FAST METHODS OF FREQUENCY ANALYSIS APPLIED IN PROCESS IDENTIFICATION AND CONTROL

By

A. Frigyes and L. Langer

Department of Process Control, Technical University, Budapest

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1. Introduction

Computers applied to the control of plants generally influence the control loops by set-point control, but direct digital control (DDC) is also increasingly used. In process control it is common practice to model the process by a linear system and to use simple digital algorithms, particularly the discrete equivalent of the proportional — plus — integral — plus — derivative (PID) controller.

There is a need for the on-line tuning of the controllers and DDC parameters to set new control loops or to improve the performance of the working control loops. The modification of control is necessary to correct the effect of parameters changing slowly due to nonlinearities and setpoint or load changes.

The control parameters can be set by the process operator or by automatic tuning procedures. A program package is necessary in both cases to perform fast analysis of the control loop, to identify the parameters of an apropriately simple model, and to determine the controller setting. A scheme for computer identification and control is demonstrated in Fig. 1.

The computer determines the input signal for the plant according to the DDC algorithm defined by the discrete transfer function $D(z) = u(z)/\varepsilon(z)$. The discrete signal $u(k)$ is stored by the zero-order hold (Z. O. H.) for the sampling period T_0. The computer measures the output signal of the plant, which is $y(k)$ at $t = kT_0$. In the knowledge of the input-output sequence the computer can identify the dynamical characteristics of the plant.

The identificating procedure can superpose an appropriate test signal upon the input signal, and the transient function of the loop can be determined in this way. In the following we assume that the step response of the plant is available.

In this paper we present an identification of two-stage type consisting of

a) determination of a nonparametric model by Fourier analysis,

b) model-fitting to obtain a simple parametric model containing equivalent time constants and time delay.

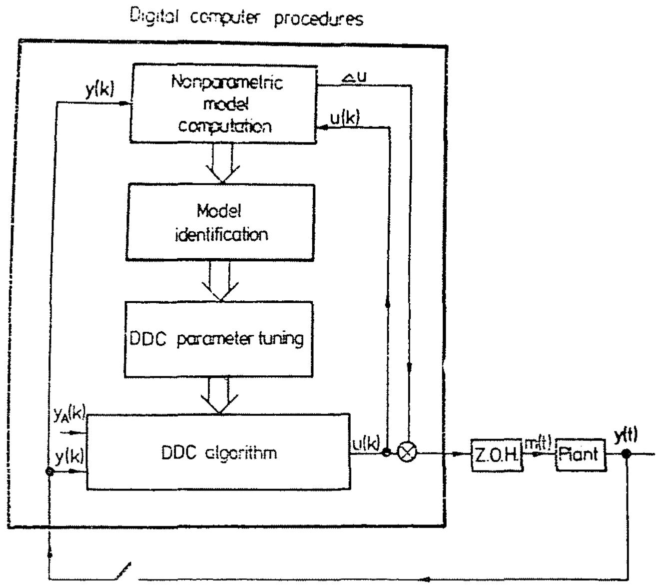

Fig. 1

A deterministic input signal can be applied to the input of most plants without considerably disturbing the working conditions. It is possible to use the frequency response analysis effectively if the noise-to-signal ratio can be reduced to a low level by proper instrumentation or by filtering procedures.

In the following chapters methods for the numerical evaluation of the frequency response will be summarized (sec. 2) and a new algorithm presented capable of determining particularly the middle frequency range (sec. 3). Then methods for the derivation of simple models will be surveyed and algorithms for fitting a second order model containing dead-time will be developed (sec. 4). Identification and control based on this method will be demonstrated by an example (sec. 5).

2. Methods for the evaluation of the frequency response from the transient response

The frequency response can be determined from the weight function $w(t)$ of the linear control loop as

$$W(j\omega) = \int_0^\infty w(t)\, e^{-j\omega t}\, dt \tag{1}$$

The problem is to compute the Fourier transform of the function $w(t)$, but only the transient function

$$v(t) = \int_0^t w(t)\,dt \tag{2}$$

is available from measurements. It is represented by its samples

$$v_k = v(kT_0) \quad \text{where} \quad k = 0, 1, 2 \ldots N-1 \tag{3}$$

The Fourier transform can be interpreted as the Laplace transform

$$W(s) = \int_0^\infty w(t)\,e^{-st}\,dt \tag{4}$$

with the substitution of $s = j\omega$.

