THE LOCAL SUPREMUM PRINCIPLE FOR OPTIMUM CONTROL PROBLEMS WITH NONSCALAR-VALUED PERFORMANCE CRITERION

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Symbols and abbreviations

$\in$	element of
$\notin$	non element of
$\Rightarrow$	implies
$\emptyset$	empty set
τ	topology
A°	interior of A
$A = \{a: \text{property of } a\}$	definition of the set A
A/B	$A\backslash B = \{a: a \in A;\ a \notin B\}$
$A \subset B$	$a \in A$ implies $a \in B$
$A \times B$	$A \times B = \{(a,b): a \in A;\ b \in B\}$
$A \cap B$	intersection of A and B
R^n	n-dimensional linear normed space
$F: E_1 \to E_2$	mapping from E_1 into E_2
$R(F)$	range of F
$F'(x)$	Fréchet derivative of F
$\mathfrak{L}(E_1 \to E_2)$	linear operators from E_1 into E_2
$\mathfrak{B}(E_1 \to E_2)$	bounded linear operators from E_1 into E_2
$<x,y>$	inner product
$\lVert x \rVert$	norm of x
$F \circ G$	composition of the mappings F and G such that $F \circ G(x) = F(G(x))$
$F(.,y)$	the mapping $F(x,y)$ for fixed y
$M_{m\times n}$	set of $m\times n$ matrices
$\lambda(A)$	Lebesgue measure of A
$C^{(n)}(t_0,t_1)$	the set of the continuous functions $F: [t_0,t_1] \to R^n$
$L_\infty^{(n)}(t_0,t_1)$	the set of the essentially bounded functions $F: [t_0,t_1] \to R^n$
$C^{(m\times n)}(t_0,t_1)$	the set of the continuous $m \times n$ matrices $F: [t_0,t_1] \to M_{m\times n}$
$L_\infty^{(m\times n)}(t_0,t_1)$	the set of the essentially bounded $m\times n$ matrices $F: [t_0,t_1] \to M_{m\times n}$

1. Introduction

In many optimization problems the quality of the process cannot be characterized by a single scalar-valued optimality criterion, because the user is simultaneously interested in several cost functionals. The scalar-valued cost functionals can be reduced to a single vector-valued performance criterion.

The linear state estimation problem also leads to an optimum problem with a nonscalar-valued performance criterion, if the covariance matrix of the error between the state and its estimation must be infimum. In this case the performance criterion is matrix-valued.

Dynamic optimization problems with nonscalar-valued performance criteria are studied in the present paper. The meaning of "better than" has to be defined, and this will be done by a partial-order relation. Partial ordering is defined by a positive cone. It is supposed that the performance criterion in the dynamic optimization problem has its range in a finite-dimensional partially ordered linear normed space (which is not necessarily an Euclidean space).

The necessary conditions of the local infimum are summarized in two theorems. These theorems establish a relation between the local maximum principle of DUBOVICKIJ, MILJUTIN and GIRSANOV [1], the infimum principle of Athans and Geering [2], and the author's results [3].

The proofs of the theorems in the appendix of the paper can be found in the author's dissertation.

2. Partial ordering

Partial ordering on a set is a reflexive, (antisymmetric) and transitive relation. If the set is a linear topological space, then it will be supposed that the partial ordering is given by a closed and convex cone having a nonempty interior.

Definition 1: Let (E, τ) be a linear topological space, and let $P \subset E$ be a closed and convex cone such that $P^\circ \neq \emptyset$. We say that $x \geq y$ if $x,y \in E$ and $x-y \in P$. A linear topological space with a relation $\geq$ defined in this way is said to be a partially ordered linear topological space. Notation: $(E, \tau, \geq)$. Since $x \in P \Leftrightarrow x \geq 0$, the cone P will be called the positive cone (defining the relation $\geq$). If $\pm z \in P \Rightarrow z = 0$, then $\geq$ is antisymmetric, i.e. $x \geq y$ and $y \geq x \Rightarrow x = y$.

Example 1: Let R^n be the usually n-dimensional Euclidean space. If $P = \{x = (x_1, \ldots, x_n) \in R^n\colon x_i \geq 0,\ i = 1, \ldots, n\}$, then P is a positive cone in R^n and so P defines a partial ordering in R^n. Notation: $(R^n, \geq)$. If $\| x \| = \| y \| = 1$ and $x, y \in P$, then $\| x + y \| \geq 1$.

