PULSE RESPONSE OF RADIAL VIBRATION OF PIEZOELECTRIC DISK

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1. Introduction

In free disks two kinds of vibration may occur, thickness and radial vibration. When the thickness vibrations of piezoelectric disks are applied the radial vibrations are considered as undesirable, parasitic vibrations. In many cases radial vibrations are used intentionally, e.g. in electromechanical filters, low-frequency ultrasonic radiators, etc. Steady-state radial vibrations of disks were studied by MASON [1].

Piezoelectric disk transducers are used in pulse-measuring and testing instruments, imposing detailed analysis of transient processes.

This paper deals with the distribution of radial stresses in a thin piezoceramic disk in response to an input signal of electrical step function. The front planes of the disk are assumed to be coated by metal electrodes and the edges are supposed to be free. Internal losses in the piezoelectric material are neglected. The analysis is carried out by solving the wave equation of the disk by Laplace transformation.

2. Calculation of Laplace-transformed radial stresses

The equation of motion of the disk in a cylindrical co-ordinate system is

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} = \frac{1}{v^2} \cdot \frac{\partial^2 u_r}{\partial t^2} \tag{1}$$

where u_r — is the radial component of elastic displacements,

v — the velocity of longitudinal waves in the piezoelectric material.

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The Laplace-transformed equation (1) can be written as

$$\frac{\partial^2 \bar{u}_r}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial \bar{u}_r}{\partial r} - \left(\frac{p^2}{v^2} + \frac{1}{r^2}\right) \bar{u}_r = 0 \tag{2}$$

where p — is the complex variable,

$\bar{u}_r = \mathfrak{L} u_r$ — the Laplace transform of radial displacements.

The solution of Eq. (2) is a Bessel function of imaginary argument called also Bessel hyperbolic or modified function

$$\bar{u}_r = A I_1 \left(\frac{pr}{v}\right). \tag{3}$$

Constant A is obtained from the boundary condition

$$T_{rr}(t, r = a) = 0 \tag{4}$$

where T_{rr} — is the radial component of stresses,

a — the radius of the disk.

The relationship between the radial stresses T_{rr}, radial displacements u_r, and exciting electric field E_z can be established by means of a piezoelectric equation deduced by MASON [1] for disks in cylindrical co-ordinate system. For a thin piezoelectric disk, subject to radial vibrations excited by axial electric field, this equation can be written in a Laplace-transformed form as

$$\bar{T}_{rr} = \frac{s_{11}}{s_{11}^2 - s_{12}^2} \cdot \frac{\partial \bar{u}_r}{\partial r} - \frac{s_{12}}{s_{11}^2 - s_{12}^2} \cdot \frac{\bar{u}_r}{r} - e_{31} \bar{E}_z \tag{5}$$

where s_{11}, s_{12} — elastic constants of the piezoelectric material, and

e_{31} — the piezoelectric constant.

The electroded surfaces form equipotential planes parallel to the direction of motion of the vibration type considered, hence the electric field E_z has been chosen as an independent variable. So the elastic constants in Eq. (5) are replaced by constants measured at a given constant electric field strength: $s_{11} = s_{11}^E$, $s_{12} = S_{12}^E$.

By substituting the expression of displacements (3) and its derivative in (5), and applying the boundary condition (4), A can be determined. Then Eq. (3) will be written as

$$\bar{u}_r = \frac{a(s_{11}^2 - s_{12}^2) e_{31} \bar{E}_z}{s_{11} \frac{pa}{v} \cdot I_0 \left(\frac{pa}{v}\right) - (s_{11} + s_{12}) I_1 \left(\frac{pa}{v}\right)} \cdot I_1 \left(\frac{pr}{v}\right) \tag{6}$$

Equ. (6) and its derivative substituted in (5) yield the final expression of radial stresses in the piezoelectric disk

$$\overline{T}_{rr} = e_{31}\overline{E}_z \left[\frac{\frac{pa}{v} \cdot I_0\left(\frac{pr}{v}\right) - \frac{a}{r}(1-\sigma) \cdot I_1\left(\frac{pr}{v}\right)}{\frac{pa}{v} \cdot I_0\left(\frac{pa}{v}\right) - (1-\sigma) \cdot I_1\left(\frac{pa}{v}\right)} - 1 \right] \tag{7}$$

where $\sigma = -s_{12}/s_{11}$ is the Poisson's ratio.