2.1. *The finite Fourier transform*

For the numerical evaluation of (4) the approximating formula

$$W(s) \approx W_2(s) = \int_0^{t_N} w(t)\,e^{-st}\,dt + \frac{w(t_N)}{s}\,e^{-st_N} + \frac{w'(t_N)}{s^2}\,e^{-st_N} \tag{5}$$

can be used which is an approximation of second order in terms of $v(t)$. The above formula is equivalent to Eq. (6)

$$W_2(s) = \frac{1}{s^2}\int_0^{t_N} w''(t)\,e^{-st}\,dt \tag{6}$$

This can be proved by partial integrations of (6) which will yield formula (7), (8) and (9), respectively

$$W_2(s) = \frac{1}{s}\int_0^{t_N} w'(t)\,e^{-st}\,dt + \frac{w'(t_N)}{s^2}\,e^{-st_N} \tag{7}$$

$$= \int_0^{t_N} w(t)\,e^{-st}\,dt + \frac{w(t_N)}{s}\,e^{-st_N} + \frac{w'(t_N)}{s^2}\,e^{-st_N} \tag{8}$$

$$= s\int_0^{t_N} v(t)\,e^{-st}\,dt + v(t_N) + \frac{w(t_N)}{s}\,e^{-st_N} + \frac{w'(t_N)}{s^2}\,e^{-st_N} \tag{9}$$

In order to have better convergence in the above equations, let us split off the assymptotes from the integrand as

$$w_0'(t) = w'(t) - w'(t_N)$$
$$w_0(t) = w(t) - w(t_N) - w'(t_N)(t - t_N)$$
$$v_0(t) = v(t) - v(t_N) - w(t_N)(t - t_N) - w'(t_N)(t - t_N)^2/2$$

to obtain the formulas (10), (11) and (12):

$$W_2(s) = \frac{1}{s}\int_0^{t_N} w_0'(t)\, e^{-st}\, dt + \frac{w'(t_N)}{s^2} \tag{10}$$

$$= \int_0^{t_N} w_0(t)\, e^{-st}\, dt + \frac{w(t_N)}{s} + \frac{w'(t_N)}{s^2} \tag{11}$$

$$= s\int_0^{t_N} v_0(t)\, e^{-st}\, dt + v(t_N) + \frac{w(t_N)}{s} + \frac{w'(t_N)}{s^2} \tag{12}$$

2.2. *Application of the Fast Fourier Transform*

Further on, substitute $s = j2\pi f$ and examine the numerical approximation to the finite Fourier integral in formula (12) of the following type:

$$X(f) = \int_0^{t_N} x(t)\, e^{-j2\pi ft}\, dt \tag{13}$$

where the function $x(t)$ is sampled in N points at intervals of length Δt, i.e. $t_N = N\Delta t$ (The notation is the one used in literature). It is assumed that

$$x(t) = 0 \text{ if } t > t_N \tag{14}$$

The simplest approximation of (13) is

$$X_p(f) = \sum_{k=0}^{N-1} x(k\Delta t)\, e^{-j2\pi fk\Delta t}\, \Delta t \tag{15}$$

which means the approximation of the integral by the rectangular rule applied to the integrand $x(t)e^{-j2\pi ft}$. If $x_0 = x_N = 0$ then this approximation is equivalent to that obtained by the trapezoidal rule.