Example 2: Let H be a Hilbert space. If $E = \{A \in \mathcal{B}(H \to H)\colon A$ is self-adjoint$\}$ and $P = \{A \in E\colon < Ax,x > \geq 0$ for all $x \in H\}$, then $E \subset \mathcal{B}(H \to H)$ is a closed subspace $\Rightarrow E$ is a Banach space and $P \subset E$ is a positive cone in E. Hence P defines a partial ordering in E. Notation: $(E, \geq)$. If $\| A \| = \| B \| = 1$ and $A,B \in P$, then $\| A + B \| \geq 1$.

Example 3: Notation is as in Example 2. Let $H = R^n$ (with fixed orthonormal basis.) Then $\mathcal{B}(R^n \to R^n)$ can be identified with the set of $n \times n$

matrices and similarly E with the set of symmetric $n \times n$ matrices. Then P is the set of positive semidefinite symmetric $n \times n$ matrices. The positive cone P defines a partial ordering in the Banach space of the symmetric $n \times n$ matrices. Notation: $(M^s_{n \times n}, \geq)$. $M^s_{n \times n}$ can be considered as a $\frac{n(n+1)}{2}$ — dimensional subspace of the linear normed space R^{n^2} (without inner product).

Remark: If $(E, \tau, \geq)$ is a partially ordered linear topological space, then $x \geq y$ and $z \in E \Rightarrow x + z \geq y + z$.

Definition 2: Let $(E, \tau, \geq)$ be a partially ordered linear topological space, $Q \subset E$ and $x_0 \in Q$. We say that

1) $x_0 = \max Q$, if there does not exist any $x \in Q$ such that $x \geq x_0$ and $x \neq x_0$,
2) $x_0 = \min Q$, if there does not exist any $x \in Q$ such that $x_0 \geq x$ and $x \neq x_0$,
3) $x_0 = \sup Q$, if $x_0 \geq x$ for all $x \in Q$,
4) $x_0 = \inf Q$, if $x \geq x_0$ for all $x \in Q$.

In general, $\max Q$ and $\min Q$ are not unique, because the partial ordering is usually not a linear ordering (Q may have elements which are not comparable). Sup Q and inf Q are always unique (supposed that they exist and the partial ordering is antisymmetric).

3. The local supremum principle

Condition (C): We say that (n,r,m,T,Φ) satisfies the condition (C), if

$$0 < T < \infty,$$

Φ: $R^n \times R^r \times [0,T] \to R^m$ is a mapping,

$$S \subset [0,T] \text{ and } \lambda([0,T] \setminus S) = 0,$$

furthermore

1) there exist $\Phi_y(y,v,t)$, $\Phi_v(y,v,t)$ for all $t \in S$, and $\{\Phi_y(\cdot,\cdot,t) : t \in S\}$, $\{\Phi_v(\cdot,\cdot,t) : t \in S\}$ are equicontinuous in (y,v) on all compact sets $F \times G \subset R^n \times R^r$;
2) $\Phi(y,v,\cdot)$, $\Phi_y(y,v,\cdot)$ and $\Phi_v(y,v,\cdot)$ are measurable functions in t for all fixed $(y,v) \in R^n \times R^r$;
3) for each fixed bounded set $F \times G \subset R^n \times R^r$, there is a real number k such that $\|\Phi(y,v,t)\| < k$, $\|\Phi_y(y,v,t)\| < k$ and $\|\Phi_v(y,v,t)\| < k$ for all $(y,v,t) \in F \times G \times S$.

Theorem 1: Suppose that (n,r,m,T,Φ) and (n,r,n,T,φ) satisfy the condition (C). Let M be a convex set in R^r and $M^0 \neq \emptyset$. Let c, $d \in R^n$. Let P_0 be a closed and convex cone in R^m such that $P_0^0 \neq \emptyset$ and $\pm z \in P_0 \Rightarrow z = 0$, and let the positive cone P_0 define a partial ordering $\geq$ in R^m (R^m is not necessarily an Euclidean space). Let the constraint Q be

$$Q = \{(x,u) \in C^{(n)}(0,T) \times L_\infty^{(r)}(0,T):$$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \varphi(x(t), u(t), t) \text{ for almost every } t \in [0,T];$$

$$x(0) = c \text{ and } x(T) = d;$$
$$u(t) \in M \text{ for almost every } t \in [0,T]\},$$

and let $(x_0,u_0) \in Q$. Suppose there is a neighbourhood V of (x_0,u_0) such that (x_0,u_0) is a solution of the problem

$$\inf \left\{ \int_0^T \Phi\big(x(t), u(t), t\big)\,\mathrm{d}t : (x, u) \in Q \cap V \right\}.$$