Expression (7) permits to calculate the radial component of stresses at any point of the thin piezoelectric disk in response of the given exciting electric field E_z.

3. Calculation of the inverse Laplace-transformed stress function

The time dependence of the stresses excited by a step function of electric field strength $E_z(t) = E \cdot 1(t)$ and $\overline{E}_z = E/p$ will be calculated. This requires the inverse Laplace transformation of function

$$\overline{T}_{rr} = e_{31}E \cdot$$

$$\cdot \frac{\frac{pa}{v} \cdot I_0\left(\frac{pr}{v}\right) - \frac{a}{r}(1-\sigma) \cdot I_1\left(\frac{pr}{v}\right) - \frac{pa}{v} \cdot I_0\left(\frac{pa}{v}\right) + (1-\sigma) \cdot I_1\left(\frac{pa}{v}\right)}{p\left[\frac{pa}{v} \cdot I_0\left(\frac{pa}{v}\right) - (1-\sigma) \cdot I_1\left(\frac{pa}{v}\right)\right]} . \tag{8}$$

In paper [2] the inverse transformation of (8) was obtained by replacing the Bessel function by the asymptotic expressions $I_n(x) \approx e^x/\sqrt{2\pi x}$ valid for high values of arguments. This kind of approximation enables us to calculate the form of stresses T_{rr} developed at the beginning of the transient, but does not expose the later history and the oscillation character of the process. This will subsequently analysed in detail.

The inverse Laplace transform of (8) is calculated by means of the expansion theorem [3].

$$T_{rr} = \mathfrak{L}^{-1}\overline{T}_{rr} = \sum_{k=1}^{n} \frac{M(p_k)}{N'(p_k)} e^{p_k t} \tag{9}$$

where $M(p)$ and $N(p)$ — are the numerator and denominator of function (8), respectively.

Let us specify the poles p_k of function (8). It should be noted that point $p = 0$ is not pole as proved in the following way. At low values of argument the Bessel function can be approximated by its power series

$$I_0(x) = 1 + \frac{x^2}{4} + \dots, \qquad I_1(x) = \frac{x}{2} + \dots. \tag{10}$$

Making use of these expressions, the limit value of function (8) is obtained, when p tends to zero:

$$\lim_{p \to 0} \overline{T}_{rr} = 0 .$$

Point $p = 0$ is seen to be the zero point and not the pole of the function. The poles of function (8) are thus given by the roots of the equation

$$\frac{pa}{v} \cdot I_0\left(\frac{pa}{a}\right) - (1 - \sigma) I_1\left(\frac{pa}{v}\right) = 0 . \tag{11}$$

Using the relationship

$$I_n(x) = j^{-n} J_n(jx) \tag{12}$$

where J_n is a first-order Bessel function, we obtain

$$\xi_k J_0(\xi_k) - (1 - \sigma) J_1(\xi_k) = 0 \tag{13}$$

with $jpa/v = \xi$ introduced. The roots ξ_k of Eq. (13) determine the poles of function (8):

$$p_k = \pm j \xi_k \frac{v}{a}, \qquad k = 1, 2, \dots \tag{14}$$

Expressing separately the numerator $M(p_k)$ of function (8), with the positive values of poles p_k substituted, and by use of (12) we get

$$M(p_k) = e_{31} E j \left\{ \left[\xi_k J_0\left(\xi_k \frac{r}{a}\right) - \frac{a}{r} (1 - \sigma) J_1\left(\xi_k \frac{r}{a}\right) \right] + \right.$$
$$\left. + \left[(1 - \sigma) J_1(\xi_k) - \xi_k J_0(\xi_k) \right] \right\} .$$

