STATE EQUATION FOR LINEAR NETWORKS

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(Received September 16, 1976)

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Introduction

A method suitable for writing the state equation for networks containing linear two-poles and two-ports that may also have extreme parameters, is described. Consequently, the method can also be applied for electronic networks with approximately linear elements. To write the equations, two-ports other than coupled inductances are modelled by equivalent circuits containing controlled generators. Equations necessary for the model are written by means of the graph theory.

The way of writing state equations is discussed in [1, 2] too. [1] models two-ports by applying nullators and norators, and [2] by chain parameters of two-ports. This paper is a further development of [3], concerned with equations for steady-state linear networks.

Modelling the network

Two-ports of the network are modelled by circuits containing controlled generators (Fig. 1). In the equivalent circuits voltage-time functions u_1, u_2, and current-time functions i_1, i_2 are related as for two-ports. Both on the primary and secondary side of the models there is a circuit consisting of a controlled source and passive two-poles.

Modelling the two-ports as given above, and considering each independent generator as consisting of several branches containing a source and separate passive two-poles, results in a network with branches of resistances (conductances), inductances, capacitances, independent, or controlled sources.

Controlled sources may include some that got into the network else than in modelling the two-ports. In such a case, if the control voltage is that of a branch other than resistance or break, a branch consisting of a break will be connected in parallel to this branch, the voltage of which is considered as control voltage. If in turn, control current is that of a branch other than resistance or short-circuit, the current of the short circuit connected in series with the branch is considered as control current. In the resulting model all the controlled sources are in branches constituting the primary or secondary side of a two-port.

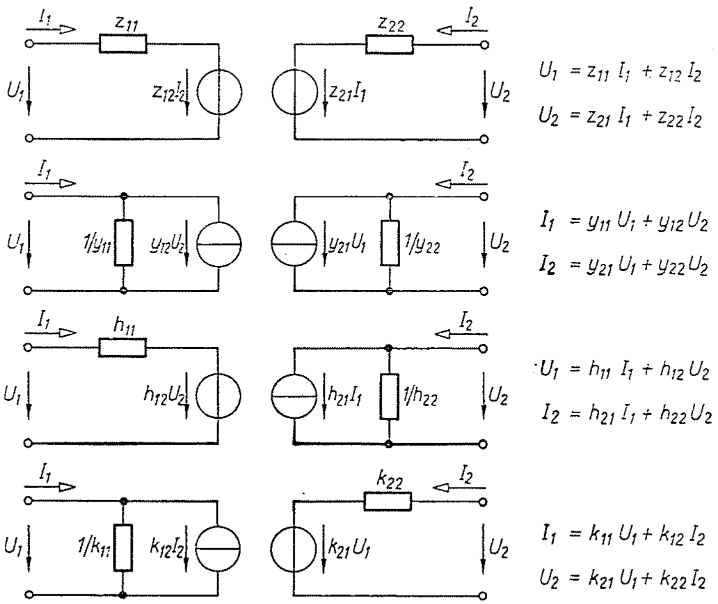

Fig. 1

The state equation

To write the state equation, the branches of the model are classified in groups.

Controlled sources and resistances represent branches of type g or r, according to the following:

g-type branches in the network are

a) all the branches containing controlled current sources;

b) all the branches containing conductances, if their voltage is a control voltage;

c) all the breaks;

d) branches of finite conductance.

r-type branches are:

a) all the branches containing controlled voltage sources;

b) all the branches containing resistances, the current of which is a control current;

c) all the short-circuits;

d) branches of finite resistance

Network branches of non-zero finite resistance are either of g- or of r-type one of the following six groups:

1. branches containing independent current sources (links);
2. g-type branches (links);
3. branches containing inductances (links);
4. branches containing capacitors (twigs);
5. r-type branches (twigs);
6. branches containing independent voltage sources (twigs).

Accordingly, branches of non-zero finite resistance are classified as g- or r-type, so that the totality of branches in Groups 4, 5, and 6 form a tree. If this is not feasible then there is a capacitive loop or an inductive cut-set in the network. This helps to detect the existence of hidden capacitive loops or inductive cut-sets due to the presence of a two-port with extreme parameters, rather difficult to demonstrate by other methods.

Numbering the branches in the order of grouping, hence by $1, 2, \ldots, b_1$ in Group 1, by $b_1 + 1, b_1 + 2, \ldots, b_1 + b_2$ in Group 2, and so on. The number and direction of loops in the fundamental loop system generated by the chosen tree will be indentical with those of the link in the loop. The cut-sets of the fundamental cut-set system generated by the same tree will be numbered in the order of generating twigs in the cut-set, further the branch and the cut-set will be of identical direction along the twig.