It is well-known that the Fourier transform of the sampled function is a periodic function

$$X_p(f) = \sum_{k=-\infty}^{\infty} \mathbf{X}(f + kF) \tag{16}$$

with the period

$$F = 1/\Delta t \ . \tag{17}$$

Thus, Δt is chosen so that the distortion, the so-called aliasing, involved in constructing $X(f)$ from $X_p(f)$ is negligible, i.e.

$$X(f) \approx X_p(f) \quad \text{for} \quad |f| \leq \frac{1}{2} F . \tag{18}$$

If we consider the values of $X_p(f)$ in N equally spaced points in the domain $[0,F]$ with

$$\Delta f = F/N = 1/N\Delta t \tag{19}$$

we obtain

$$X_p(r\Delta f) = \sum_{k=0}^{N-1} x(k\Delta t)\, e^{-j2\pi kr/N}\, \Delta t \tag{20}$$

If the assumption (14) is not valid, then the Fourier transform of the sequence x_k is

$$X_p(f) = \sum_{k=0}^{\infty} x(k\Delta t)\, e^{-j2\pi fk\Delta t}\, \Delta t \tag{21}$$

It can be proved that in this case

$$X_p(r\Delta f) = \sum_{k=1}^{N-1} x_p(k\Delta t)\, e^{-j2\pi kr/N}\, \Delta t \tag{22}$$

where

$$x_p(t) = \sum_{l=-\infty}^{\infty} x(t + lT) \tag{23}$$

is periodic with $T = N\Delta t$.

It is to be noted that the limitation in the summing causes an aliasing in the time domain. Formula (22) is called the discrete Fourier transform (DFT) of the sequence $x_p(k\Delta t)$, $k = 0, 1, \ldots N-1$.

The procedure to be followed at the application of the DFT to compute the Fourier transform can be summarized as:

a) Δt must be chosen to make the frequency $F = 1/\Delta t$ large enough to encompass the region where $X(f)$ is significantly different from zero.

b) N is chosen to get the frequency resolution $\Delta f = F/N$ required.

c) The aliased sequence $x_p(k\Delta t)$ must be formed and it is generally sufficient to consider only the term with $l = -1$. If N is chosen to avoid the aliasing of $x(k\Delta t)$ it will generally lead to greater frequency resolution than required, i.e. to more computation work.

The Fast Fourier Transform (FFT) is a method for efficiently computing the discrete Fourier transform (DFT);

$$X_r = \sum_{k=0}^{N-1} x_k W^{-kr} \quad r = 0, 1 \dots N-1 \tag{24}$$

where

$$W = e^{j2\pi/N}$$

Specifically, if the time series consists of $N = 2^n$ samples then $2Nn$ real multiplications will be necessary to evaluate all N coefficients. In the case of a large N, the above number is very small compared with the number N^2 of the operations required for the straightforward calculation of the DFT coefficients. This fact made possible the real-time applications of transform methods.

2.3. *Approximations of different order*

In practice the sampling rate of the computer is constrained, so the sampling period can not be chosen small enough to avoid the aliasing of the frequency response. Therefore, it is necessary to use some kind of interpolation for the sampled function.

The integral of the form

$$I = \int_0^{t_N} v^{(n)}(t)\, e^{-st}\, \mathrm{d}t \tag{25}$$

will be partitioned and the function $v^{(n)}(t)$ will be approximated by the constant $v_k^{(n)}$ in the k-th interval, i.e. $v(t)$ is composed from polynomials of n-th order. So we obtain

$$I = \sum_{k=0}^{N-1} v_k^{(n)} \int_{t_k}^{t_{k+1}} e^{-st}\, \mathrm{d}t + \sum_{i=0}^{n-1} v_0^{(n-1-i)}\, s^i \tag{26}$$

The correction is necessary if the function $v(t)$ and its derivatives have jump at $t = 0$. Let the partition be equidistant with the interval length MT_0, i.e.

$$t_k = kMT_0$$

where T_0 is the sampling period.

So the transfer function can be determined by the formula

$$W_n(s) = \frac{1}{s^n} s \frac{1 - e^{-sMT_0}}{s} \sum_{k=0}^{N-1} v_k^{(n)} e^{-skMT_0} + \\ + \sum_{i=0}^{n-1} v_0^{(n-1-i)} s^{i-n+1} + v_N^{(n)} \frac{e^{-stN}}{s^n} \tag{27}$$

where n denotes the order of the approximation in terms of $v(t)$.

a) Zero-order approximation:

with $n = 0$, and $M = 1$ we get

$$W_0(s) = \frac{1 - e^{-sT_0}}{s} s \sum_{k=0}^{N-1} v_k e^{-skT_0} + v_N e^{-sNT_0} \tag{28}$$

This formula contains the discrete Fourier transform of the sequence v_k multiplied by the transfer function of the zero-order hold.

