ON SOME PROBLEMS OF THE STATIC AND DYNAMIC ACCURACY OF LOGARITHMIC MULTIPLIERS

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Introduction

Some natural electrical processes can be characterized by logarithmic or exponential curves. This fact offers the possibility of building electric circuits with output signals varying proportional to the logarithm of the input signals. Circuits having such static characteristics are termed as logarithmic function generators. It is similarly possible to build circuits in which the logarithm of the output signal is proportional to the input signal: such circuits are called exponential function generators.

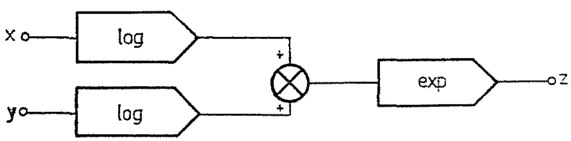

Fig. 1

Logarithmic and exponential circuits are widely used in analogue multiplier circuits in a way that the input signals to be multiplied are led through logarithmic function generators to a summator, the output signal of which is connected to an exponential function generator (see Fig. 1). The operation of such a multiplier is based on the well-known identity

$$x \cdot y = e^{(\ln x + \ln y)} \tag{1}$$

The input signals to be multiplied are represented by continuously varying voltage levels. The amplitude scaling can be performed by the method of normalized variables:

$$x = \frac{X - X_0}{X_{\max} - X_0} = \frac{E_x}{E_{x\max}} \tag{2}$$

where

X = actual value of the represented signal

X_0 = zero point value of the represented signal

X_{max} = maximum value of the represented signal
E_x = actual value of the voltage representing the signal
E_{xmax} = permissible maximum value of the voltage representing the signal, "machine unit"
x = normalized variable scaled to the signal.

The variables y and z have similar interpretation. The values X_0, X_{max}, Y_0, Z_0 and Z_{max} have to be chosen in a way that the dimensionless relative quantities x, y and z may change in the interval -1, $+1$ [1].

Questions of the accuracy, stability and operational speed of logarithmical function generators have been analyzed in detail in the surveying articles of RISLEY and SHEINGOLD [2], [3]. The theory presented by them starts from the assumption that the level of the input signal of the logarithmic element may change by several orders of magnitude during operation.

However, the logarithmic multipliers have different properties. The operating range of the input signals never exceeds two or three decades and therefore, the theories mentioned above cannot be applied to these.

The analysis presented below aims at defining relations useful in dimensioning logarithmic multipliers meeting the requirements of static accuracy and operational speed by optimal solutions.

1. The accuracy of logarithmic multipliers built from logarithmic and exponential function generators

The real static characteristics will serve as starting points to the error analysis of the circuit.

In the case of natural laws described by logarithmic or exponential functions, the argumentum of the function as always a dimensionless number. Therefore, the output voltage of the circuits can always be written as a function of the quotient of the input signal and the reference voltage (E_{REF}). The non-negligible shape error of the actual circuit can always be taken into account by stating the input (E_{os1}) and output offset voltage (E_{os2}) (see Fig. 2). Shape errors of other nature of the nonlinear static characteristics will be disregarded both here and further on. Thus the logarithmic characteristic can be written as

$$E_{out} = K \cdot \ln \frac{E_{in} - E_{os1}}{E_{REF}} + E_{os2} \tag{3}$$

and the exponential characteristic:

$$E_{out} = K \cdot \exp \frac{E_{in} - E_{os1}}{E_{REF}} + E_{os2} \tag{4}$$

For the logarithmic elements of the multiplier circuit of Fig. 1 the indices of x or y, and for the exponential elements the index z can be used:

$$E_{\text{out}\,x} = K_x \cdot \ln \frac{E_x - E_{\text{os}\,1x}}{E_{\text{REF}\,X}} + E_{\text{os}\,2x} \tag{5}$$

$$E_{\text{out}\,y} = K_y \cdot \ln \frac{E_y - E_{\text{os}\,1y}}{E_{\text{REF}\,Y}} + E_{\text{os}\,2y} \tag{6}$$

$$E_{\text{out}\,z} = K_z \cdot \exp \frac{E_{\text{in}\,z} - E_{\text{os}\,1z}}{E_{\text{REF}\,Z}} + E_{\text{os}\,2z} \tag{7}$$

After summation:

$$E_{\text{in}\,z} = a_x \cdot E_{\text{out}\,x} + a_y \cdot E_{\text{out}\,y} + E_{\text{oss}} \tag{8}$$

where a_x and a_y are the weight factors of summation, and E_{oss} is the offset voltage of the summator.