Then there exist a $n \times n$ constant matrix T_0 and $m \times n$ matrix functions $\psi(t)$ such that

i) $T_0 y \geq 0$ for all $y \geq 0$ i.e. $T_0(P_0) \subset P_0$;

ii) either $T_0 \neq 0$ or $\psi(t) \not\equiv 0$;

iii) $\dfrac{\mathrm{d}\psi(t)}{\mathrm{d}t} = -\psi(t)\,\varphi_y\big(x_0(t), u_0(t), t\big) + T_0\Phi\big(x_0(t), u_0(t), t\big)$ for almost every $t \in [0,T]$;

iv) $\big(-\psi(t)\varphi_v(x_0(t), u_0(t), t) + T_0\,\Phi_v\,(x_0(t), u_0(t), t)\big).\ \big(u - u_0(t)\big) \in P_0$ for all $u \in M$ and for almost every $t \in [0,T]$;

v) specially if the system

$$\frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} = \varphi_y\big(x_0(t), u_0(t), t\big)\,\bar{x}(t) + \varphi_v\big(x_0(t), u_0(t), t\big)\,\big(u(t) - u_0(t)\big)$$

for almost every $t \in [0,T]$, $\bar{x}(0) = \bar{x}(T) = 0$,
$u(t) \in M^0$ for almost every $t \in [0,T]$

has a solution $(x,u) \in C^{(n)}(0,T) \times L_\infty^{(r)}(0,T)$, then $T_0 = I$ is the $m \times m$ identity matrix. If in addition $\Phi(y,v,t)$ is a convex mapping and the dynamical system is linear, i.e. $\varphi(y,v,t) = A(t)y + B(t)v$, then the local infimum in (x_0,u_0) is also a global infimum on Q, furthermore, iii) and iv) are sufficient conditions of the global infimum.

Proof: We will use the following notations:

$$F_0\colon C^{(n)}(0,T) \times L_\infty^{(r)}(0,T) \to R^m,$$
$$F_0(x,u) = \int_0^T \Phi(x(t),u(t),t)\,\mathrm{d}t,$$
$$F_1\colon C^{(n)}(0,T) \times L_\infty^{(r)}(0,T) \to C^{(n)}(0,T),$$
$$F_1(x,u)(t) = x(t) - c - \int_0^t \varphi(x(\tau), u(\tau),\tau)\,\mathrm{d}\tau,$$

$F_2 : C^{(n)}(0,T)\times L_\infty^{(r)}(0,T) \to R^n$,
$F_2(x,u) = x(T) - d,$
$A = \{(x,u) \in C^{(n)}(0,T)\times L_\infty^{(r)}(0,T) : u(t) \in M$ for almost every $t \in [0,T]\}$.

Then

$$Q = \{(x,u) \in C^{(n)}(0,T)\times L_\infty^{(r)}(0,T) : F_1(x,u) = 0;\ F_2(x,u) = 0; \tag{1}$$

$(x,u) \in A\}$ and $(x_0,u_0) \in Q$ is a solution of the problem

$$\inf\{F_0(x,u) : (x,u) \in Q\cap V\}. \tag{2}$$

Since $A = C^{(n)}(0,T)\times B$ and M is a convex set such that $M^\circ \neq \emptyset$, it follows that $B \subset L_\infty^{(r)}(0,T)$ such that B is a convex set and $B^\circ \neq \emptyset$. Hence A is a convex set and $A^\circ \neq \emptyset$. Since (n,r,m,T,Φ) and (n,r,n,T,φ) satisfy the condition (C), it follows that F_0 and F_1 are continuously Fréchet differentiable and

$$F_0'(x_0,u_0)(\bar{x},\bar{u}) = \int_0^T \left[\Phi_y(x_0(t), u_0(t),t)\bar{x}(t) + \Phi_v(x_0(t), u_0(t),t)\,\bar{u}(t)\right] dt, \tag{3}$$

$$F_1'(x_0,u_0)(\bar{x},\bar{u})(t) = x(t) - \int_0^t [\varphi_y(x_0(\tau),u_0(\tau),\tau)\bar{x}(\tau) +$$

$$+ \varphi_v(x_0(\tau),u_0(\tau),\tau)\bar{u}(\tau)]d\tau, \tag{4}$$

furthermore, by Lemma 3A (in Appendix)

$$R\big(F_1'(x_0,u_0)\big) = C^{(n)}(0,T) \qquad (5$$

is satisfied. Evidently, F_2 is continuously Fréchet differentiable and

$$F_2'(x_0,u_0)\,(\bar{x},\bar{u}) = \bar{x}(T)\,, \tag{6}$$

$$R\big(F_2'(x_0,u_0)\big) = R^n\,. \tag{7}$$

By Lemma 4A and by (1)—(7), Theorem 1A can be applied. Hence there are bounded linear operators

