THE THEORY OF SENSITIVITY INVARIANTS AND THEIR APPLICATION TO OPTIMIZATION OF TOLERANCES AND NOISES

By

K. GÉHER

Institute of Telecommunication and Electronics, Technical University Budapest

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Presented by Prof. Dr. I. BARTA

1. Introduction

The network characteristic $y = y(x_1, \ldots x_i, \ldots x_N)$ depends upon the circuit parameters x_i, where N is the number of parameters. The absolute sensitivity of the network characteristic y is

$$S_i = \frac{\partial y}{\partial x_i} \tag{1}$$

and its relative sensitivity:

$$S_i^r = \frac{\partial \ln y}{\partial \ln x_i} = \frac{x_i}{y} S_i . \tag{2}$$

The sum of relative sensitivities is known to be invariant, i.e

$$\sum_{i=1}^{n} S_i^r = M \tag{3}$$

where n is the number of the dimensional circuit parameters. For non-dimensional transfer functions:

$$\sum_{i=1}^{n} S_i^r = 0 \tag{4}$$

and for impedances:

$$\sum_{i=1}^{n} S_i^r = 1 . \tag{5}$$

2. Demonstration of the invariants

Using the definition $S_i^r = \frac{x_i}{y} S_i = \frac{x_i}{y} \frac{\partial y}{\partial x_i}$, Eq. (4) can be written in the following form:

$$\sum_{i=1}^{n} x_i S_i = \sum_{i=1}^{n} x_i \frac{\partial y}{\partial x_i} = 0 \tag{6}$$

and Eq. (5):

$$\sum_{i=1}^{n} x_i S_i = \sum_{i=1}^{n} x_i \frac{\partial y}{\partial x_i} = y. \tag{7}$$

Introducing the vectors

$$\mathbf{x} = [x_1, \ldots x_i \ldots x_n]$$
$$\mathbf{s} = [S_1, \ldots S_i \ldots S_n] = \operatorname{grad} y. \tag{8}$$

we obtain for transfer functions

$$\mathbf{xs} = \mathbf{x} \operatorname{grad} y = 0 \tag{9}$$

and for impedances

$$\mathbf{xs} = \mathbf{x} \operatorname{grad} y = y\,. \tag{10}$$

Considering the dimensional circuit parameters as vector **x** and the network characteristic y as scalar-vector function, the level surfaces and the gradient vector **s** can be introduced. Thereby the summed sensitivity invariants can simply be demonstrated. For non-dimensional transfer functions the vectors **x** and **s** are perpendicular to each other, for impedances the scalar product of **x** and **s** gives exactly the network characteristic y, i.e. the impedance. Naturally, the functions obtained by integrating the partial differential equations (6) and (7) can only be considered as network functions if they are bilinear or biquadratic functions of the circuit parameters. In the following the relationships will be shown for the case of resistance networks with two and three variables.

The transfer function of the circuit shown in Fig. 1a is

$$K = \frac{U_2}{U_1} = \frac{x_2}{x_1 + x_2} = y\,. \tag{11}$$

For $K_1 < K_2 < K_3$ the level lines are shown in Fig. 1b. The value of the summed sensitivity invariant is zero.

The resistance shown in Fig. 2a is

$$R = x_1 + x_2 = y\,. \tag{12}$$

For $R_1 < R_2 < R_3$ the level lines are shown in Fig. 2b. The value of the summed sensitivity invariant is R, according to Eq. (7).