By splitting the assimptote v_n we obtain

$$W_0(s) = (1 - e^{-sT_0}) \sum_{k=0}^{N-1} (v_k - v_N) e^{-skT} + v_N \tag{29}$$

Applying the following equality

$$e^{-sT_0} \sum_{k=0}^{N-1} v_k e^{-skT_0} = \sum_{k=1}^{N} v_{k-1} e^{-skT_0} \tag{30}$$

Eq. (28) can be put into the form

$$W_0(s) = v_0 + \sum_{k=1}^{N} (v_k - v_{k-1}) e^{-skT_0} \tag{31}$$

$$= v_0 + \sum_{k=0}^{N-1} (v_{k+1} - v_k) e^{-skT_0} \cdot e^{-sT_0} \tag{32}$$

which is the DFT of the first difference of the sequence, i.e.

$$W_0 = \mathrm{DFT}(\nabla v_k), \; k = 0, 1, 2, \ldots N \tag{33}$$

where

$$\nabla v_0 = v_0, \quad \nabla v_k = v_k - v_{k-1}, \quad k = 1, 2, \ldots N \tag{34}$$

b) First-order approximation:
with $n = 1$ and $M = 1$ the following formula can be derived

$$W_1 = v_0 + \frac{1 - e^{-sT_0}}{s} \sum_{k=0}^{N-1} v_k' e^{-skT_0} + v_N' \frac{e^{-sNT_0}}{s} \tag{35}$$

The polygonal approximation to $v(t)$ means that

$$v_k' = \frac{v_{k+1} - v_k}{T_0} \tag{36}$$

Substituting this into formula (35), and using the procedure described at zero-order approximation, we obtain the formulas [3]

$$W_1 = v_0 + \frac{1 - e^{-sT_0}}{s} \sum_{k=0}^{N-1} \frac{v_{k+1} - v_k}{T_0} e^{-skT_0} + \frac{v_{N+1} - v_N}{T_0} \frac{e^{-sNT_0}}{s} \tag{37}$$

$$= v_0 + \frac{v_1 - v_0}{sT_0} + \frac{1}{sT_0} \sum_{k=1}^{N} (v_{k+1} - 2v_k + v_{k-1}) e^{-skT_0} \tag{38}$$

This can be expressed as the DFT of the second central difference of the sequence, i.e.

$$W_1 = v_0 + \frac{1}{sT_0} DFT(\nabla_c^2 v_k), \quad k = 0, 1, 2, \ldots N \tag{39}$$

where

$$\nabla_c^2 v_0 = v_1 - v_0, \quad \nabla_c^2 v_k = v_{k+1} - 2v_k + v_{k-1}, \quad k = 1, 2, \ldots N \tag{40}$$

Formula (37) can be written also in the form

$$W_1 = v_0 + \frac{e^{sT_0}}{T_0} \left(\frac{1 - e^{-sT_0}}{s} \right)^2 s \sum_{k=0}^{N-1} v_k e^{-skT_0} \tag{41}$$

which contains the DFT of the sequence v_k multiplied by the transfer function of the polygonal hold:

$$\frac{e^{sT_0}}{T^0} \left(\frac{1 - e^{-sT_0}}{s} \right)^2 \tag{42}$$

Substituting $s = j\omega$, we obtain

$$W_1 = v_0 + \left(\frac{\sin \omega T_0/2}{\omega T^0/2} \right)^2 j\omega \sum_{k=0}^{N-1} v_k e^{-jk\omega T_0} \tag{43}$$

which is the formula published in [5].

The approximation of second order can be derived from (27) with $n = 2$ and $M = 2$, so the formula proposed by Göldner [4] can be obtained.