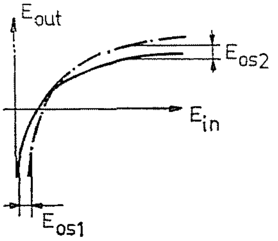

Fig. 2

Substituting (5), (6) and (7) into equation (8), then multiplying both sides by $E_{x\,\max} \cdot E_{y\,\max} \cdot E_{z\,\max}$, after reduction we obtain

$$\frac{E_{\text{out}\,z}}{E_{z\,\max}} = \frac{E_{\text{os}\,2z}}{E_{z\,\max}} + \frac{K_z}{E_{z\,\max}} \cdot \exp \frac{a_x \cdot E_{\text{os}\,2x} + a_y \cdot E_{\text{os}\,2y} + E_{\text{oss}} - E_{\text{os}\,1z}}{E_{\text{REF}\,Z}} \cdot$$

$$\cdot \frac{1}{\left(\frac{E_{\text{REF}\,X}}{E_{x\,\max}}\right)^{\frac{a_x \cdot K_x}{E_{\text{REF}\,Z}}} \cdot \left(\frac{E_{\text{REF}\,Y}}{E_{y\,\max}}\right)^{\frac{a_y \cdot K_y}{E_{\text{REF}\,Z}}}} \cdot \left(\frac{E_x - E_{\text{os}\,1x}}{E_{x\,\max}}\right)^{\frac{a_x \cdot K_x}{E_{\text{REF}\,Z}}} \cdot \left(\frac{E_y - E_{\text{os}\,1y}}{E_{y\,\max}}\right)^{\frac{a_y \cdot K_y}{E_{\text{REF}\,Z}}} \tag{9}$$

As can be seen, the form of the resulting characteristic of the multiplier is

$$\frac{E_{\text{out}\,z}}{E_{z\,\max}} = K_R \cdot \left(\frac{E_x - E_{\text{os}\,1x}}{E_{x\,\max}}\right)^p \cdot \left(\frac{E_y - E_{\text{os}\,1y}}{E_{y\,\max}}\right)^q + \frac{E_{\text{os}\,2E}}{E_{z\,\max}} \tag{10}$$

where K_R is the resulting transfer factor;
$E_{\text{os}\,1x}$ and $E_{\text{os}\,1y}$ are the reduced offset values characteristic of the inputs;
p and q are real exponents;
$E_{\text{os}\,2\text{R}}$ is the offset output voltage.

The multiplier is statically accurate if $E_{os\,1x} = E_{os\,1y} = E_{os\,2R} = 0$ and

$$p = q = K_R = 1. \tag{11}$$

The static error is small if the offset voltages have low values, p, q and K_R are constant and (11) is optimally satisfied.

Thus the sources of static errors are as follows:

a) A comparison of equations (9) and (10) shows that the multiplier "inherits" the input offset error of the logarithmic inputs, and the offset resultant of the output has the same value as the offset of the exponential circuit.

b) A further error source may be the change of the transfer factor. Assume in the examination that in the basic condition the multiplier is free of any linearity error, i.e.:

$$p = \frac{a_x \cdot K_x}{E_{\mathrm{REF}\,Z}} = q = \frac{a_y \cdot K_y}{E_{\mathrm{REF}\,Z}} = 1 \tag{12}$$

In this case:

$$K_R = \frac{K_z \cdot E_{x\,\max} \cdot E_{y\,\max}}{E_{\mathrm{REF}\,X} \cdot E_{\mathrm{REF}\,Y} \cdot E_{z\,\max}} \cdot \exp \frac{a_x \cdot E_{os\,2x} + a_y \cdot E_{os\,2y} + E_{oss} - E_{os\,1z}}{E_{\mathrm{REF}\,Z}} \tag{13}$$

and the whole change of the resulting transfer factor caused by small changes around the working point M is:

$$\begin{aligned} \Delta K_R = {} & \left.\frac{\partial K_R}{\partial K_z}\right|_M \cdot \Delta K_z + \left.\frac{\partial K_R}{\partial E_{\mathrm{REF}\,X}}\right|_M \cdot \Delta E_{\mathrm{REF}\,X} + \left.\frac{\partial K_R}{\partial E_{\mathrm{REF}\,Y}}\right|_M \cdot \Delta E_{\mathrm{REF}\,Y} + \\ & + \left.\frac{\partial K_R}{\partial E_{\mathrm{REF}\,Z}}\right|_M \cdot \Delta E_{\mathrm{REF}\,Z} + \left.\frac{\partial K_R}{\partial a_x}\right|_M \cdot \Delta a_x + \left.\frac{\partial K_R}{\partial a_y}\right|_M \cdot \Delta a_y + \\ & + \left.\frac{\partial K_R}{\partial E_{os\,2x}}\right|_M \cdot \Delta E_{os\,2x} + \left.\frac{\partial K_R}{\partial E_{os\,2y}}\right|_M \cdot \Delta E_{os\,2y} + \\ & + \left.\frac{\partial K_R}{\partial E_{oss}}\right|_M \cdot \Delta E_{oss} + \left.\frac{\partial K_R}{\partial E_{os\,1z}}\right|_M \cdot \Delta E_{os\,1z} \end{aligned} \tag{14}$$

