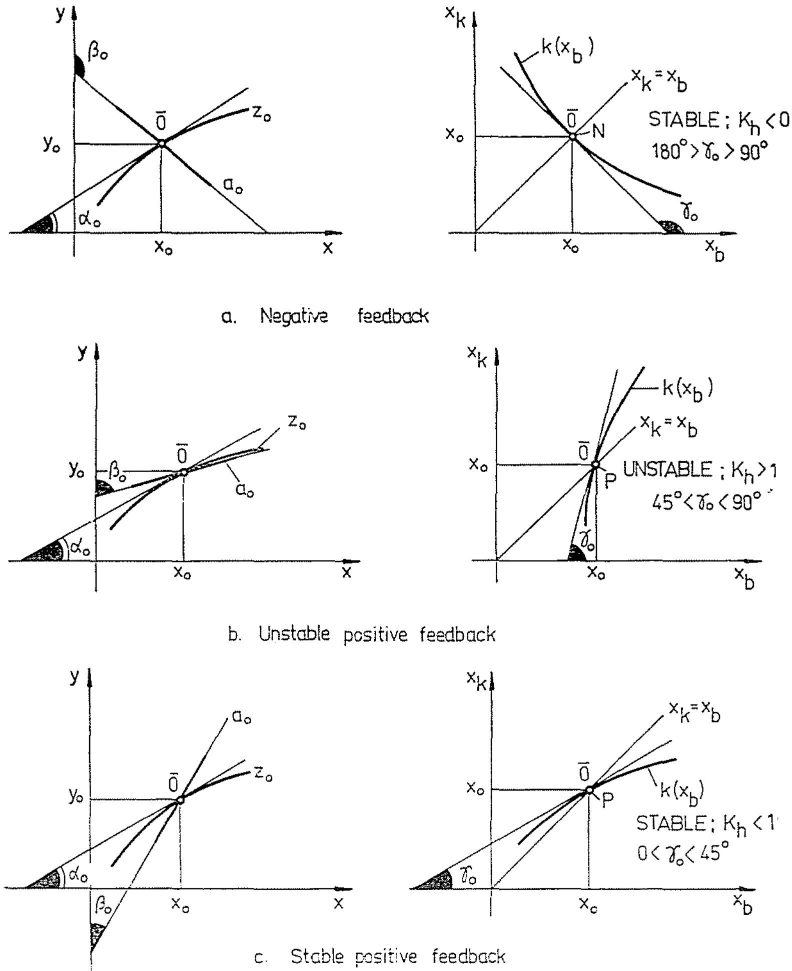


Fig. 9. Typical steady-state and loop factor curves of various systems

6. The Feedback System having More than one Equilibrium Point

Nevertheless, the functions $g(x,y,a_0) = 0$ and $f(x,y,z_0) = 0$ are both single-valued, and they satisfy the conditions mentioned in Part 3, the characteristic curves of the controller and process may intersect each other in several points (Fig. 10). In this case, evidently, also the characteristic curve of the loop factor $x_k = k(x_b)$ will intersect the straight $x_k = x_b$ in several points, and the intersection points correspond to the possible equilibrium positions of the feedback system in the state plane x, y (Fig. 11). The stability conditions of the individual equilibrium points can easily be determined from the state matrix of the differential equation system giving a linear approximation, or from the angles α_{i0} and β_{i0} of the tangents drawn to the operating points or from the angles γ_{i0} of the characteristic curve of the loop factor.

When the stability conditions of one of the equilibrium points have been decided, the other points do not require any detailed analysis for stability, since it becomes evident that the equilibrium points follow each other