As it was shown above, the frequency response of the plant can be determined by means of the discrete Fourier transform, but it generally needs some preprocessing of the sequence, and postprocessing of the DFT coefficients. Using the computational advantage of the FFT algorithm the Fourier transform can be computed in real-time.

3. Derivation of new Transform Methods

It is possible to get new formulas for the real and imaginary parts of the Fourier transform by a polynominal approximation of the sine and cosine function instead of the function $v(t)$.

The transfer function can be approximated by

$$W_n(s) = \frac{1}{s^n} \int_0^{t_N} v^{(n+1)}(t)\, f(t) \mathrm{d}t \tag{44}$$

where

$$f(t) = e^{-st} \tag{45}$$

It is an approximation of n-th order in terms of $v(t)$.

After partial integration m times we obtain the general formula

$$W_{n,m}(s) = \frac{1}{s^n} I_{n,m} + \frac{1}{s^n} \sum_{i=0}^{m-1} (-1)^i v^{(n-i)}(t_N)\, f^{(i)}(t_N) \tag{46}$$

where

$$I_{n,m} = (-1)^m \int_0^{t_N} v^{(n-m+1)}(t)\, f^{(m)}(t)\, \mathrm{d}t \tag{47}$$

The second subscript of W denotes the number of partial integrations, and the order of the derivate of $f(t)$ in Eq. (47). The expressions belonging to the same n are theoretically equivalent, but they differ from the view-point of numerical computation according to m.

The formulas (6), (7), (8) and (9) can be derived from (46) with $f(t) = e^{-st}$, $n = 2$ and $m = 0, 1, 2, 3$ respectively.

With the combination of n and m new formulas can be obtained.

Let either of the components of the function

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

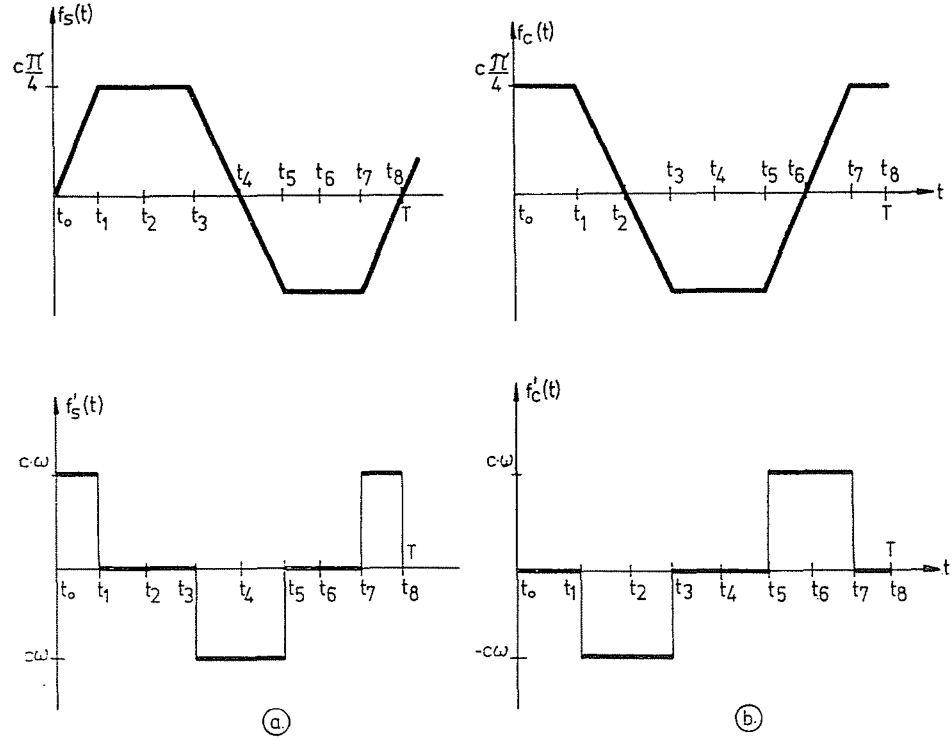

Fig. 2

be denoted by $f(t)$ in the following. The first-order approximation to $f(t)$ mean, that it is composed of straight lines, i.e. $f'(t)$ has a constant value $f'(t_k)$ in the interval $[t_k,t_{k+1}]$.