According to (13) the second term in square brackets is zero.
The derivative of denominator $N(p)$ is

$$N'(p) = \frac{\partial}{\partial p} N(p) = \frac{pa}{v} (1 + \sigma) \cdot I_0\left(\frac{pa}{v}\right) + \left(\frac{pa}{v}\right)^2 \cdot I_1\left(\frac{pa}{v}\right) .$$

Substituting the positive values of poles and replacing I by J according to (12) we obtain

$$N'(p_k) = j[\xi_k J_0(\xi_k)(1+\sigma) - \xi_k^2 J_1(\xi_k)] .$$

To reduce the expression we substitute $\xi_k J_0(\xi_k) = (1-\sigma)J_1(\xi_k)$ according to (13)

$$N'(p_k) = j(1 - \sigma^2 - \xi_k^2)J_1(\xi_k) .$$

Substituting the negative values of poles p_k we obtain the same kind of expressions

$$\frac{M(-p_k)}{N'(-p_k)} = \frac{-M(p_k)}{-N'(p_k)} = \frac{M(p_k)}{N'(p_k)} .$$

The expressions obtained for $M(p_k)$ and $N'(p_k)$ are substituted in (9):

$$T_{rr} = e_{31}E \sum_{k=1}^{\infty} \frac{\xi_k J_0\left(\xi_k \dfrac{r}{a}\right) - \dfrac{a}{r}(1-\sigma)J_1\left(\xi_k \dfrac{r}{a}\right)}{(1-\sigma^2-\xi_k^2)J_1(\xi_k)} \left(e^{j\xi_k \cdot \frac{v}{a} \cdot t} + e^{-j\xi_k \cdot \frac{v}{a} \cdot t}\right)$$

or

$$T_{rr} = 2e_{31}E \sum_{k=1}^{\infty} \frac{\xi_k J_0\left(\xi_k \dfrac{r}{a}\right) - \dfrac{a}{r}(1-\sigma)J_1\left(\xi_k \dfrac{r}{a}\right)}{(1-\sigma^2-\xi_k^2)J_1(\xi_k)} \cos\left(\xi_k \frac{v}{a} t\right) . \qquad (15)$$

Expression (15) is a sum of cosinusoidals with frequencies determined by roots ξ_k of the transcendent equation (13). These roots are no integer multiples of each other, hence the process described by (15) is a sum of time periodic functions of frequencies that are no integer multiples of a basic frequency. With the number of k increasing, the amplitudes of cosinusoidals decrease because the exciting step function E_z also shows a decaying spectrum $|S(\omega)| \sim \sim 1/\omega$. Thereby the form of vibration expressed by (15) can be calculated by summing some of the first members of the sum.

4. Numerical calculations

Eq. (13) has been solved for a Poisson's ratio of $\sigma = 0.33$ corresponding to the piezoelectric ceramic PZT-4 [4]. The first twenty roots ξ_k of Eq. (13) are given in Table 1.

The indefinability of the numerator in the case $r/a = 0$ prevents the direct use of formula (15) for calculating the radial stresses in the centre of the disk. The use of approximation (10) gives the following expression of the mid-disk stress:

$$T_{rr}(r=0) = 2\, e_{31} E \sum_{k=1}^{\infty} \frac{\frac{1}{2}\xi_k(1+\sigma)}{(1-\sigma^2-\xi_k^2)J_1(\xi_k)} \cdot \cos\left(\xi_k \cdot \frac{v}{a} \cdot t\right). \tag{16}$$

Table 1

k	ξ_k
1	2.067299
2	5.395099
3	8.545399
4	11.73436
5	14.88586
6	18.03386
7	21.17996
8	24.32492
9	27.46908
10	30.61271
11	33.75597
12	36.89894
13	40.04169
14	43.18427
15	46.32672
16	49.46907
17	52.61131
18	55.75349
19	58.89559
20	62.03767

The forms of radial stresses calculated by means of formulae (15) and (16) are shown in Fig. 1. The constant factor preceding the sign of summation was omitted.