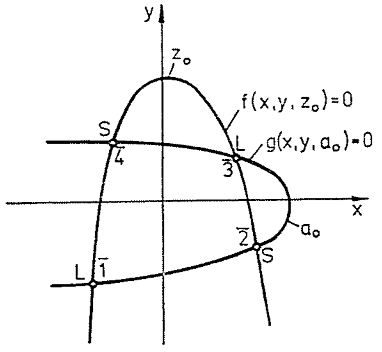


Fig. 10. Steady-state curves of controller and process

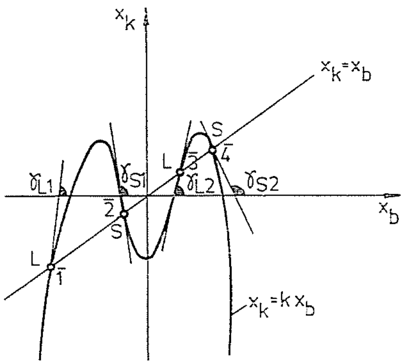
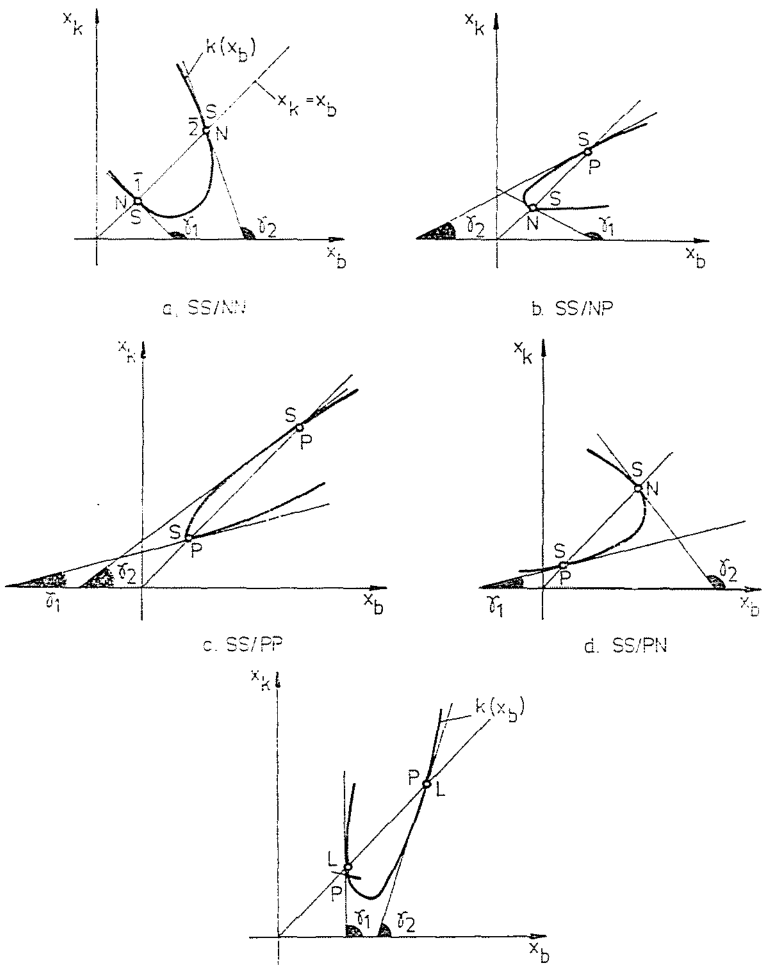


Fig. 11. Loop factor curve

in the order: ... stable, unstable, stable, unstable etc. This simply follows from the curve of the loop factor $x_k = k(x_b)_{\bar{0}}$ (Fig. 12) from which it can be read that two subsequent points with identical stability condition would be possible in multivalued functions, a case excluded from our considerations.

The positive or negative feedback of a full feedback system with several equilibrium points cannot be spoken of in the usual sense. In the stable point S, in the system with the equilibrium points.........SLSL........ the loop factor is negative ($K_h < 0$; point SN), or, in the interval (0.1) a positive value ($0 < K_h < 1$; point SP). The block diagram of the environment of the operating point then exhibits a negative or positive, resp., feedback.

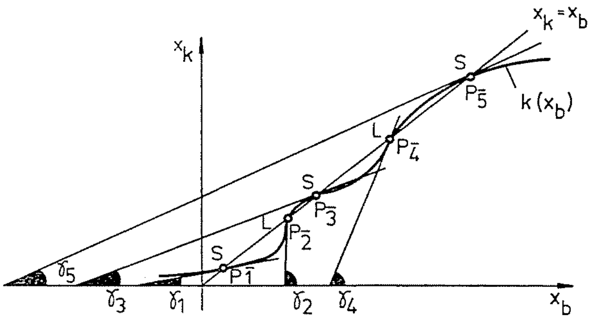


e. LL/PP

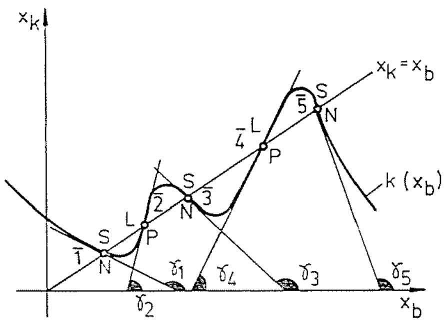
Fig. 12. Multivalued loop factor curves of systems having two equilibrium points of identical type concerning stability