Thus, the following equation can be obtained from Eq. (46)

$$W_{n,1}(s) = -\frac{1}{s^n}\sum_{k=0}^{N-1} f'(t_k)\int_{t_k}^{t_{k+1}} v^{(n)}(t)\,\mathrm{d}t + v^{(n)}(t_N)\frac{f(t_N)}{s^n} \tag{48}$$

Substituting $n = 1$ into Eq. (48) we get

$$W_{1,1}(s) = -\frac{1}{s}\sum_{k=0}^{N-1} f'(t_k)\left[v(t_{k+1}) - v(t_k)\right] + v'(t_N)\frac{f(t_N)}{s} \tag{49}$$

which is a simple formula since the samples of the transient function $v(t)$ are used.

The sine and cosine function can be approximated appropriately by a trapezoid [1].

The period T belonging to the actual frequency ω is divided into M equidistant intervals according to Fig. 2, where $M = 8$. Thus both the sin ωt and cos ωt functions are approximated by straight lines.

The approximation of sine function is defined as

$$f_s(t) = \begin{cases} c\omega t & \text{if } 0 \leq t \leq t_1 \\ c\pi/4 & \text{if } t_1 \leq t \leq t_3 \\ c\pi/4 - c\omega(t - t_3) & \text{if } t_3 \leq t \leq t_5 \\ -c\pi/4 & \text{if } t_5 \leq t \leq t_7 \\ -c\pi/4 + c\omega(t - t_7) & \text{if } t_7 \leq t \leq t_8 \end{cases} \tag{50}$$

where

$$t_1 = T/M = \pi/(4\omega) \tag{51}$$

$$t_k = kt_1 \tag{52}$$

The cosine function can be expressed as

$$f_c(t) = f_s(t + T/4) \tag{53}$$

The slope of the sides is $c\ \omega$ where

$$c = 1.11$$

which is a correcting factor to make the amplitude of the fundamental harmonic of the trapezoid function equal to unity, but c was modified to the value

$$c = 1.13 \tag{54}$$

on the basis of test runs.

We shall compute the frequency response at frequencies

$$\omega_r = 2\pi/T_r \tag{55}$$

where the actual period time is selected to be an integer multiple of the minimal period time defined as

$$T_1 = MT_0 \tag{56}$$

So the following relations are obtained

$$T_r = rT_1 = rMT_0 \tag{57}$$

$$\omega_r = 2\pi/T_r = 1/r \cdot 2\pi/T_1 \tag{58}$$

This means that the equidistant intervals of $f(t)$ are increasing continuously with decreasing frequencies.

We shall approximate the integral (44) by taking the integer multiple of the actual period time, i.e.

$$t_N = KT = K\,2\pi/\omega \tag{59}$$

From Eq. (49) we obtain

$$W_{1,1}(s) = -\frac{1}{s}\sum_{r=0}^{K-1}\sum_{k=rM}^{(r+1)M-1} f'(t_k)\left[v(t_{k+1}) - v(t_k)\right] + v'(t_N)\frac{f(t_N)}{s} \tag{60}$$

with the limitation

$$t_{k+1} \leq NT_0$$

since the summation can not be extended over the measured interval.

Substituting $K = 1$, $M = 8$ into Eq. (60) we obtain for the approximation by the given trapezoids:

$$R_{1,1,1}(\omega) = c\left[v(t_1) + v(t_3) - v(t_5) - v(t_7) + v(t_8)\right] \tag{61}$$

$$Q_{1,1,1}(\omega) = c\left[v(t_1) - v(t_3) - v(t_5) + v(t_7) - v'(t_8)\frac{\pi}{4\omega}\right] \tag{62}$$

where R denotes the real part and Q the imaginary part of the frequency response function.

The first subscript gives the order of the approximation in terms of $v(t)$, the second subscript defines the order of the approximation of $f(t)$, and the third subscript refers to the upper limit of the integral.