$T_0 \in \mathcal{B}(R^m \to R^m)$,
$T_1 \in \mathcal{B}(C^{(n)}(0,T) \to R^m)$,
$T_2 \in \mathcal{B}(R^n \to R^m)$,
$\widetilde{T} \in \mathcal{B}(C^{(n)}(0,T)\times L_\infty^{(r)}(0,T) \to R^m)$

such that $T_0(P_0) \subset P_0$ and $T_i \neq 0$ for at least one $i \in \{0,1,2\}$ and

$$\widetilde{T} = T_0 \circ F_0(x_0,u_0) + T_1 \circ F_1'(x_0,u_0) + T_2 \circ F_2'(x_0,u_0)\,, \tag{8}$$

$$\widetilde{T}(\bar{x},\bar{u}) \geq \widetilde{T}(x_0,u_0) \text{ for all } (\bar{x},\bar{u}) \in A\,. \tag{9}$$

In a fixed basis the bounded linear operators T_0, T_1 and T_2 can be identified with a $m \times m$ constant matrix, with a $m \times n$ matrix function and with a $m \times n$ constant matrix, respectively. In the special case T_0 is the $m \times m$ identity matrix. Since $\tilde{T}(\bar{x},\bar{u}) = \tilde{T}(\bar{x},0) + \tilde{T}(0,\bar{u})$ and $\pm z \in P_0 \Rightarrow z = 0$, it follows from the form of the set A, that

$$\tilde{T}(x,o) = 0 \text{ for all } x \in C^{(n)}(0,T)\,; \tag{10}$$

$$\hat{T}(\bar{u}) = \tilde{T}(\bar{x},\bar{u}) = \tilde{T}(0,\bar{u})\,. \tag{11}$$

Let $\bar{u} \in L_\infty^{(r)}(0,T)$ be fixed and let $\bar{x} = \bar{x}(\bar{u})$ be the solution of the equation $F_1'(x_0,u_0)(\bar{x},\bar{u}) = 0$, i.e.

$$\begin{aligned} \frac{\mathrm{d}\bar{x}(t)}{\mathrm{d}t} &= \varphi_y\big(x_0(t), u_0(t), t\big)\,\bar{x}(t) + \varphi_v\big(x_0(t), u_0(t), t\big)\,\bar{u}(t) \\ &\text{for almost every } t \in [0,T]\,, \\ &\bar{x}(0) = 0\,, \end{aligned} \tag{12}$$

which has a unique solution by Lemma 3A. It follows from (8) that

$$\hat{T}(\bar{u}) = T_0 \int_0^T \big(\Phi_y|_t\,\bar{x}(t) + \Phi_v|_t\,\bar{u}(t)\big)\,\mathrm{d}t + T_2\bar{x}(T) \tag{13}$$

for all $(\bar{x}(\bar{u}), \bar{u})$. Let $\psi\colon [0,T] \to \mathfrak{B}(R^n \to R^m)$ be the solution of

$$\begin{aligned} \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} &= -\psi(t)\,\varphi_y\big(x_0(t), u_0(t), t\big) + T_0\Phi_y\big(x_0(t), u_0(t), t\big) \\ &\text{for almost every } t \in [0,T]\,, \\ &\psi(T) = -T_2\,, \end{aligned} \tag{14}$$

then in a fixed basis $\psi(t)$ can be identified with a $m \times n$ matrix function and the solution $\psi(t)$ is unique by Lemma 3A. If both T_0 and $\psi(\cdot)$ are zero, then T_2 is also zero by $\psi(T) = -T_2$. Hence $\hat{T}$ is zero by (13) and $T_1 \circ F_1'(x_0,u_0)$ is also zero by (11) and (8). Since $R\big(F_1'(x_0,u_0)\big) = C^{(n)}(0,T)$, it follows that T_1 is also zero. But this contradicts iii) in Theorem 1A. This contradiction proves that either $T_0 \neq 0$ or $\psi(t) \not\equiv 0$. By (14) and (12) and through integration by parts it follows

$$T_0 \int_0^T \Phi_y|_t\,\bar{x}(t)\,\mathrm{d}t = -T_2\bar{x}(T) - \int_0^T \psi(t)\,\varphi_v|_t\,\bar{u}(t)\,\mathrm{d}t\,. \tag{15}$$