The resultant of three parallel-connected resistances (Fig. 3a) is

$$R = \frac{x_1 x_2 x_3}{x_1 x_2 + x_2 x_3 + x_1 x_3} = y\,. \tag{13}$$

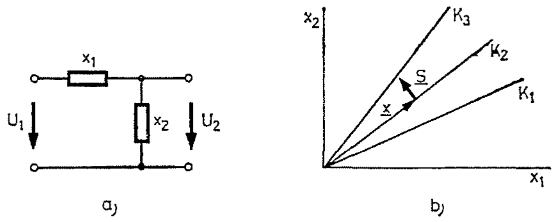

Fig. 1

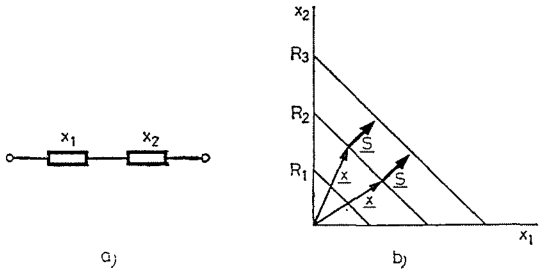

Fig. 2

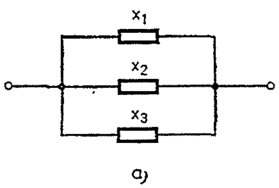

Fig. 3a

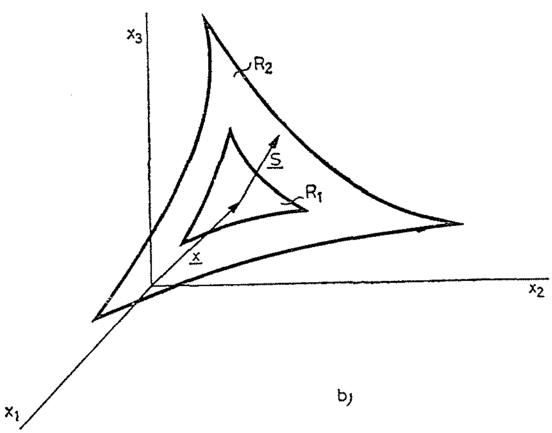

Fig. 3b

For $R_1 < R_2$ the level surfaces are shown in Fig. 3b which demonstrates also the relationship between **x** and **s**.

As last illustrations let be given Figs 4a and 4b. Here

$$K = \frac{U_2}{U_1} = \frac{x_1 x_2 + x_2 x_3}{x_1 x_2 + x_1 x_3 + x_2 x_3} = y \qquad (14)$$

and the summed sensitivity invariant is zero.

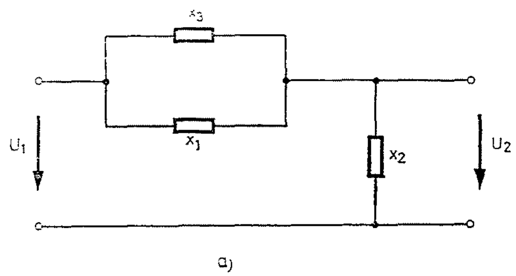

Fig. 4a

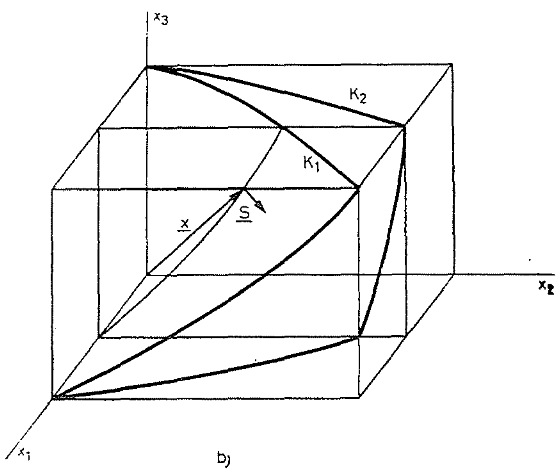

Fig. 4b

3. Theoretical limitations of optimization

From among the problems of optimum-sensitivity linear networks it is interesting to examine the consequences of the summed sensitivity invariance when networks of minimum sensitivity are sought for.

(i) The minimization of the cost function:

$$\varphi = \sum_{i=1}^{N} | S_i^r |^2 \tag{15}$$

with the accessory condition

$$\sum_{i=1}^{n} S_i^r = M \tag{16}$$

is known to be given by the Lagrange method of the limited extremum problem. Here S^r is the relative sensitivity, N the number of circuit parameters, M the summed sensitivity invariant and n the number of dimensional circuit elements occurring in the expression of invariance.

Thus, our task is to determine the minima

$$\varphi = \sum_{i=1}^{N} \left[(Re\, S_i^r)^2 + (Im S_i^r)^2 \right] \tag{17}$$

with the accessory conditions

$$\begin{aligned} &\sum_{i=1}^{n} Re\, S_i^r - Re\, M = 0 \\ &\sum_{i=1}^{n} Im S_i^r - Im\, M = 0\,. \end{aligned} \tag{18}$$

Considering the real and imaginary parts of the relative sensitivities to be independent variables, the calculation results at the minimum in:

$$\begin{aligned} &Re\, S_i^r = 0 \\ &Im\, S_i^r = 0 \end{aligned} \qquad n < i \leq N$$

$$\begin{aligned} &Re\, S_i^r = \frac{Re\, M}{n} \\ &Im S_i^r = \frac{Im\, M}{n} \end{aligned} \qquad 1 \leq i \leq n \,. \tag{19}$$

The minimum value is

$$\varphi_{\text{min}} = \frac{Re\, M^2 + Im\, M^2}{n} = \frac{|M|^2}{n}\,. \tag{20}$$

The summed sensitivity invariance can be stated to limit the possible absolute minimum. From this point, to achieve absolute minimum is conditioned by zero relative sensitivity of the non-dimensional circuit elements and validity of the relationship $S_i^r = M/n$ for the relative sensitivities of the dimensional circuit elements. As value of minimum $|M|^2/n$ is obtained.