The formulas for period K are

$$R_{1,1,K}(\omega) = c\sum_{r=0}^{K-1}\left[v(8r+1) + v(8r+3) - v(8r+5) - v(8r+7)\right] + cv(8K) \tag{63}$$

$$Q_{1,1\,K}(\omega) = c\sum_{r=0}^{K-1}\left[v(8r+1) - v(8r+3) - v(8r+5) + v(8r+7)\right] - cv'(8K)\frac{\pi}{4\omega} \tag{64}$$

The derivate of $v(t)$ at the last point can be approximated by the backward difference. With the second-order approximation of $v(t)$ a correction term to the real part of the frequency function can be determined, the imaginary part does not change:

$$R_{2,1,K}(\omega) = R_{1,1,K}(\omega) - v''(8K)\frac{1}{\omega^2} \tag{65}$$

$$Q_{2,1,K}(\omega) = Q_{1,1,K}(\omega) \tag{66}$$

The transformation based on these formulas is called the middle frequency transformation (MFT), since it has been demonstrated [1] that the approximation to the frequency response is appropriate in the decade around the frequency belonging to the phase shift of $-180°$. This domain is decisive for tuning a controller. Thus, the middle frequency portion of the frequency response computed by MFT can be used for fitting a model which is a basis of the controller setting.

The accuracy of the MFT is detailed in [2] with several test examples. Now, only the effect of the choice of k and of the correction term will be illustrated.

Let the plant be defined by the transfer function

$$W(s) = \frac{10.112}{(1 + 0.354s)^2\,(1 + 2.828s)^2} \tag{67}$$

The gain is set to the critical value, thus, on the exact Bode plot of the plant the amplitude curve a_p intersects the 0 dB axis where the phase curve φ_p has the value $-180°$ (see Fig. 3). The amplitude and phase curves computed by MFT (Eq. (63), (64)) with different values of K are indexed with K.

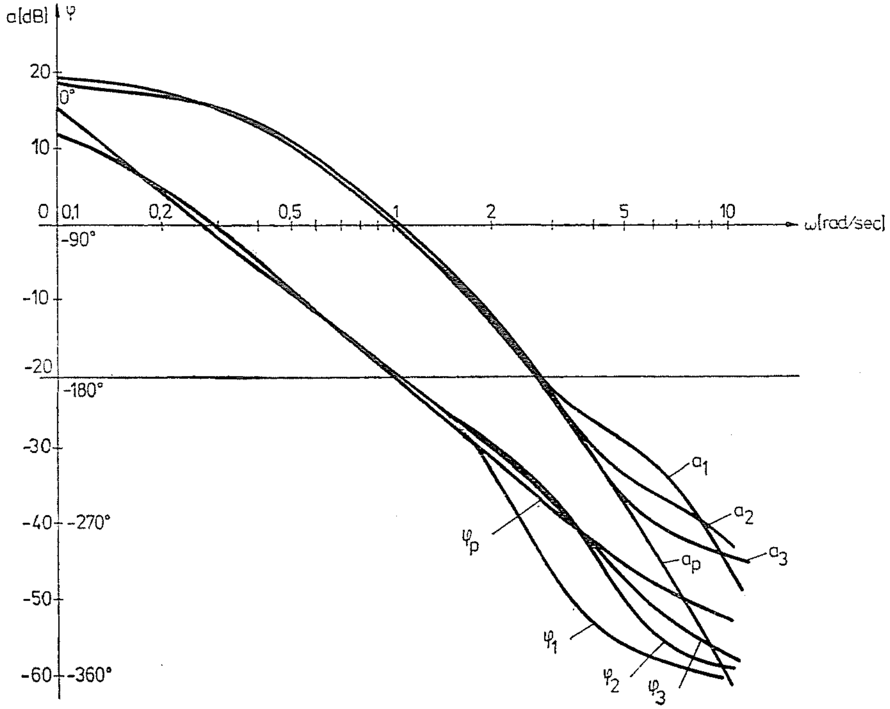

Fig. 3

It can be seen that the approximation with $K = 1$ is appropriate from $\varphi = -45°$ until $\varphi = -225°$, i.e. the error in phase is less than 5°, and in amplitude is less then 1dB in this domain. This accuracy is sufficient for model fitting from the view-point of closed-loop control. With increasing K the approximation improves in the range of high frequencies.