Hence it follows from (13) and (15) that

$$\hat{T}(\overline{u}) = \int_0^T \left(-\psi(t)\, \varphi_v|_t + T_0 \Phi_v|_t\right) \overline{u}(t)\, \mathrm{d}t\,. \tag{16}$$

By (9), (11) and (16)

$$\hat{T}(\overline{u} - u_0) = \int_0^T \left(-\psi(t)\, \varphi_v|_t + T_0 \Phi_v|_t\right)\left(\overline{u}(t) - u_0(t)\right) \mathrm{d}t \in P_0 \tag{17}$$

is satisfied for all $\overline{u} \in B = \{\overline{u} \in L_\infty^{(r)}(0,T)\colon \overline{u}(t) \in M$ for almost every $t \in [0,T]\}$. Hence by Lemma $2A$

$$\left(-\psi(t)\, \varphi_v|_t + T_0 \Phi_v|_t\right)\left(u - u_0(t)\right) \in P_0 \tag{18}$$

for all $u \in M$ and for almost every $t \in [0,T]$.

In the special case v) of the theorem let now $\Phi(y,v,t)$ be a convex function and let the dynamical system be linear, i.e. $\varphi(y,v,t) = A(t)y + B(t)v$. Let $(x,u) \in Q$ and use the notations $\overline{x} = x - x_0$ and $\bar{u} = u - u_0$. Then

$$\dot{\overline{x}} = A(t)\,\overline{x} + B(t)\,\overline{u}\,, \tag{19}$$

$$\overline{x}(0) = \overline{x}(T) = 0$$

is satisfied, i.e. $\overline{x}$ has the form $\overline{x} = \overline{x}(\overline{u})$. If iii) and iv) in the theorem are satisfied, then (14) and (18) are also satisfied with $T_0 = I$ and $T_2 = -\psi(T)$. Since P_0 is a closed and convex cone and $\overline{x}$ has the form $\overline{x} = \overline{x}(\overline{u})$, it follows from (18), (17), (16), (15), (14), (13), (11) and $\overline{x}(T) = 0$, that

$$\hat{T}(u) = F_0'(x_0,u_0)\,(\overline{x},\overline{u}) = F_0'(x_0,u_0)\,(x - x_0, u - u_0) \in P_0\,, \tag{20}$$

thus ii′) in Theorem $1A$ is satisfied, which is the sufficient condition of the global infimum.

Remark: Define $H(x,u,\psi,t) = \psi\varphi(x,u,t) - T_0\Phi(x,u,t)$. Then iv) is equivalent to

$$-H_u\left(x_0(t),\, u_0(t),\, \psi(t), t\right)\left(u - u_0(t)\right) \in P_0 \tag{21}$$

for all $u \in M$ and for almost every $t \in [0,T]$. By Theorem $1A$ (13) is the necessary condition for the function $-H(x_0(t),\, u, \psi(t), t)$ to attain local infimum on the set M in the point $u = u_0(t)$. Thus, if (x_0,u_0) is a solution of the optimum control problem in Theorem 1, then the function $H(x_0(t),u,\psi(t),t)$ satisfies the necessary condition of the local supremum on the set M for almost every $t \in [0,T]$ in the point $u = u_0(t)$. Hence Theorem 1 will be called a local supremum principle.

If the performance criterion is given not by an integral but $F_0(x,u) = F(x(T))$, where $F(\cdot)$ is a differentiable function and the final state $x(T)$ is free $(x(T) \in R^n)$, then with $\psi(T) = -F'(x(T))$ and $\Phi \equiv 0$ (formally) in the proof, Theorem 1 remains still valid, furthermore, T_0 is the $m \times m$ identity matrix.