In the points L the loop factor is larger than unity ($K_h > 1$, point LP), the unstable point can be only a result of positive feedback.

Figure 13 shows the two possible plots of the characteristic curve $x_k = k(x_b)_{\bar{0}}$. In Fig. 13/a the stable points are related to positive, in Fig. 13/b to negative feedback in the environment of the operating point. While the curve of the loop factor in Fig. 13/a increases monotonically and thus the positive feedback is decisive here, the monotonicity vanishes in Fig. 13/b and the positive and negative feedback appear alternatively. In the latter case the system as a whole has neither positive nor negative feedback.



a. Points S have P feedback



b. Points S have N feedback

Fig. 13. Loop factor curves of systems having more than one equilibrium point

A similar observation can be made already in the case when at least one point of the set of equilibrium points is an equilibrium point of SN type. The curves of four characteristic loop factors of a system having SLSL-type equilibrium points can be seen in Fig. 14.

7. Classification of the Feedback System according to First and Last Equilibrium Points

The curve $x_k = k(x_b)_{\bar 0}$ of a feedback system having n equilibrium positions intersects the straight of unity slope in n points. The first and last points of this series of points may define a position of either stable (S) or unstable (L) equilibrium. The theoretically possible loop factor curves corresponding to these positions are shown in Fig. 15. According to the properties of the extremal points of the series of equilibrium points, feedback systems can be classified as follow:

a) SS-type (stable) systems.