The effect of the correction term in Eq. (65) is illustrated by Fig. 4. The exact curves are indexed with p, the MFT without correction has the index 1 and the corrected MFT has the index 2, both are computed with $K = 1$. There is a considerable improvement of the approximation for higher frequencies, as the correction means the second-order extrapolation of $v(t)$, which is significant on the initial portion. The importance of the correction decreases with increasing time, (i.e. at low frequencies) or K.

It can be seen that in the middle frequency range the frequency response of the plant can be determined with sufficient accuracy by the appropriate formula of MFT. Formulas (61) and (62) are very simple, but they give good approximation in the decade, that is the most important for controller tuning.

The computational requirements of MFT are minimal, even compared with the FFT algorithm. The disadvantage of MFT is the great noise sensivity, because of the few points used for computation. That is why we have assumed

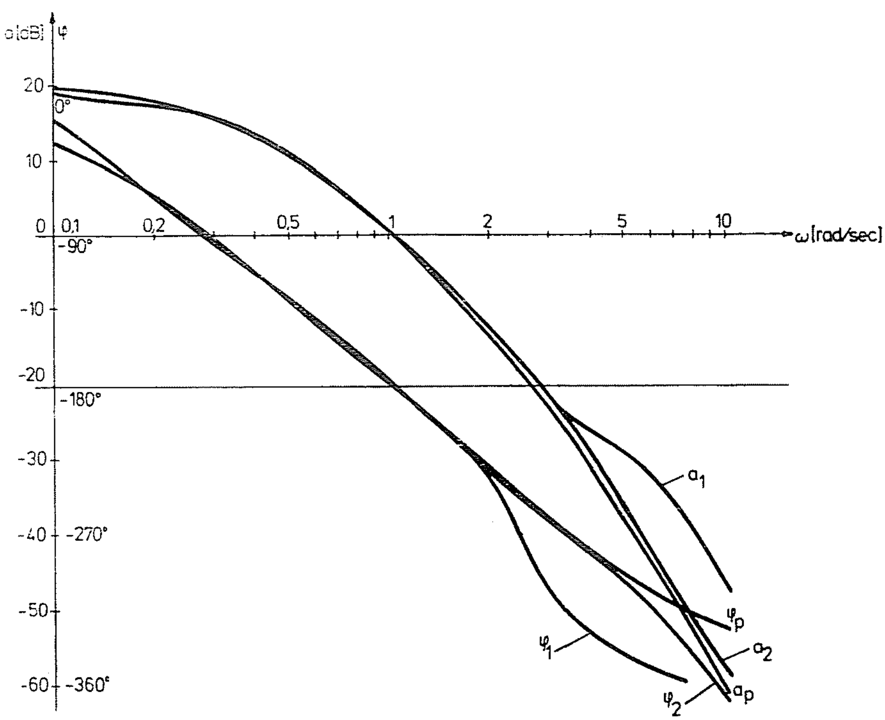

Fig. 4

that the noise-to-signal ratio is reduced by hardware and software means to a small level. The smoothing procedures are implicit components of the control packages.

Considerable measure time can be saved by the fact, that the MFT computes the Fourier integral with an upper limit which depends on the actual frequency. Thus, the knowledge of the stationary state is not necessary for the middle frequency analysis.

In the next part of this paper methods for fitting models to the frequency response will be described and the control application of MFT will be presented.

Summary

Methods for the numerical evaluation of the frequency response from the transient function are reviewed. The application of the Fast Fourier Transform algorithm to compute Fourier integral by various approximations to the transient function is described. New formulas can be derived by means of generalization of the approximating methods. The so-called middle-frequency transformation is presented, which enables the real-time frequency analysis of the control loop in the middle-frequency range.

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Prof. Dr. Andor FRIGYES
László LANGER } H-1521 Budapest