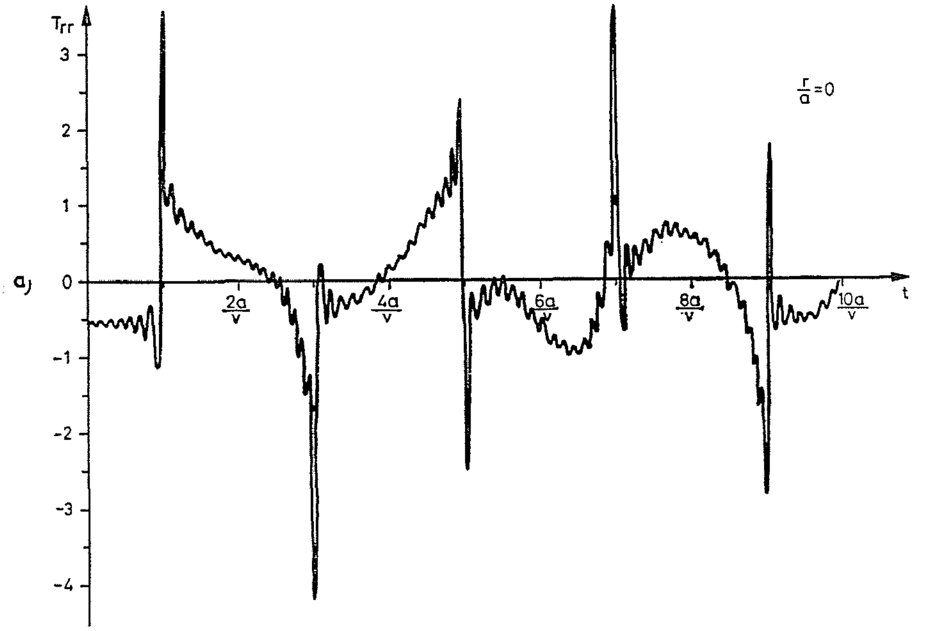
5. Evaluation of results, conclusions

Eq. (13) is seen to coincide with the known condition of disk resonance [1], with $p = j\omega$ substituted. Hence the frequencies of cosinusoidals in formula (15) correspond to the radial eigenfrequencies of the disk. The calculated values of roots ξ_k permit to determine the resonance frequencies of the radial vibration mode of the thin disk [1]

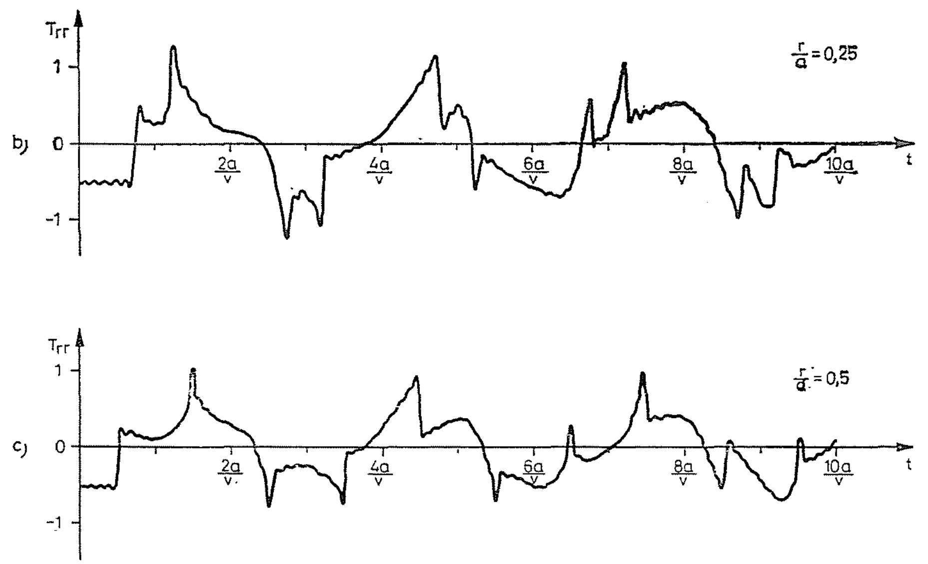
$$f_k = \frac{\xi_k}{2\pi a}\sqrt{\frac{1}{\varrho s_{11}^E(1-\sigma^2)}}$$