where the values of the partial derivatives in M can be determined from (9). Since the small changes around the working point can be considered to be independent probability variables, the resulting transfer factor can be expected to have the uncertainty

$$\frac{\Delta K_R}{K_R}=\Bigg[\left(\frac{\Delta K_z}{K_z}\right)^2+\left(\frac{\Delta E_{\mathrm{REF}\,X}}{E_{\mathrm{RFF}\,X}}\right)^2+\left(\frac{\Delta E_{\mathrm{REF}\,Y}}{E_{\mathrm{REF}\,Y}}\right)^2+ \\ +\left(\frac{E_{\mathrm{os}\,2x}+E_{\mathrm{os}\,2y}+E_{\mathrm{oss}}-E_{\mathrm{os}\,1z}}{E_{\mathrm{REF}\,Z}}\right)^2\cdot\left(\frac{\Delta E_{\mathrm{REF}\,Z}}{E_{\mathrm{REF}\,Z}}\right)^2+ \\ +\left(\frac{a_x\cdot\Delta E_{\mathrm{os}\,2x}}{E_{\mathrm{REF}\,Z}}\right)^2+\left(\frac{a_y\cdot\Delta E_{\mathrm{os}\,2y}}{E_{\mathrm{REF}\,Z}}\right)^2+\left(\frac{E_{\mathrm{os}\,2x}\cdot\Delta a_x}{E_{\mathrm{REF}\,Z}}\right)^2+\left(\frac{E_{\mathrm{os}\,2y}\cdot\Delta a_y}{E_{\mathrm{REF}\,Z}}\right)^2+ \\ +\left(\frac{E_{\mathrm{oss}}}{E_{\mathrm{REF}\,Z}}\right)^2+\left(\frac{E_{\mathrm{os}\,1z}}{E_{\mathrm{REF}\,Z}}\right)^2\Bigg]^{\frac{1}{2}} \tag{15}$$

With the usual solution of summation the expectable changes of the weight factors of summation can be neglected as compared with the other factors. From (15) it is seen that, if the resultant of the offset voltages is zero, an alteration of the voltage $E_{\mathrm{REF}\,Z}$ does not affect directly the transfer factor. The relative change of the transfer factor is the sum of the relative change of the parameters K_z, $E_{\mathrm{REF}\,X}$, $E_{\mathrm{REF}\,Y}$, and of the offset voltage changes (drifts) with respect to $E_{\mathrm{REF}\,Z}$.

c) It is a peculiar problem of the logarithmic multiplier that a considerable linearity error may occur [4]. The change of the linearity error of multipliers as a function of exponent p is shown in Table 1.

Table I

p	0.8	0.9	0.95	0.99	1.0	1 01	1.05	1.1	1.2
h_{lin} [%]	8.19	3.87	1.89	0.37	0	−0.37	−1.79	−3.50	−6.70

The linearity error can be eliminated by stabilizing the quotients

$$p=\frac{a_x\cdot K_x}{E_{\mathrm{REF}\,Z}}\quad\text{and}\quad q=\frac{a_y\cdot K_y}{E_{\mathrm{REF}\,Z}} \tag{16}$$

composed from the transfer factors of the logarithmic circuits, the input weight factors of the summator and the reference voltage of the exponential circuit, further by continuously adjusting one of the parameters influencing the quotient.

Stability of the values of p and q can be ensured either by the stability of the singular factors or by a design in which the values of the numerator and the denominator change in the same manner upon the influence of disturbing signals.

Since, e.g., the nominal value of p is $p_0 = 1$, small changes near the working point will produce, according to (16):

$$\Delta p = \frac{\Delta a_x}{a_x} + \frac{\Delta K_x}{K_x} - \frac{\Delta E_{\mathrm{REF}\,Z}}{E_{\mathrm{REF}\,Z}} \approx \frac{\Delta K_x}{K_x} - \frac{\Delta E_{\mathrm{REF}\,Z}}{E_{\mathrm{REF}\,Z}} \tag{17}$$

Similarly

$$\Delta q \approx \frac{\Delta K_y}{K_y} - \frac{\Delta E_{\mathrm{REF}\,Z}}{E_{\mathrm{REF}\,Z}} \tag{18}$$

Thus, the compensation must be designed in a manner that external disturbances produce relative changes of identical extent in the parameters K_x, K_y and $E_{\mathrm{REF}\,Z}$.