Theorem 2: Suppose that (n,r,n,T,φ) satisfies the condition (C). Let $c \in R^n$. Let P_0 be a closed and convex cone in R^m such that $P_0^0 \neq \emptyset$ and $\pm z \in P_0 \Rightarrow z = 0$, and let the positive cone P_0 define a partial ordering $\geq$ in R^m(R^m is not necessarily an Euclidean space). Let $F : R^n \to R^m$ be a differentiable mapping and let the constraint Q be

$$Q = \{(x, u) \in C^{(n)}(0, T) \times L_\infty^{(r)}(0, T) :$$

$$\frac{dx(t)}{dt} = \varphi(x(t), u(t), t) \text{ for almost every } t \in [0, T] ;$$

$$x(0) = c\} ,$$

and let $(x_0,u_0) \in Q$. Suppose there is a neighbourhood V of (x_0,u_0) such that (x_0,u_0) is a solution of the problem

$$\inf \{F(x(T)) : (x,u) \in Q \cap V\}.$$

Then there exists a $m \times n$ matrix function $\psi(t)$ such that

i) $\dfrac{d\psi(t)}{dt} = -\psi(t)\, \varphi_y(x_0(t), u_0(t), t)$ for almost every $t \in [0, T]$;

ii) $\psi(T) = -F'(x_0(T))$;

iii) $\psi(t)\varphi_v(x_0(t),u_0(t),t) = 0$ for almost every $t \in [0,T]$;

iv) specially if $m = n$ and $F'(x_0(T))$ has an inverse, than $\varphi_v(x_0(t),u_0(t),t) = 0$ for almost every $t \in [0,T]$.

If F is a convex function and the dynamic system is linear, i.e. $\varphi(y,v,t) = A(t)y + B(t)v$, then the local infimum in (x_0,u_0) is also a global infimum on Q, furthermore i)—iii) or iv) are the sufficient conditions of the global infimum.

Proof: By the remark after Theorem 1, i) and ii) are satisfied and

$$-\psi(t)\, \varphi_v(x_0(t),u_0(t),t)\,(u - u_0(t) \in P_0 \tag{22}$$

for all $u \in R^r$ and for almost every $t \in [0,T]$. On the contrary, suppose iii) is not valid. Then there exists $\hat{S} \subset [0,T]$ such that $\lambda(\hat{S}) > 0$ and

$$-\psi(t)\, \varphi_v(x_0(t),u_0(t),t) \neq 0 \tag{23}$$

for all $t \in \hat{S}$. Hence for all $t \in \hat{S}$ there exists $\tilde{u}(t) \in R^r$ such that

$$-\psi(t)\,\varphi_v(x_0(t),u_0(t),t)\,\tilde{u}(t) \neq 0 \text{ and } \tilde{u}(t) \neq 0\,. \tag{24}$$

Let $u_1(t) = \tilde{u}(t) + u_0(t)$ and $u_2(t) = -\tilde{u}(t)+u_0(t)$. By (22) there is a $\tilde{S} \subset \hat{S}$ such that $\lambda(\tilde{S}) > 0$ and

$$\pm\psi(t)\,\varphi_v\big(x_0(t),u_0(t),t\big)\;\tilde{u}(t) \in P_0 \tag{25}$$

for all $t \in \tilde{S}$. Since $\pm z \in P_0 \Rightarrow z = 0$, hence

$$\psi(t)\;\varphi_v\big(x_0(t),u_0(t),t\big)\,\tilde{u}(t) = 0 \tag{26}$$

for all $t \in \tilde{S} \subset \hat{S}$ and so (26) contradicts (24). This contradiction proves iii).

Specially if $m = n$ and $F'\big(x(T)\big)$ has an inverse, then by Lemma $3A$

$$-F'\big(x_0(T)\big)\,\Phi\big(t,T\big)\,\varphi_v(x_0(t),u_0(t),t\big) = 0 \tag{27}$$

for almost every $t \in [0,T]$, where the $n\times n$ matrix $-F'\big(x_0(T)\big)\Phi(t,T)$ has an inverse for all $t \in [0,T]$ and so it follows from (27), that

$$\varphi_v\big(x_0(t),u_0(t),t\big) = 0 \tag{28}$$

for almost every $t \in [0,T]$. The proof of the sufficience part is analog to that in Theorem 1.

Remark: Let $H(x,u,\psi,t) = \psi\varphi(x,u,t)$, then the function $H\big(x_0(t),u,\psi(t),t\big)$ satisfies the necessary condition of the local supremum (condition iii)) on R^r for almost every $t \in [0,T]$ in the point $u = u_0(t)$. Hence Theorem 2 is also a local supremum principle.