Designating the column matrix formed of the voltages and currents in each group by $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_6$, and $\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_6$ respectively, current of g-type branches depends on their voltages and on the current of r-type branches as follows:

$$\mathbf{i}_2 = \boldsymbol{G}\mathbf{u}_2 + \boldsymbol{K}\mathbf{i}_5 \tag{1}$$

where the elements in the main diagonal of $\boldsymbol{G}$ are the conductances of branches in Group 2, while the elements outside the main diagonal are the conductance parameters representing the relationship between currents of voltage-controlled current sources and control voltages. $\boldsymbol{K}$ is composed of proportionality factors between source currents of current-controlled current sources and control currents in Group 2. Rows of $\boldsymbol{K}$ correspond to branches of Group 2, while columns to those of Group 5.

Voltage of branches in Group 5 can be expressed in terms of voltages of branches in Group 2 and of currents of branches in Group 5:

$$\mathbf{u}_5 = \boldsymbol{M}\mathbf{u}_2 + \boldsymbol{R}\mathbf{i}_5 \ . \tag{2}$$

Elements of $\boldsymbol{M}$ are multiplying factors between source voltages of voltage-controlled voltage sources and control voltages. Rows of $\boldsymbol{M}$ are ordered in accordance with branches of Group 5, columns with those of Group 2. $\boldsymbol{R}$ is a quadratic matrix, in which main diagonal elements are the resistances of r-type branches, while the elements outside the main diagonal are multiplying factors between source voltages of current-controlled voltage sources and control current.

Voltages and currents of branches in Group 3 are related as:

$$\mathbf{u}_3 = \boldsymbol{L}\dot{\mathbf{i}}_3 \,. \tag{3}$$

The main diagonal of $\boldsymbol{L}$ contains self-induction coefficients of the branches, corresponding mutual induction coefficients outside the main diagonal.

Currents and voltages of branches in Group 4 are related as:

$$\mathbf{i}_4 = \boldsymbol{C}\dot{\mathbf{u}}_4 \,. \tag{4}$$

$\boldsymbol{C}$ is a diagonal matrix with capacitances of condensers as elements.

Loop matrix $\boldsymbol{B}$ and cut-set matrix $\boldsymbol{Q}$ partitioned according to the grouping of branches will be used for writing the Kirchhoff equations of the network:

$$\begin{bmatrix} 1 & 0 & 0 & \boldsymbol{F}_{11} & \boldsymbol{F}_{12} & \boldsymbol{F}_{13} \\ 0 & 1 & 0 & \boldsymbol{F}_{21} & \boldsymbol{F}_{22} & \boldsymbol{F}_{23} \\ 0 & 0 & 1 & \boldsymbol{F}_{31} & \boldsymbol{F}_{32} & \boldsymbol{F}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \\ \mathbf{u}_5 \\ \mathbf{u}_6 \end{bmatrix} = \mathbf{0} \tag{5}$$

and

$$\begin{bmatrix} -\boldsymbol{F}_{11}^{+} & -\boldsymbol{F}_{21}^{+} & -\boldsymbol{F}_{31}^{+} & 1 & 0 & 0 \\ -\boldsymbol{F}_{12}^{+} & -\boldsymbol{F}_{22}^{+} & -\boldsymbol{F}_{32}^{+} & 0 & 1 & 0 \\ -\boldsymbol{F}_{13}^{+} & -\boldsymbol{F}_{23}^{+} & -\boldsymbol{F}_{33}^{+} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \\ \mathbf{i}_4 \\ \mathbf{i}_5 \\ \mathbf{i}_6 \end{bmatrix} = \mathbf{0} \tag{6}$$

yielding:

$$\mathbf{u}_1 + \boldsymbol{F}_{11}\mathbf{u}_4 + \boldsymbol{F}_{12}\mathbf{u}_5 + \boldsymbol{F}_{13}\mathbf{u}_6 = \mathbf{0} \tag{7}$$

$$\mathbf{u}_2 + \boldsymbol{F}_{21}\mathbf{u}_4 + \boldsymbol{F}_{22}\mathbf{u}_5 + \boldsymbol{F}_{23}\mathbf{u}_6 = \mathbf{0} \tag{8}$$

$$\mathbf{u}_3 + \boldsymbol{F}_{31}\mathbf{u}_4 + \boldsymbol{F}_{32}\mathbf{u}_5 + \boldsymbol{F}_{33}\mathbf{u}_6 = \mathbf{0} \tag{9}$$