2. The accuracy of a logarithmic multiplier built from logarithmic elements

A compensated construction in accordance with the above considerations is described by the implicit form of the basic identity (1) of operation:

$$\ln x + \ln y - \ln z = 0 \tag{19}$$

The construction of the logarithmic multiplier working according to (19) requires only one type of nonlinear basic element (Fig. 3). With appropriate design, the corresponding parameters of the logarithmic circuits applied

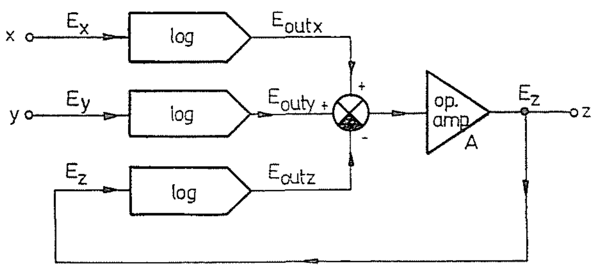

Fig. 3

agree with each other. Thus the multiplier may be free of errors deriving from the alteration of the exponent.

Write the operational equations for the static analysis of the circuit. In accordance with the foregoing, the characteristics of the logarithmic elements will be:

$$E_{\text{out}\,x} = K_x \cdot \ln \frac{E_x - E_{\text{os}\,1x}}{E_{\text{REF}\,X}} + E_{\text{os}\,2x} \tag{20}$$

$$E_{\text{out}\,y} = K_y \cdot \ln \frac{E_y - E_{\text{os}\,1y}}{E_{\text{REF}\,Y}} + E_{\text{os}\,2y} \tag{21}$$

$$E_{\text{out}\,z} = K'_z \cdot \ln \frac{E_z - E'_{\text{os}\,1z}}{E'_{\text{REF}\,Z}} + E'_{\text{os}\,2z} \tag{22}$$

The amplifier's output voltage will be

$$E_z = A \cdot (a_x \cdot E_{\text{out}\,x} + a_y \cdot E_{\text{out}\,y} - E_{\text{out}\,z} + E_{\text{oss}}) \tag{23}$$

where $A = \dfrac{\partial E_z}{\partial E_{\text{out}\,z}}$ is the gain of the amplifier.

A comparison of the equations, after reduction, will yield:

$$\frac{E_z}{E_{z\,\max}} = \frac{E'_{\text{os}\,1z}}{E_{z\,\max}} + \frac{E_{\text{REF}\,Z}}{E_{z\,\max}} \cdot \exp \frac{a_x \cdot E_{\text{os}\,2x} + a_y \cdot E_{\text{os}\,2y} - E'_{\text{os}\,2z} + E_{\text{oss}} - \dfrac{E_z}{A}}{K'_z} \cdot$$
$$\cdot \frac{\left(\dfrac{E_x - E_{\text{os}\,1x}}{E_{x\,\max}}\right)^{\frac{a_x \cdot K_x}{K'_z}} \cdot \left(\dfrac{E_y - E_{\text{os}\,1y}}{E_{y\,\max}}\right)^{\frac{a_y \cdot K_y}{K'_z}}}{\left(\dfrac{E_{\text{REF}\,X}}{E_{x\,\max}}\right)^{\frac{a_x \cdot K_x}{K'_z}} \cdot \left(\dfrac{E_{\text{REF}\,Y}}{E_{y\,\max}}\right)^{\frac{a_y \cdot K_y}{K_z}}} \tag{24}$$

Considering the similarity of forms and the differences of contents between Eqs (24) and (9), the following can be stated of static characteristics of the resulting multiplier looked for in the form (10):

a) This multiplier, too, "inherits" the input offset error of the logarithmic circuits belonging to x and y; the value of the output offset, however, agrees with the input offset error of the logarithmic circuit feeding back the signal E_z.

b) The resulting transfer factor is:

$$K_R = \frac{E'_{\text{REF}\,Z} \cdot E_{x\,\max} \cdot E_{y\,\max}}{E_{z\,\max} \cdot E_{\text{REF}\,X} \cdot E_{\text{REF}\,Y}} \cdot \exp \frac{a_x \cdot E_{\text{os}\,2x} + a_y \cdot E_{\text{os}\,2y} - E'_{\text{os}\,2z} + E_{\text{oss}} - \dfrac{E_z}{A}}{K'_z} \tag{25}$$

Due to the similarities between the logarithmic circuits applied, it can be supposed that both in the basic state, and in its vicinity the reference voltages and the transfer factors of the particular circuits, respectively, agree with each other.