4. Applications

A generalization of Pontryagin's principle in the form of a global supremum principle can be derived from the local supremum principle with the same technique as used by Dubovickij, Miljutin and Girsanov ([1], pp. 83—92).

Theorem $5A$ shows that infimizing the error covariance matrix, all scalar-valued performance crtiteria used practically will be simultaneously minimized.

In [2] a global infimum principle was reported and the applicability of the theory to the analysis of dynamic vector estimation problems and to a class of uncertain optimal control problems was demonstrated. However, all problems examined in [3] can also be easily solved applying the local supremum principle (Theorem 2, case iv)).

Appendix

Theorem 1A: Suppose that the following conditions are satisfied:

1) E and E_i are Banach spaces, $i = 1,\ldots, n+k$; E_0 is a reflexive Banach space;

2) $P_i \subset E_i$ is a closed and convex cone, $0 \notin P_i^0 \neq \emptyset$ and P_i (as a positive cone) defines a partial ordering $\geq$ in the Banach space E_i, $i = 0,\ldots, n$; furthermore, there is a real number $\delta > 0$ such that for all $y_1, y_2 \in P_0$ and $\|y_1\| = \|y_2\| = 1$ it is $\|y_1 + y_2\| \geq \delta$;

3) $F_i : E \to E_i$ is a mapping which has a Fréchet derivative $F_i'(x_0)$ in the point x_0, $i = 0,\ldots, n$ and for which $R(F_i'(x_0))$ is closed in E_i, $i = 1,\ldots, n$;

4) $F_i : E \to E_i$ is a mapping, which is continuously Fréchet differentiable in a neighbourhood of x_0 and for which $R(F_i'(x_0))$ is closed in E_i, $i = n+1,\ldots, n+k$;

5) $A \subset E$ is a convex set and $A^0 \neq \emptyset$;

6) $Q = \{x \in E : -F_i(x_0) \geq 0,\ i = 1,\ldots, n;\ F_i(x) = 0,\ i = n+1,\ldots, n+k;\ x \in A\}$ is a constraint, $x_0 \in Q$ and there exists a neighbourhood V of x_0 such that

$$\inf \{F_0(x) : x \in Q \cap V\} = F_0(x_0).$$

Then there are linear mappings $T_i \in \mathfrak{L}(E_i \to E_0)$, which are continuous on $R(F_i'(x_0))$, $i = 0,\ldots, n+k$ and for which

i) $T_0 y_0 \in P_0$, i.e. $T_0 y_0 \geq 0$ for all $y_0 \in P_0$; furthermore, $T_i y_i \in P_0$ i.e. $T_i y_i \geq 0$ for all $y_i \in R(-F_i'(x_0)) \cap (P_i + F_i(x_0))$, $i = 1,\ldots, n$,

ii) with the notation $T = \sum_{i=0}^{n+k} T_i \circ F_i'(x_0)$ the inequality $Tx \geq Tx_0$, i.e. $T(x - x_0) \in P_0$ holds for all $x \in A$,

iii) $T_i \neq 0$ for at least one i,

iv) if $i \in \{1,\ldots, n\}$ and $-F_i(x_0) \in P^0$, then $T_i = 0$,

v) if the system

$$R(F_i'(x_0)) = E_i, \quad i = n+1,\ldots, n+k,$$
$$R(-F_0'(x_0)) \cap P_0^0 \neq \emptyset,$$
$$i \in \{1,\ldots, n\} \text{ and } -F_i(x_0) \notin P_i^0 \Rightarrow R(-F_i'(x_0)) \cap (P + \{\lambda F_i(x_0) : \lambda > 0\}) \neq \emptyset$$

can be satisfied, then $T_i \circ F_i'(x_0) \neq 0$ for at least one i,

vi) specially if $R(F_i'(x_0)) = E_i$, $i = n+1,\ldots, n+k$ and the system

$$F_i'(x_0)(x - x_0) = 0, \quad i = n+1,\ldots, n+k,$$
$$i \in \{1,\ldots, n\} \text{ and } -F_i(x_0) \notin P_i^0 \Rightarrow -F_i'(x_0)(x - x_0) \in P_i^0 + \{\lambda F_i(x_0) : \lambda > 0\}$$
$$x \in A^0$$

has a solution in x, then $T_0 = I$ is the identity operator. If in addition F_i is a convex mapping, $i = 0,\ldots, n$; $F_i(x) = B_i x + b_i$, where B_i is a bounded linear operator and $b_i \in E_i$, $i = i = n+1,\ldots, n+k$, then the local infimum is also a global infimum on Q and i), ii) or ii'), iv), vi) are the sufficient conditions of the global infimum, where ii') $x \in Q \Rightarrow F_0'(x_0)(x - x_0) = T(x - x_0) - \sum_{i=1}^{n} T_i \circ F_i'(x_0)(x - x_0) \in P_0$; specially for $n = 0$, ii') has the form $x \in Q \Rightarrow T(x - x_0) \in P_0$.