The first and last points are points defining a stable position with

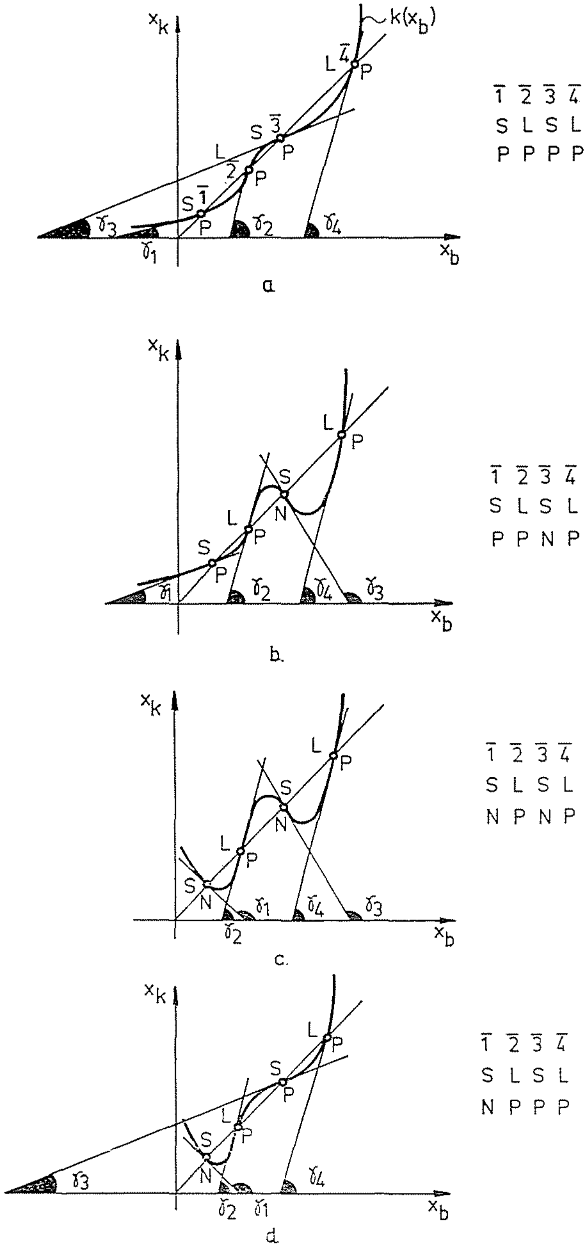


Fig. 14. Possible loop factor curves of a system having four equilibrium points

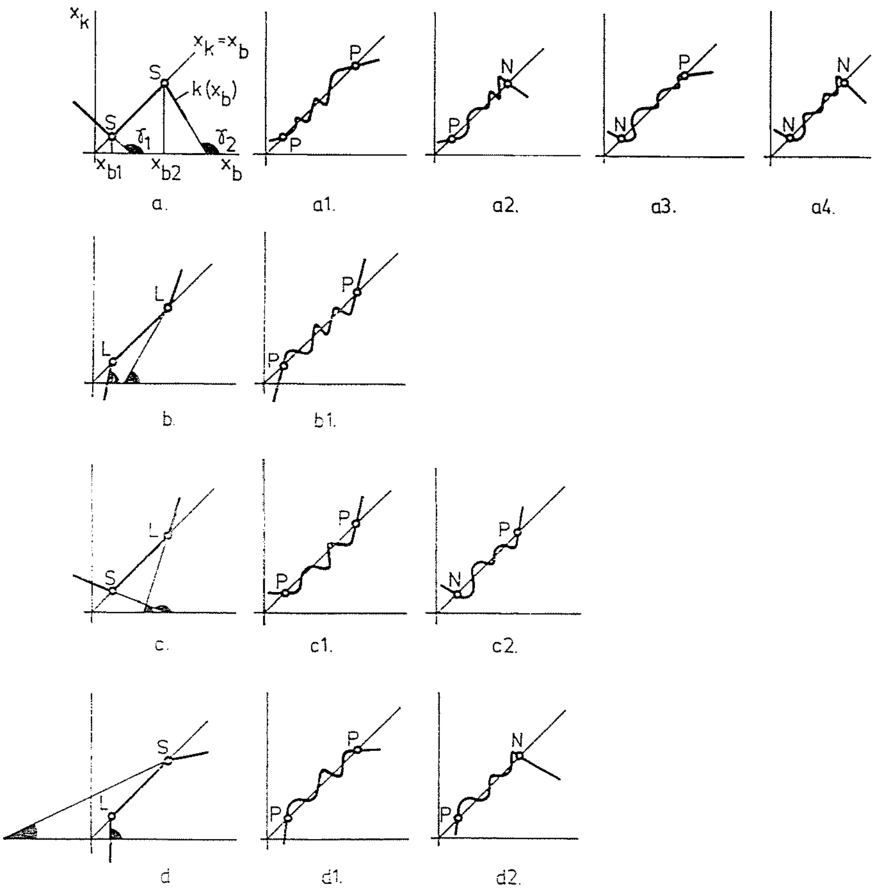


Fig. 15. Classification of feedback systems, based on their loop factor curves. a: stable (type SS); b: unstable (type LL); c: unstable at the top (type SL) and d: unstable at the bottom (type LS)

positive or negative feedback (systems of SS/PP, SS/PN, SS/NP, and SS/NN, see Fig. 15/a).

b) LS-type system (unstable at the bottom).
The first point is an unstable equilibrium point with positive feedback, the last point is a stable equilibrium point with either positive or negative feedback (LS/PP, SL/NP Fig. 15/d).

c) SL-type system (unstable at the top).
The first point is a stable equilibrium point with either positive of negative feedback, the last point is an unstable equilibrium point with positive feedback (SL/PP and SL/NP systems, see Fig. 15/c).

d) LL-type (unstable) system.
The first and last points are unstable equilibrium points with positive feedback (LL/PP system, see Fig. 15/b).