$$E_{\mathrm{REF}\,X} = E_{\mathrm{REF}\,Y} = E'_{\mathrm{REF}\,Z} = E_{\mathrm{REF}} \tag{26}$$

$$K_x = K_y = K'_z = K \tag{27}$$

In this case, however,

$$p = a_x \quad \text{and} \quad q = a_y\,, \tag{28}$$

and thus a_x and a_y may have only the value 1 in the basic state. Consider also our earlier statement on the exceptable change of the summation weight factors, further assume that in the basic state the resultant of the offset voltages can be set to zero. Using all this, let us write the relative change of the resulting transfer factor that can be expected to be caused by small changes of the parameters around the working point determined by the values of the basic state.

$$\frac{\Delta K_R}{K_R} = \sqrt{\left(\frac{\Delta E_{\mathrm{REF}}}{E_{\mathrm{REF}}}\right)^2 + \left(\frac{\Delta(E_{\mathrm{os}\,2x} + E_{\mathrm{os}\,2y} - E_{\mathrm{os}\,2z} + E_{\mathrm{oss}})}{K}\right)^2} \tag{29}$$

A most typical form of additional errors is the temperature dependence of the circuit characteristics. If the temperature dependence of the resulting transfer factor depends to the same extent on the temperature dependence of the reference voltage and of the resultant of the offset voltages, then the following dimensioning relationships will be obtained from (29):

$$|\alpha_{\mathrm{EREF}}| \le \frac{\sqrt{2}}{2} \cdot \alpha_{KR\,\max} \tag{30}$$

$$|\alpha_{\mathrm{OSAMP}} + \alpha_{\mathrm{OSLOG}}| \le \frac{\sqrt{2}}{2} \cdot K \cdot \alpha_{KR\,\max} \tag{31}$$

where α_{EREF} = temperature coefficient of the reference voltage of the basic logarithmic circuit [%/°C];

$\alpha_{KR\,\max}$ = permissible temperature coefficient of the resulting transfer factor of the logarithmic multiplier [%/°C];

α_{OSAMP} = temperature coefficient of the input offset voltage of the operational amplifier, in other words: drift of the amplifier [V/°C];

α_{OSLOG} = output drift of the basic logarithmic circuit [V/°C];

K = transfer factor of the basic logarithmic circuit [V].

Writing (31), it was presumed that, due to the similarity of the circuits,

$$E_{\mathrm{os}\,2y}(\vartheta) - E_{\mathrm{os}\,2z}(\vartheta) \approx 0 \tag{32}$$

c) It can further be seen from (24) that due to the gain A being finite, a product error appears as well. If the gain is large, it will be sufficient to consider only the first term of the Taylor series of the exponential expression. So

$$h = x \cdot y \cdot \frac{E_{z\,\max}}{A \cdot K} \cdot 100\% \tag{33}$$

The minimum value of the gain belonging to the permissible maximum product error will be:

$$A_{\min} = \frac{E_{z\,\max}}{K} \cdot \frac{100\%}{h_{\max}} \tag{34}$$

d) Following from earlier considerations the linearity error of such a multiplier is dependent only on the values of a_x and a_y (see (26), (27) and (28)). The linearity error can be eliminated if a_x and a_y can continuously be adjusted.

3. Accuracy of the logarithmic multiplier from exponential elements

Following from the considerations made at the end of Sec. 1, also the logarithmic multiplier built from purely exponential elements according to Fig. 4 can be expected to work without linearity errors.

To the analysis of accuracy, let us write the operational equations of the singular circuits. For simplicity presume that the transfer factors and reference voltages of the singular exponential circuits and the gains of the operational amplifiers agree with each other and also the summation weight factors have been set identical.

Using the notations of Fig. 4:

$$\Phi_x = C \cdot \exp \frac{\Phi_{lnx} - \Phi_{os\,1x}}{\Phi_{REF}} + \Phi_{os\,2x} \tag{35}$$

$$\Phi_{lnx} = A \cdot (E_x - \Phi_x + E_{osAMP\,x}) \tag{36}$$

$$\Phi_y = C \cdot \exp \frac{\Phi_{lny} - \Phi_{os\,1y}}{\Phi_{REF}} + \Phi_{os\,2y} \tag{37}$$

$$\Phi_{lny} = A \cdot (E_y - \Phi_y - E_{osAMP\,y}) \tag{38}$$

$$\Phi_{lnz} = \Phi_{lnx} + \Phi_{lny} + E_{oss} \tag{39}$$

$$E_z = C \cdot \exp \frac{\Phi_{lnz} - \Phi_{os\,1z}}{\Phi_{REF}} + \Phi_{os\,2z} \tag{40}$$