where ϱ — is the material density.

Fig. 1 shows that the excitation of a piezoelectric disk by a step function electric field starts complicated vibrations in it with shapes depending on the



a)



b)

Fig. 1. The time function of radial stresses at co-ordinates $r/a = 0$, $r/a = 0.25$ and $r/a = 0.5$

radial co-ordinate of the given point. In the centre of the disk (Fig. 1a) the first sharp pulse lags by a/v behind the instant of excitation, that is, by a time interval required for the wave to travel from the edge to the centre of the disk. At points $r/a = 0.25$ and $r/a = 0.5$ the value of this delay is $0.75a/v$, and $0.5a/v$, respectively. This agrees with JACOBSEN's principle [5] namely that ultrasonic waves are produced at those parts of the piezoelectric transducer where the product $E_z \cdot e_{31}$ has a gradient.

At mid-disk (Fig. 1a) a short pulse of very large amplitude occurs because of the coincidence of waves starting from all round the edges. This phenomenon of internal focussing can be utilized for emitting short pulses [2]. When moving off the centre (Fig. 1b and c) this pulse becomes wider and its amplitdue smaller.

High frequency oscillation — best seen in Fig. 1a — is obtained because of the fact that only finite number (twenty) of member have been taken into account in the sums of Eqs (15) and (16). It follows from the approximation by partial sums, similarly to the GIBBS phenomenon known in connection with the Fourier series. From the centre away, convergence of the sum (15) is accelerated.

Due to the relationship between the longitudinal and transversal deformation of materials in a real piezo transducer, the radial stresses produce a normal (axial) deformation component in any point which induces thickness displacements normal to the faces of the disk. The radial component of vibrations highly influences the form of the normal displacements for both thin and thick disks [2], consequently, normal displacements are also highly dependent on the co-ordinates of the considered point. With the pulse excitation of the piezoelectric disk no uniformly distributed mechanical displacement independent of the radius can be expected on the disk surface. This has to be taken into account e.g. when establishing the pulse radiation pattern of piezo transducers.

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Summary

Solution of wave equation of piezoelectric disk in response of a step function of exciting electric field strength is presented. The time dependence of the radial stresses for several radial co-ordinates is calculated.

References

1. MASON, W.: Piezoelectric Crystals and their Application to Ultrasonics. D. Van Nostrand Co., New York 1950.
2. Машонис, А.: Кажис Р. И.: Работа дискового пьезопреодразователя в импульсном режиме. Материалы 2-ой Всесоюзной конференции по ультразуковой спектроскопии (18-20 сентябвя 1973 г.. Вильнюс), Каунас, 1973, 167–170.
3. FODOR, G.: Laplace Transforms in Engineering. Akadémiai Kiadó Budapest, 1965.
4. MASON, W.: Physical Acoustics, Vol. 1, A. Academic Press, New York—London 1964.
5. JACOBSEN, E. H.: Sources of Sound in Piezoelectric Crystals. J. Acoust. Soc. Amer. **32**, No. 8. 1960. 1. pp. 949—953.

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