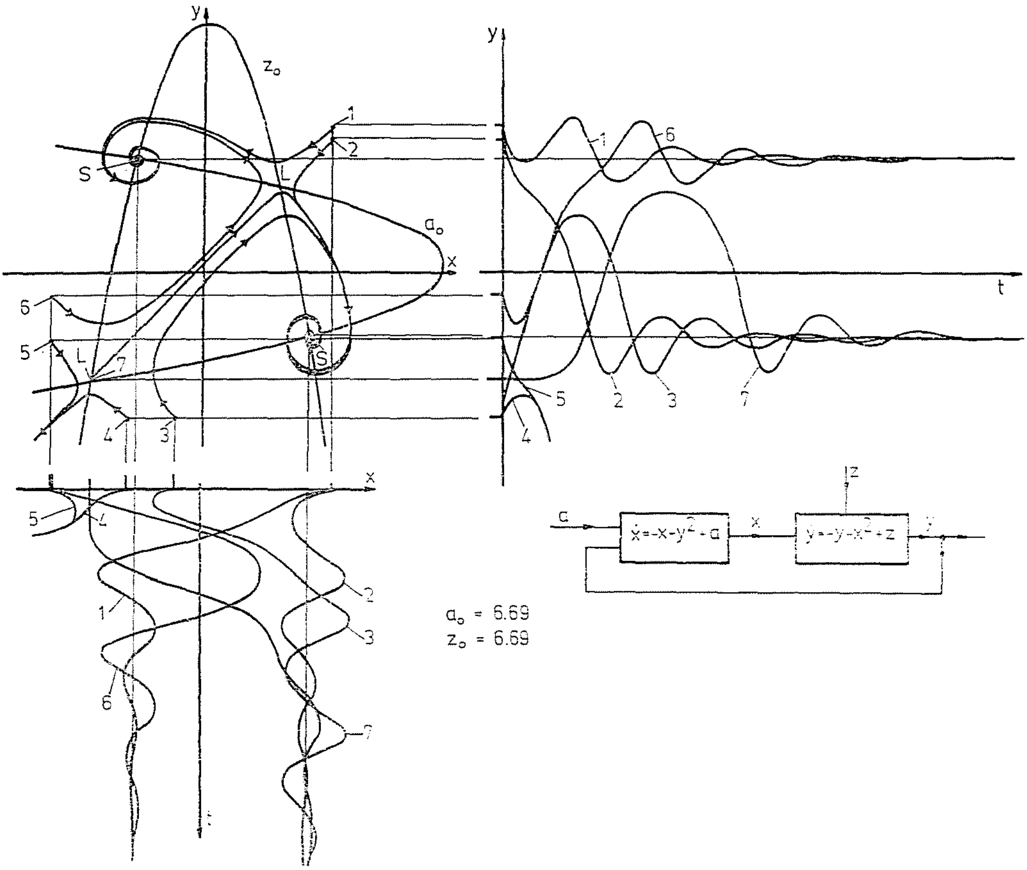


Fig. 16. Trajectories of state and time responses of an LSLS-type nonlinear system. (Results obtained by measurement)

Experiments carried out by analog and digital simulation have demonstrated that the trajectory $\bar{q}(t)$ starting from any point $\bar{q}(0)$ of the state plane of an SS-type stable system with $t \to \infty$ and constant excitation $\bar{u}(t) = \bar{u}_0$ — tends into one of the stable equilibrium points $\bar{q}_0$. The state planes of the partly unstable (LS and SL) or unstable (LL) systems, on the other hand, have such a domain from which the state vector of motions tends toward the infinite.

Separation of the state plane into stable and unstable domains involves considerable difficulties. Therefore, instead of calculating the separating curves analytically, it appeared more practical to divide the state plane $x \sim y$ by a "raster network" and to start a trajectory from each of its corner points as the points defining the initial conditions. The network obtained in this way demonstrates the stability conditions of the equilibrium points and the separating curves of the stability domains.

8. An example

Let the differential equations of the controller and of the process be

$$\dot{x}(t) = -x(t) - y^2(t) + a(t)$$
$$\dot{y}(t) = -y(t) + x^2(t) + z(t)$$

In this case the steady-state characteristic curve and the curve of the loop factor will be obtained in accordance with Figs. 10 and 11. After having performed the examination in an analog and digital computer, the trajectories of Figs. 16 and 17 will be obtained as measuring results on the analog computer and as numerical solutions on the digital computer.