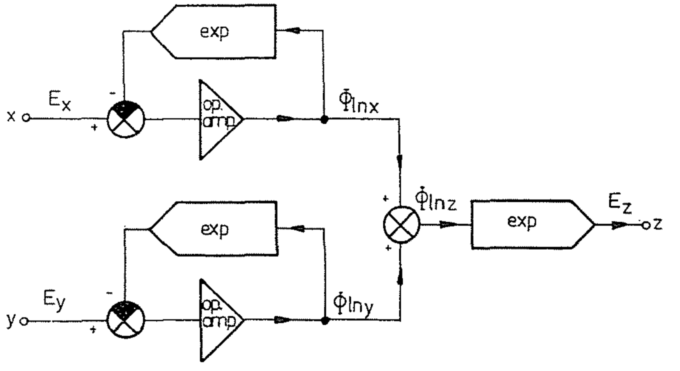

Fig. 4

After substitution and reduction:

$$\frac{E_z}{E_{z\max}} = \frac{E_{x\max} \cdot E_{y\max}}{E_{z\max} \cdot C} \cdot \exp\left(\frac{\Phi_{\mathrm{os}\,1x} + \Phi_{\mathrm{os}\,1y} + E_{\mathrm{oss}} - \Phi_{\mathrm{os}\,1z}}{\Phi_{\mathrm{REF}}}\right) \cdot$$

$$\cdot \frac{E_x + \Phi_{\mathrm{os}\,2x} + E_{\mathrm{osAMP}\,x} - \dfrac{\Phi_{lnx}}{A}}{E_{x\max}} \cdot \tag{41}$$

$$\cdot \frac{E_y + \Phi_{\mathrm{os}\,2y} + E_{\mathrm{osAMP}\,y} - \dfrac{\Phi_{lny}}{A}}{E_{y\max}} + \frac{\Phi_{\mathrm{os}\,2z}}{E_{z\max}}$$

Based on (41), the following statements can be made on the multiplier built from exponential basic elements:

a) The input offset of the multiplier is the resultant of the input offset value of the exponential circuits belonging to the inputs x and y on the one hand, and of the input offset value of the amplifiers belonging to the inputs x and y, on the other.

b) It can be seen that, with finite gains, a linearity error depending on x and y arises, respectively.

$$h_{\mathrm{LX}}(x, y_0) = \frac{\Phi_{\mathrm{REF}} \cdot \ln \left| x \cdot \dfrac{E_{x\max}}{C} \right|}{A \cdot E_{x\max}} \cdot y_0 \cdot 100\% \tag{42}$$

$$h_{\mathrm{LY}}(x_0, y) = \frac{\Phi_{\mathrm{REF}} \cdot \ln \left| y \cdot \dfrac{E_{y\max}}{C} \right|}{A \cdot E_{y\max}} \cdot x_0 \cdot 100\% \tag{43}$$

Eqs (42) and (43) deliver dimensioning data for the design of multipliers built from basic exponential elements. If the permissible linearity errors $h_{LX\max}$ and $h_{LY\max}$ depending on x and y, respectively, are given in advance, then it is possible to determine the required values of the gains A_x and A_y of the amplifiers:

$$A_x \geq \frac{\Phi_{REF} \cdot \ln \frac{E_{x\max}}{C}}{E_{x\max}} \cdot \frac{100\%}{h_{LX\max}} \tag{44}$$

$$A_y \geq \frac{\Phi_{REF} \cdot \ln \frac{E_{y\max}}{C}}{E_{y\max}} \cdot \frac{100\%}{h_{LY\max}} \tag{45}$$

where Φ_{REF} is the reference voltage of the exponential circuit and C is the transfer factor of the exponential circuit.

c) Based on a comparison of (41) and (10), the transfer factor of the logarithmic multiplier built from exponential elements is

$$K_R = \frac{E_{x\max} \cdot E_{y\max}}{E_{z\max} \cdot C} \cdot \exp \frac{\Phi_{os\,1x} + \Phi_{os\,1y} + E_{oss} - \Phi_{os\,1z}}{\Phi_{REF}} \tag{46}$$

If we presume that, in the basic state the resultant of the offset voltages can be adjusted to zero, i.e.,

$$\Phi_{os\,1x} + \Phi_{os\,1y} + E_{oss} - \Phi_{os\,1z} = 0 \tag{47}$$

then the expected value of the relative change of K_R upon the effect of small changes of the particular parameters will be

$$\frac{\Delta K_R}{K_R} = \sqrt{\left(\frac{\Delta C}{C}\right)^2 + \left(\frac{\Delta(\Phi_{os\,1x} + \Phi_{os\,1y} - \Phi_{os\,1z} + E_{oss})}{\Phi_{REF}}\right)^2} \tag{48}$$

