DERIVING A GENERAL ROTATING-TO-STATIONARY TRANSFORMATION

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The problem of a priori deriving a special transformation for transforming a time-dependent differential equation of a uniform air-gap balanced two-phased machine to a time-invariant one, was discussed in a previous paper [1]. A similar method yields a more general transformation which transforms the time-dependent differential equations of a general machine [3] to time-invariant ones. A general transformation matrix independent of the machines' impedance is derived and with special assumptions this theory is shown to lead to γ, δ and other transformations.

1. Stating the problem

The differential equation of a general electric machine [3] can be written as

$$\dot{x}(t) = \mathbf{A}(t)x(t) + u(t) \tag{1}$$

where $\mathbf{A}$ is a time-dependent matrix depending on $\mathbf{R}$ and $\mathbf{L}$ the resistance and inductance matrix of the machine, respectively, x depends on $\mathbf{L}$ and the column vector of currents i and u is the column vector of the voltages. Time dependence of the differential equation is due to the machine itself, so it is useful to transform the original machine into a dq machine with stationary axes with differential equations time-invariant.

By transforming (1) and substituting

$$y = \mathbf{T}x \tag{2}$$

where $\mathbf{T}$ is the transformation matrix, we get:

$$\dot{y}(t) = (\dot{\mathbf{T}}(t) \cdot \mathbf{T}^{-1}(t) + \mathbf{T}(t)\mathbf{A}(t)\mathbf{T}(t)^{-1}y(t) + \mathbf{T}u(t)\,. \tag{3}$$

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From (3) it is evident that the differential equation of the machine is time-invariant if

$$\mathbf{B} = \dot{\mathbf{T}}\mathbf{T}^{-1} + \mathbf{TAT}^{-1} \tag{4}$$

is a constant matrix.

To derive a transformation applicable both for uniform air-gap machines and salient-pole machines the transformation matrix must not depend on the machines impedance matrix. From (4) it is clear that this is the case if the impedance matrix has a special form. Thus not all types of differential equations of a general machine can be changed from time-dependent to a time-invariant one by an impedance-independent transformation.

2. The generalized transformation

From (2) and (4) it follows that

$$\dot{\mathbf{T}} = \mathbf{BT} - \mathbf{TA}\,, \tag{5}$$

is a solution meeting the boundary condition $\mathbf{T}(t_0) = \mathbf{T}_0$. If $\varphi_1(t, t_0)$ is the transition matrix of

$$\dot{\mathbf{T}} = \mathbf{BT} \tag{5a}$$

and $\varphi_2(t, t_0)$ is the transition matrix of

$$\dot{\mathbf{T}} = -\mathbf{A}_t\mathbf{T} \tag{5b}$$

where the subscript t denotes the transpose,

$$\mathbf{T}(t) = \varphi_1(t, t_0)\mathbf{T}(t_0)\varphi_{2t}(t, t_0)\,. \tag{6}$$

Using the Peano—Baker series expansion, the transformation matrix becomes

$$\mathbf{T}(t) = [\mathbf{E} + \mathbf{B}(t - t_0) + \mathbf{B}^2(t - t_0)/2! + \ldots]\,\mathbf{T}_0\Big\{\mathbf{E} - \Big[\int_{t_0}^{t} \mathbf{A}_t(\tau_1)d\tau_1 + \int_{t_0}^{t} \mathbf{A}_t(\tau_1)\int_{t_0}^{\tau_1} \mathbf{A}(\tau_2)d\tau_2 d\tau_1 + \ldots\Big]\Big\} \tag{7}$$

where $\mathbf{E}$ is an identity matrix and $\mathbf{B}$ is a real constant matrix. This transformation can also be used if the impedance matrix consists of harmonic fields. The transformed differential equations become time-invariant and are easy to solve.

For $t_0 = 0$

$$\mathbf{T}(t) = \exp [\mathbf{B}t]\mathbf{T}_0\{\mathbf{E} - [\int_0^t \mathbf{A}_t(\tau_1)d\tau_1 + \int_0^t \mathbf{A}_t(\tau_1) \int_0^t \mathbf{A}_t(\tau_2)d\tau_2 d\tau_1 + \ldots]\}, \quad (8)$$

and the corresponding impedance invariant transformation:

$$\mathbf{T}(t) = \exp [\mathbf{k}_1 t]\mathbf{T}_0 \quad (9)$$

where

$$\mathbf{k}_1 = \dot{\mathbf{T}} \cdot \mathbf{T}^{-1} \quad (10)$$

is a constant matrix.

With the assumptions made above, according (9) not all transformations are impedance-independent in view of that in general case **B** can be also a constant matrix if

$$\dot{\mathbf{T}}(t) \cdot \mathbf{T}(t)^{-1} = \mathbf{k}_1(t) \quad (11)$$

$$\mathbf{T}(t)\,\mathbf{A}(t)\,\mathbf{T}(t)^{-1} = \mathbf{k}_2(t) \quad (12)$$

where $\mathbf{k}_1(t)$ and $\mathbf{k}_2(t)$ are time-dependent matrices.