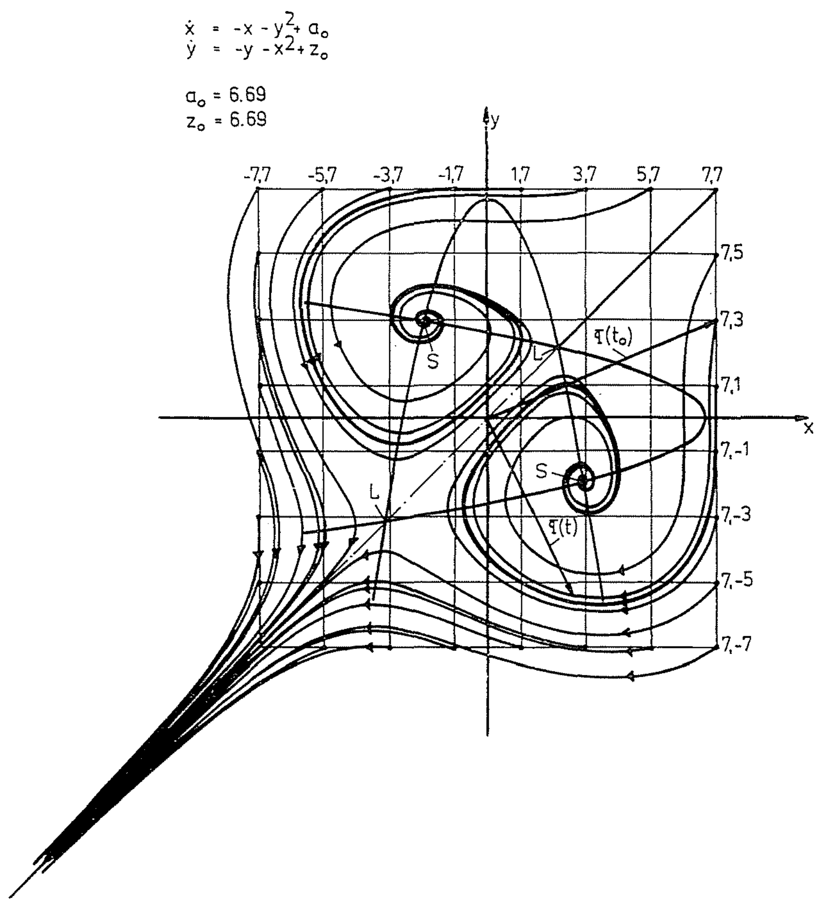


Fig. 17. Trajectories of state and time responses of an LSLS-type nonlinear system. (Results obtained by measurement)

9. Summary

The equilibrium points in a second-order, self-regulating feedback system consisting of "process" and "controller" and subjected to constant excitation lie in the intersection points of the static curves plotted in the state plane of the process and the controller. In the case of single-valued steady-state curves the successive equilibrium points of either curve follow each other in the order of stable (S), unstable (L), stable (S), etc.

Depending on the stable or unstable character of the first and last points of the series of equilibrium points, the feedback system is stable (type SS), partly unstable (type SL or LS) or unstable (LL). The stable system alone has the property that, if $t \to \infty$, the trajectory started from any point of the state plane ends in a point of stable equilibrium. In the state plane of a partly unstable or unstable system there is a domain, from which the trajectory tends to the infinite.

The curve of the loop factor $x_k = k(x_b)_{\bar{0}}$ intersects the line of unity slope generally several times. The number of the intersections is equal to the number of the equilibrium points. If the curve increases monotonically, then all equilibrium points relate to the positive feedback. The system as a whole then operates in positive feedback. If in a series of equilibrium points consisting of several points there is a stable equilibrium point in the vicinity of which the loop factor is negative, the monotonicity of the curve $x_k = k(x_b)_{\bar{0}}$ vanishes, and then neither the positive nor the negative feedback of the system as a whole can be interpreted.

The stable equilibrium points characterize a behaviour around the operating point corresponding either to the negative ($K_0 < 0$) or to the positive ($0 < K_h < 1$) feedback, while the unstable points correspond exclusively to the positive feedback ($K_h > 1$).

References

1. Szilágyi, B.: Szabályozási rendszer jelterjedésének vizsgálata statikus jelleggörbék alapján. Mérés és Automatika, 1968. 3. szám, 1968. 9. szám.
2. Csáki, F.: Korszerű szabályozáselmélet. Akadémiai Kiadó, Budapest, 1970.
3. C. N., Kerr: The stability of systems with multiplicative feedback. Automatica, 1968. Vol. 4.
4. Leon, O. Chua: Introduction to Nonlinear Network Theory. McGraw-Hill, 1969.

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