The formal identity and the similarity of contents in Eqs (48) and (29) allow conclusion to the two dimensioning relationships

$$|\alpha_C| \leq \frac{\sqrt{2}}{2} \cdot \alpha_{KR\max} \tag{49}$$

$$|\alpha_{oss} + \alpha_{osEXP}| \leq \frac{\sqrt{2}}{2} \cdot \Phi_{REF} \cdot \alpha_{KR\max} \tag{50}$$

where α_C = temperature coefficient of the transfer factor of the exponential circuit [%/°C];

$\alpha_{KR\,max}$ = permissible temperature coefficient of the result transfer factor of the multiplier [%/°C];

α_{oss} = temperature coefficient of the offset voltage, in other words: drift of the summator circuit [V/°C];

$\alpha_{os\,EXP}$ = drift of the exponential circuit [V/°C].

d) The linearity error of the circuit can be eliminated by adjustment of the weight factors. Because of the previous assumptions, relationships (41) expresses the conditions present after elimination of such errors.

4. Open-loop gain and dynamic properties of the logarithmic multiplier built from logarithmic elements

The multiplier built from purely logarithmic elements contains a closed control loop (Fig. 3). This loop is nonlinear, but using the principle of working point linearization, the open-loop gain can be interpreted in each working point:

$$H = A \cdot \frac{\partial E_{out\,z}}{\partial E_z} \tag{51}$$

After substitution:

$$H(E_z) = A \cdot \frac{K}{E_z} \tag{52}$$

With the constraints (34) made on the gain, and with regard to (2):

$$H(z_0) = b \cdot \frac{1}{z_0} \cdot \frac{100\%}{h_{max}} \tag{53}$$

where $b > 1$ is a safety factor.

Thus the open-loop gain is a hyperbolic function of the working point value of the output signal of the multiplier, and the minimum value of the open-loop gain required to keep the product error within a previously fixed value is a quantity independent of the parameters of the circuit elements.

Knowing the open-loop gain, one can perform the dynamic examination of the linearized model of the multiplier. It must be taken into consideration that the nonlinear feedback is connected to an amplifier having frequency-dependent signal transfer. Suppose that the amplifier applied can be considered to be a proportional element with three simple lags, and the logarithmic element to be a proportional element with level-dependent transfer factor and without time lag (Fig. 5).

With the above assumptions the loop gain will be level-dependent and also frequency-dependent.

$$H(z_0, \omega) = b \cdot \frac{100\%}{h_{max}} \cdot \frac{1}{z_0} \cdot \frac{1}{(1 + j\omega T_1) \cdot (1 + j\omega T_2) \cdot (1 + j\omega T_3)} \tag{54}$$

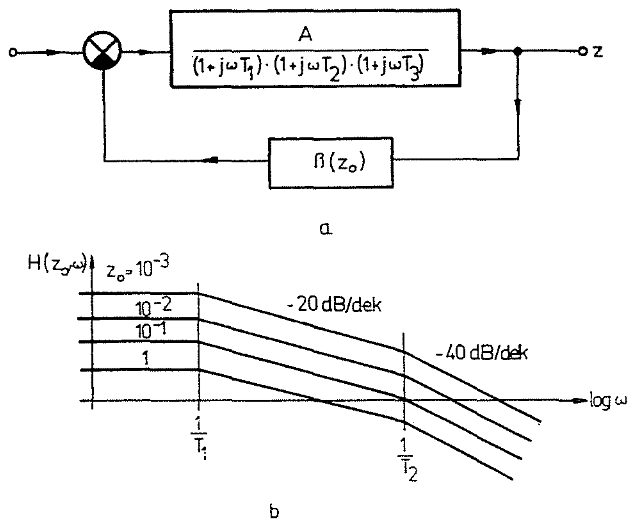

Fig. 5

The circuit will be stable at the values z_0, with which the crossover frequency is lower than the reciprocal of the second time-constant. In order to obtain a stable circuit for all the values of

$$\varepsilon \leq z_0 \leq 1$$

with $\varepsilon > 0$ given in advance, dynamic compensation must be provided for the control loop. A compensation is required which ensures that the frequency function of the open-loop gain decreases to the 0 dB-level with a slope of 20 dB/decade even in the case of $z_0 = \varepsilon$.