Hence the general impedance-dependent transformation is:

$$\mathbf{T}(t) = \{\mathbf{E} + \int_{t_0}^t [\mathbf{k}_1(\tau_1) + \mathbf{k}_2(\tau_2)]d\tau_1 + \int_{t_0}^t [\mathbf{k}_1(\tau_1) + \mathbf{k}_2(\tau_1)] \int_{t_0}^{\tau_1} [\mathbf{k}_1(\tau_2) +$$
$$+ \mathbf{k}_2(\tau_2)]d\tau_2 d\tau_1 + \ldots\}\,\mathbf{T}(t_0) \cdot \{\mathbf{E} - [\int_{t_0}^t \mathbf{A}_t(\tau_1)d\tau_1 +$$
$$+ \int_{t_0}^t \mathbf{A}_t(\tau_1) \int_{t_0}^{\tau_1} \mathbf{A}_t(\tau_2)d\tau_2\, d\tau_1 + \ldots]\} \quad (13)$$

and the general impedance-independent similarity transformation is:

$$\mathbf{T}(t) = [\mathbf{E} + \int_{t_0}^t \mathbf{k}_1(\tau_1)d\tau_1 + \int_{t_0}^t \mathbf{k}_1(\tau_1) \int_{t_0}^{\tau_1} \mathbf{k}_1(\tau_2)d\tau_2\, d\tau_1 + \ldots]\mathbf{T}_0. \quad (14)$$

In a specific case where $\mathbf{k}_1(\tau_1)$ and its integral commute

$$\mathbf{T}(t) = \exp [\int_{t_0}^t \mathbf{k}_1(\tau_1)d\tau_1]\mathbf{T}_0 \quad (15)$$

which results (9) if $\mathbf{k}_1$ is a constant matrix, and $t_0 = 0$. Considering (12) we get for $\mathbf{A}(t)$ in it case governed by (15):

$$\mathbf{A}(t) = \exp [- \int_{t_0}^t \mathbf{k}_1(\tau)d\tau]\mathbf{k}_2(\tau) \exp [\int_{t_0}^t \mathbf{k}_1(\tau)d\tau]\,. \quad (16)$$

3. The generalized γ, δ rotating real transformation

Let us consider a balanced two-phase salient-pole machine, with symmetrical two-phase stationary windings and α, β rotating windings, where the stator is of the salient-pole type. Let Ω designate the speed of the machine looked at from an arbitrary reference frame rotating at speed α_1 referred to the stationary windings dq. If the rotating reference frame $\alpha\beta$ rotates at 2_1 speed referred to axes dq, then $\Omega(t) = d/dt\ \xi(t)$ where $\xi(t) = \alpha_1(t) - \psi(t)$. For simplicity, a two-pole machine is taken into consideration, but the analysis applies to any number of pole-pairs The magnetic-flux densities are assumed to be proportional to the mmfs so superposition can be applied, i.e. saturation is neglected. Also the space harmonics of mmf are neglected, only the fundamental wave is taken into consideration.

From (1) we get

$$\frac{d}{dt}[\mathbf{L}(t)i(t)] = -\mathbf{R}\frac{1}{k_{11}}[l_1, l_2, l_3, l_4]\mathbf{L}(t)i(t) + u(t) \tag{17}$$

where the phase current and source vectors are

$$i = \begin{bmatrix} i_a(t) \\ i_b(t) \\ i_\gamma(t) \\ i_\delta(t) \end{bmatrix} \qquad u = \begin{bmatrix} u_a(t) \\ u_b(t) \\ u_\gamma(t) \\ u_\delta(t) \end{bmatrix} \tag{18}$$

and the inductance matrix and resistance matrices are

$$\mathbf{L}(t) = \begin{bmatrix} L_{aa} & 0 & L_{am}\cos\xi & L_{am}\sin\xi \\ 0 & L_{bb} & -L_{bm}\sin\xi & L_{bm}\cos\xi \\ L_{am}\cos\xi & -L_{bm}\sin\xi & L_d\cos^2\xi + L_q\sin^2\xi & (L_d - L_q)\cos\xi\sin\xi \\ L_{am}\sin\xi & L_{bm}\cos\xi & (L_d - L_q)\cos\xi\sin\xi & L_d\sin^2\xi + L_q\cos^2\xi \end{bmatrix} \tag{19}$$

respectively where

L_{aa} and L_{bb} are the self-inductances of windings a and b
L_{am} is the mutual inductance between winding a and α
L_{bm} is the mutual inductance between winding a and β
L_d and L_q are constant inductances

and

$$\mathbf{R} = \begin{bmatrix} R_s & 0 & 0 & 0 \\ 0 & R_s & 0 & 0 \\ 0 & 0 & R_r & 0 \\ 0 & 0 & 0 & R_r \end{bmatrix} \tag{20}$$

where R_s and R_r are resistances of the stator coils, and of the rotor coils, respectively.