Lemma 2A: Let $M \subset R^r$, $Q = \{x \in L^{(r)}_\infty(0,T) : x(t) \in M$ for almost every $t \in [0,T]\}$, $x_0 \in Q$, $A \in L_\infty^{(n\times r)}(0,T)$ and let P be a closed (and not necessarily convex) cone in R^n. Suppose that

$$\int_0^T A(t)\,(x(t) - x_0(t))\,dt \in P$$

for all $x(\cdot) \in Q$. Then

$$A(t)\,(x - x_0(t)) \in P$$

for all $x \in M$ and for almost every $t \in [0,T]$.

Lemma 3A: Let $\tau \in [0,T]$ and let $A(\cdot) \in L_\infty^{(n\times n)}(0,T)$. Then

i) the problem

$$\frac{d\Phi(t,\tau)}{dt} = \Phi(t,\tau)\,A(t) \text{ for almost every } t \in [0,T]\,,$$
$$\Phi(\tau,\tau) = I_{n\times n}$$

has exactly one solution $\Phi(\cdot,\tau) \in C^{(n\times n)}(0,T)$;

ii) the problem

$$\frac{d\varphi(t)}{dt} = \varphi(t)\, A(t) \text{ for almost every } t \in [0,T] ,$$

$$\varphi(\tau) \text{ is given,}$$

has exactly one solution $\varphi(\cdot) \in C^{m\times n}(0,T)$, and the solution is

$$\varphi(t) = \varphi(\tau)\, \Phi(t,\tau) ;$$

iii) for all $t, \tau \in [0,T]$, the inverse matrix $\Phi^{-1}(t,\tau)$ exists and

$$\Phi^{-1}(t,\tau) = \Phi(\tau,t).$$

Lemma 4A: Let P be a closed and convex cone in the linear normed space, R^n, such that $P^0 \neq \emptyset$ and $\pm z \in P \Rightarrow z = 0$. Then there exists a real number $\delta > 0$ such that for all $y_1, y_2 \in P$ and $\|y_1\| = \|y_2\| = 1$ it is $\|y_1 + y_2\| \geq \delta$.

Theorem 5A: Let $Q \subset R^n$ and let A be a positive semidefinite symmetric $n \times n$ matrix. Let $(\Omega, \mathcal{A}, p)$ be a probability space, let $\mathcal{F} \subset \{x : \Omega \to R^n$ is a random variable: $Ex = 0$ and there exists $E(xx^*)\}$, and let $\mathcal{G} = \{x \in \mathcal{F} : p(\{\omega) : x(\omega) \in Q\}) = 1$. Use the following notations:

$$\begin{aligned} &F: \mathcal{F} \to M_{,n\times n} && F(x) = E(xx^*);\\ &F_1 : \mathcal{F} \to R^1, && F_1(x) = E(x^* A x);\\ &F_2 : \mathcal{F} \to R^1, && F_2(x) = \text{trace } E(xx^*);\\ &F_3 : \mathcal{F} \to R^1, && F_3(x) = \det E(xx^*). \end{aligned}$$

If $F(x) \geq F(y)$, i.e. $F(x) - F(y)$ is a positive semidefinite symmetric matrix, then $F_i(x) \geq F_i(y)$, $i = 1,2,3$. If $x_0 \in \mathcal{G}$ and $\inf\{F(x) : x \in \mathcal{G}\} = F(x_0)$, then $\min\{F_i(x) : x \in \mathcal{G}\} = F_i(x_0)$, $i = 1,2,3$.

Summary

The principal aim of this paper is to give the necessary condition of the optimum (infimum) in form of the local supremum principle for optimum control problems with nonscalar-valued performance criterion. The performance criterion has its range in a finite-dimensional partially ordered linear normed (not necessarily Euclidean) space. The local supremum can be applied to the analysis of dynamic vector estimation problems and to uncertain optimal control problems.

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