Denoting the new time constants of the frequency function of the compensated circuit by T_1' and T_2' ($T_1' > T_2'$), the above requirement can be written as

$$\frac{T_1'}{T_2'} \geq H(z_0 = \varepsilon, \quad \omega = 0) \tag{55}$$

However, as it is known from control engineering, T_2' cannot be arbitrarily small, since

$$T_2' > T_3 \tag{56}$$

Thus, with consideration to (54), the greatest time-constant in the forward branch of the system compensated stable will be:

$$T_1' \geq b \cdot \frac{100\%}{h_{\max}} \cdot \frac{1}{\varepsilon} \cdot T_3 \tag{57}$$

Determine now the resulting transfer function of the stabilized system. Since

$$T_1' \gg T_2' \quad \text{and} \quad T_1' \gg T_3 \tag{58}$$

the forward branch will be modelled by a single-lag proportional element with time-constant T_1' (Fig. 6).

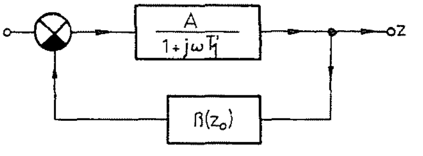

Fig. 6

The resultant frequency response of the system calculated by means of this model will be

$$W_R = \frac{A_R}{1 + j\omega T_R} = \frac{\dfrac{A}{1 + A \cdot \beta(z_0)}}{1 + j\omega \dfrac{T_1'}{1 + A \cdot \beta(z_0)}} \tag{59}$$

where A_R is the resulting transfer factor and T_R is the resulting time-constant, both dependent on the working point, in the case of small changes around the working point.

As can be seen,

$$T_R = \frac{T_1'}{H(z_0 = z_0, \quad \omega = 0)} \tag{60}$$

and, with the prescribed dynamic compensation, the settling time of 1 per cent accuracy in the cases of small signal changes around a given working point, using (54) and (57), will be

$$T(1\%) = 4{,}606 \cdot \frac{z_0}{\varepsilon} \cdot T_3 \tag{61}$$

5. Open-loop gain and dynamic properties of the logarithmic multiplier built from exponential elements

The input signals x and y of this type of multiplier are handled by nonlinear control circuits (Fig. 4).

In the circuit belonging to the input x, the open-loop gain will be similar

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to the previous ones.

$$H_x = A_x \cdot \frac{\partial \Phi_x}{\partial \Phi_{\ln x}} \tag{62}$$

After substitution:

$$H(\Phi_x) = A_x \cdot \frac{\Phi_x}{\Phi_{\mathrm{REF}}} \tag{63}$$

With the constraint (44) made on the gain, and with regard to (2):

$$H(x_0) = b_x \cdot x_0 \cdot \ln \left(\frac{E_{x\,\max}}{C} \right) \cdot \frac{100\%}{h_{\mathrm{LX\,max}}} \tag{64}$$

Similarly

$$H(y_0) = b_y \cdot y_0 \cdot \ln \left(\frac{E_{y\,\max}}{C} \right) \cdot \frac{100\%}{h_{\mathrm{LY\,max}}} \tag{65}$$

where $b_x > 1$ and $b_y > 1$ are safety factors.

In the closed control loops of the exponential multiplier the open-loop gains are linear functions of the working point values of the corresponding input signals. The minimum value of open-loop gain required for keeping the linearity error within a given limit depends also on the quotient of the machine unit chosen and of the transfer factors of the exponential circuits being applied.

The value of open-loop gain is maximum in the case of $x_0 = 1$.

$$H_{x\,\max} = b_x \cdot \ln \left(\frac{E_{x\,\max}}{C} \right) \cdot \frac{100\%}{h_{\mathrm{LX\,max}}} \tag{66}$$

Let again an amplifier model with three lags be used and the nonlinear element be frequency independent during the dynamic examination.

The circuit is stable with certainty if after compensation

$$\frac{T_1'}{T_2'} \geq H_{x\,\max} \tag{67}$$

Condition (56) is valid again, and thus:

$$T_1' \geq b_x \cdot \ln \left(\frac{E_{x\,\max}}{C} \right) \cdot \frac{100\%}{h_{\mathrm{LX\,max}}} \cdot T_3 \tag{68}$$

Calculating according to Fig. 6, the resulting time constant will be:

$$T_R \geq \frac{1}{x_0} \cdot T_3 \tag{69}$$

The settling time of 1% accuracy around a working point in the case of small signal changes:

$$T(1\%) = 4{,}606 \cdot \frac{1}{x_0} \cdot T_3$$

Similar results can be obtained from (65) for the control circuit belonging to the input y.

Summary

The behaviour of logarithmic multipliers may be analyzed starting from the static characteristics of the component circuits. The static accuracy of such multipliers is influenced by input and output offset further the stability of the transfer factor of the elements chosen, and also there are errors due to the finite gain of the op amps applied.

After clarifying the relations between the characteristics of the elements and the resultant parameters of the multiplier, some dimensioning relationships may be established. The analysis of the stability of closed control loops applied in the multipliers leads to further dynamical dimensioning relationships.

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