$$-\boldsymbol{F}_{11}^{+}\mathbf{i}_1 - \boldsymbol{F}_{21}^{+}\mathbf{i}_2 - \boldsymbol{F}_{31}^{+}\mathbf{i}_3 + \mathbf{i}_4 = \mathbf{0} \tag{10}$$

$$-\boldsymbol{F}_{12}^{+}\mathbf{i}_1 - \boldsymbol{F}_{22}^{+}\mathbf{i}_2 - \boldsymbol{F}_{32}^{+}\mathbf{i}_3 + \mathbf{i}_5 = \mathbf{0} \tag{11}$$

$$-\boldsymbol{F}_{13}^{+}\mathbf{i}_1 - \boldsymbol{F}_{23}^{+}\mathbf{i}_2 - \boldsymbol{F}_{33}^{+}\mathbf{i}_3 + \mathbf{i}_6 = \mathbf{0} \tag{12}$$

where $\mathbf{i}_3$ and $\mathbf{u}_4$, and $\mathbf{i}_1$ and $\mathbf{u}_6$ are column matrices of state variables, and of excitations, respectively. To write the state equation, variables $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_5$, $\mathbf{i}_2$, $\mathbf{i}_4$, $\mathbf{i}_5$, $\mathbf{i}_6$ have to be eliminated from equations given above. The calculation yields the state equation

$$\begin{bmatrix} \dot{\mathbf{i}}_3 \\ \dot{\mathbf{u}}_4 \end{bmatrix} = \begin{bmatrix} L^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} \left\{ \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \mathbf{i}_3 \\ \mathbf{u}_4 \end{bmatrix} + \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} \mathbf{i}_1 \\ \mathbf{u}_6 \end{bmatrix} \right\} \tag{13}$$

where

$$D_{11} = -F_{32}[1 + MF_{22} + RF^{+}_{22}(1 - KF^{+}_{22})^{-1} GF_{22}]^{-1} [RF^{+}_{22}(1 - KF^{+}_{22})^{-1} KF^{+}_{32} + RF^{+}_{32}] \tag{14}$$

$$D_{12} = F_{32}[1 + MF_{22} + RF^{+}_{22}(1 - KF^{+}_{22})^{-1} GF_{22}]^{-1} [MF_{21} + RF^{+}_{22}(1 - KF^{+}_{22})^{-1} GF_{21}] - F_{31} \tag{15}$$

$$D_{21} = F^{+}_{31} + F^{+}_{21}[1 + GF_{22}(1 + MF_{22})^{-1} RF^{+}_{22} - KF^{+}_{22}]^{-1} [KF^{+}_{32} - GF_{22}(1 + MF_{22})^{-1} RF^{+}_{32}] \tag{16}$$

$$D_{22} = F^{+}_{21}[1 + GF_{22}(1 + MF_{22})^{-1} RF^{+}_{22} - KF^{+}_{22}]^{-1} [GF_{22}(1 + MF_{22})^{-1} MF_{21} - GF_{21}] \tag{17}$$

$$E_{11} = -F_{32}[1 + MF_{22} + RF^{+}_{22}(1 - KF^{+}_{22})^{-1} GF_{22}]^{-1} [RF^{+}_{12} + RF^{+}_{22}(1 - KF^{+}_{22})^{-1} KF^{+}_{12}] \tag{18}$$

$$E_{12} = F_{32}[1 + MF_{22} + RF^{+}_{22}(1 - KF^{+}_{22})^{-1} GF^{+}_{22}]^{-1} [MF_{23} + RF^{+}_{22}(1 - KF^{+}_{22})^{-1} GF_{23}] - F_{33} \tag{19}$$

$$E_{21} = F^{+}_{11} + F^{+}_{21}[1 + GF_{22}(1 + MF_{22})^{-1} RF^{+}_{22} - KF^{+}_{22}]^{-1} [KF^{+}_{12} - GF_{22}(1 + MF_{22})^{-1} RF^{+}_{12}] \tag{20}$$

$$E_{22} = F^{+}_{21}[1 + GF_{22}(1 + MF_{22})^{-1} RF^{+}_{22} - KF^{+}_{22}]^{-1} [GF_{22}(1 + MF_{22})^{-1} MF_{23} - GF_{23}] \tag{21}$$

In knowledge of the state variables, the other variables can be expressed on the basis of equations written previously. The relevant relationship will not be described here.

Summary

A graph theory method will be presented for writing the state equation of a network consisting of linear two-poles and two-ports. Two-ports and two-poles may also have extreme parameters, thus the calculation can be applied also for electronic circuits with linear elements. To write the equations, branches are grouped in six groups to simplify writing of the relationship between voltages and currents of branches in each group.

References

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