Our result defines a theoretical limit and hints to the practical difficulties of achieving absolute minimum. Indeed, in case of bilinear relationship, zero relative sensitivity is obtained for circuit element values $-x_i = 0$ or $x_i = \infty$. If M is real, i.e. $Im\, M = 0$, then at the absolute minimum the sensitivities must also be real. With $M = 0$ zero values are obtained both for the relative sensitivities and the absolute minimum.

These conditions do not concern, of course, local minima. They do not hold, either, if as a further accessory condition, the invariability of network function $y(p)$ is stipulated.

(ii) Let us examine what limits are obtained for the minimization of the sum of absolute sensitivity values, with summed sensitivity invariants as accessory condition. Accordingly, the minimum of

$$\varphi = \sum_{i=1}^{N} \left[(Re\, S_i^r)^2 + (Im\, S_i^r)^2\right]^{1/2} \tag{21}$$

with the accessory conditions

$$\sum_{i=1}^{n} Re\, S_i^r - Re\, M = 0 \tag{22}$$

$$\sum_{i=1}^{n} Im\, S_i^r - Im\, M = 0 \tag{23}$$

has to be found.

At the theoretical minimum, the relative sensitivity of the dimensional circuit elements is

$$Re\, S_i^r = \frac{Re\, M}{n}$$
$$Im\, S_i^r = \frac{Im\, M}{n} \tag{24}$$

the minimum value being

$$\varphi_{\min} = |M|\,. \tag{25}$$

The results obtained are simple to illustrate geometrically. Fig. 5a shows that the sum of relative sensitivities is invariant. Fig. 5b shows the condition of minimum to be that the phase angle of every relative sensitivity equals the

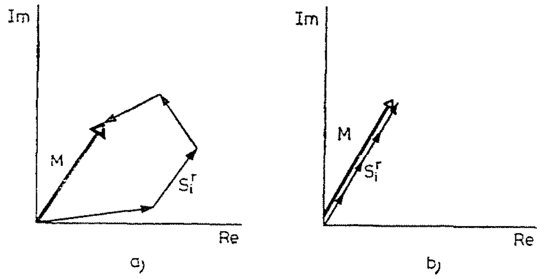

Fig. 5

phase angle of M and the relative sensitivities mutually agree. Obviously, the value of the sum cannot be lower than the absolute minimum $|M|$. Contrary to the former case, the absolute minimum value is independent of the number n of the dimensional circuit elements. Thus, the sum of the absolute sensitivity values cannot be reduced by introducing additional circuit elements (i.e. by increasing n).

(iii) The weighted sum of the squares of absolute values can be written in the form

$$\varphi = \sum_{i=1}^{N} k_i^2 \, |S_i^r|^2 = \sum_{i=1}^{N} k_i^2 \left[(Re\, S_i^r)^2 + (Im\, S_i^r)^2 \right] \tag{26}$$

where k_i is the weighting factor (e.g. variance of the circuit parameter). The minimization of the cost function with the invariance as accessory condition leads to:

$$\left.\begin{aligned} Re\, S_i^r &= 0 \\ Im\, S_i^r &= 0 \end{aligned}\right. \qquad n < i \leq N \tag{27}$$

$$Re\, S_i^r = \frac{Re\, M}{k_i^2 \sum_{i=1}^{n} \frac{1}{k_i^2}} \qquad 1 \leq i \leq n$$

$$Im\, S_i^r = \frac{Im\, M}{k_i^2 \sum_{i=1}^{n} \frac{1}{k_i^2}}$$

the minimum value being:

$$\varphi_{\min} = \frac{|M|^2}{\sum_{i=1}^{n} \frac{1}{k_i^2}} . \tag{28}$$

For $k_i = 1$ the relationships for case (i) come back. In the weighted case, however, both the conditions for theoretical minimum and the minimum value differ from those in the case without weighting. An important consequence of this will be dealt with in section 4.

(iv) One of the possible generalizations of what has been said above is by examining the cost function

$$\varphi = \sum_{i=1}^{N} [Re\, S_i^r)^2 + (Im\, S_i^r)^2]^{m/2}, \tag{29}$$

resulting in:

$$\left.\begin{array}{l} Re\, S_i^r = 0 \\ Im\, S_i^r = 0 \end{array}\right\} \quad n < i \leq N \tag{30}$$

$$\left.\begin{array}{l} Re\, S_i^r = \dfrac{Re\, M}{n} \\ Im\, S_i^r = \dfrac{Re\, M}{n} \end{array}\right\} \quad 1 \leq i \leq n$$

the minimum value being:

$$\varphi_{\min} = \frac{|M|^m}{n^{m-1}} . \tag{31}$$

Substituting $m = 1$ results in the case considered under (ii), while substituting $m = 2$ leads to case (i). At the theoretical minimum the sensitivities of the dimensional circuit elements agree and in each case the restriction

$$\frac{Im\, S_i^r}{Re\, S_i^r} = \frac{Im\, M}{Re\, M} \tag{32}$$

holds.

A further possibility of generalization is by determining the minimum of function:

$$\varphi = \sum_{i=1}^{N} k_i^m [(Re\, S_i^r)^2 + (Im\, S_i^r)^2]^{m/2} \tag{33}$$

resulting in:

$$\left.\begin{array}{l} Re\, S_i^r = 0 \\ Im\, R_i^r = 0 \end{array}\right\} \quad n < i \leq N$$

$$Re\, S_i^r = \frac{ReM}{k_i^m} \frac{1}{\sum_{i=1}^{n} \frac{1}{k_i^m}} \tag{34}$$

$$Im\, S_i^r = \frac{Im M}{k_i^m} \frac{1}{\sum_{i=1}^{n} \frac{1}{k_i^m}} \quad 1 \leq i \leq n$$

the minimum value being:

$$\varphi_{\min} = \frac{|M|^m}{\left(\sum_{i=1}^{n} \frac{1}{k_i^m}\right)} \sum_{i=1}^{n} \frac{1}{(k_i^m)^{m-1}} . \tag{35}$$

The substitutions $k_i = 1$, $m = 2$ lead to case (i), $k_i = 1$ and $m = 1$ to case (ii), $m = 2$ to case (iii), while the substitution $k_i = 1$ to (29) to (31) dealt with above.

4. Relationship between optimum sensitivity and optimum noise

The theoretical limitations stated above are not only valid for the examination of relative tolerance

$$\frac{\Delta y}{y} = \sum_{i=1}^{N} S_i^r \frac{\Delta x_i}{x_i} \tag{36}$$

but also for the output noise/signal ratio. The output noise/signal ratio of an active RC circuit is known to be

$$\frac{|U_z|^2}{|U_2|^2} = \Phi = 4kT\Delta_f \sum_{i=1}^{N_R} \frac{|S_i^r|^2}{P_i} + \sum_{i=1}^{N_A} |S_i^r|^2 \frac{|U_{zi}|^2}{|U_{2i}|^2} \tag{37}$$

where U_z the output noise voltage,
U_2 the output voltage,
$k = 1.38 \cdot 10^{-23}$ Ws/K° the Boltzmann constant,
T absolute temperature,
Δ_f frequency range,
N_R number of resistances,
P_i the power dissipated across the resistance,
N_A number of voltage-controlled voltage sources,
U_{zi} equivalent noise voltage of voltage-controlled voltage sources,
U_{2i} output voltage of voltage-controlled voltage sources.

Minimizing the output noise/signal ratio according to Eq. (37) requires the minimization of cost function of the type:

$$\Phi = \sum_{i=1}^{N} k_i^2 |S_i^r|^2 \tag{38}$$

The theoretical limitations of this problem have been dealt with in connection with case (iii). As to the optimization of sensitivity, in this concern the minimization problem of type

$$\Phi = \sum_{i=1}^{N} |S_i^r|^2 \tag{39}$$

dealt with in (i) is usual. In each case, the theoretical minimum is obtained with different relative sensitivities and thus with different values of the circuit elements. This is a theoretical confirmation to the empirical fact that in case of circuits with identical structure the optimization for sensitivity and the optimization for noise/signal ratio lead to different results.

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Summary

The sensitivities S_i of the network characteristic $y = y(x_1, \ldots x_i \ldots x_n)$ for various circuit parameters are not independent of each other. Namely, the sum of relative sensitivities S_i^r is invariant. Considering the dimensional circuit parameters as vector x and the network characteristic y as a scalar-vector function, the level surfaces and the gradient vector s can be introduced. They permit simple demonstration of the summed sensitivity invariants.

In sensitivity optimization the summed sensitivity invariance means limitation. The theoretical limitation can be determined by the Lagrange multiplier method, usual for the solution of limited extremum problems. Several cost functions are here distinguished: (i) the sum of the squares of absolute values; (ii) the sum of absolute values; (iii) the weighted sum of squares of absolute values; (iv) the general case. The results accessible to mathematical interpretation give the condition of the existence of minimum and its value. The conditions obtained are theoretical limitations and are impossible in real circumstances of practical importance.

Beside statistic dimensioning and worst-case dimensioning of tolerances the chosen cost functions occur also in calculating the output noise/signal ratio of active RC circuits. The results obtained prove theoretically that the network which is optimum for sensitivity differs from the circuit which is optimum for noise.

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Dr. Károly GÉHER, H-1521 Budapest