All the other quantities are defined in Appendix (1). Our purpose is to transform (17) into a time-invariant differential equation.

From (17) the necessary decomposition of hipermatrix is seen to exist, as

$$-\mathbf{A}(t) = \begin{bmatrix} \mathbf{E}_2 & \mathbf{O}_2 \\ \mathbf{O}_2 & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix} \begin{bmatrix} \mathbf{E}_2 & \mathbf{O}_2 \\ \mathbf{O}_2 & \mathbf{F}_t \end{bmatrix} \tag{21}$$

is in the desired form (16). Submatrices are found in Appendix (2).

As the transformation $\mathbf{T}$ changed the rotating components into stationary ones, and in electrical machine theory the transformation matrix is defined in a way to change the new components into old ones, the general rotating real transformation is

$$\mathbf{C}_{\gamma\delta} = \mathbf{T}^{-1}. \tag{22}$$

The transformation matrix $\mathbf{T}$ is obtained by considering (15) and Appendix (2), therefore using (22):

$$\mathbf{C}_{\gamma\delta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\xi & -\sin\xi \\ 0 & 0 & \sin\xi & \cos\xi \end{bmatrix}. \tag{23}$$

Considering (1), (2), (17) and (23), the transformations for the currents and voltages are

$$\begin{bmatrix} u_a \\ u_b \\ u_\gamma \\ u_\delta \end{bmatrix} = \mathbf{C}_{\gamma\delta}^{-1} \begin{bmatrix} u_a \\ u_b \\ u_d \\ u_q \end{bmatrix} \qquad \begin{bmatrix} i_a \\ i_b \\ i_\gamma \\ i_\delta \end{bmatrix} = \mathbf{C}_{\gamma\delta}^{-1} \begin{bmatrix} i_a \\ i_b \\ i_d \\ i_q \end{bmatrix}. \tag{24}$$

4. The commutator and special brush-shifting transformation

From (23), considering $\xi(t) = \alpha_1(t) - \psi_1(t)$, the rotating real transformation can be decomposed into the product of two matrices:

$$\mathbf{C}_{\gamma\delta} = \mathbf{C}_2 \mathbf{C}_{4s} \tag{25}$$

where $\mathbf{C}_2$ is the commutator transformation matrix:

$$\mathbf{C}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha_1 & -\sin\alpha_1 \\ 0 & 0 & \sin\alpha_1 & \cos\alpha_1 \end{bmatrix}. \tag{26}$$

and $\mathbf{C}_{4S}$ is the special case of the brush-shifting transformation matrix:

$$\mathbf{C}_{4S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\psi_1 & \sin\psi_1 \\ 0 & 0 & -\sin\psi_1 & \cos\psi_1 \end{bmatrix}. \tag{27}$$

Assuming the elements of $\mathbf{C}_{4S}$ to be constant, it follows from the interpretation of $\mathbf{C}_{\gamma\delta}$ that the commutator transformation changes the machine with rotating axes α, β into one with stationary axes dq.

5. Conclusions

A generalized transformation has been derived for transforming the time-dependent differential equation of a general machine. Also by setting up the equations for a two-phase salient-pole machine and using the special case of the derived transformation, γ, δ transformation has been derived.

With other assumptions it can be shown how this theory can lead to other transformations used in electrical machines theory.

6. Appendix 1

Elements of the inverse inductance matrix are:

$$k_{11} = L_{aa}L_{bb}L_dL_q - L_{aa}L_{bm}^2L_d - L_{am}^2L_{bb}L_q + L_{am}^2L_{bm}^2$$

$$l_1 = \begin{bmatrix} L_{bb}L_dL_q - L_{bm}^2L_d \\ 0 \\ \cos\xi(-L_{am}L_{bb}L_q + L_{am}L_{bm}^2) \\ \sin\xi(-L_{bb}L_{am}L_q + L_{am}L_{bm}^2) \end{bmatrix}; \quad l_2 = \begin{bmatrix} 0 \\ L_{aa}L_dL_q - L_{am}^2L_q \\ \sin\xi(-L_{aa}L_{bm}L_d + L_{am}^2L_{bm}) \\ \cos\xi(-L_{aa}L_{bm}L_d + L_{am}^2L_{bm}) \end{bmatrix};$$

$$l_3 = \begin{bmatrix} \cos\xi(-L_{bb}L_{am}L_q + L_{bm}^2L_{am}) \\ \sin\xi(-L_{aa}L_{bm}L_d + L_{am}^2L_{bm}) \\ \cos^2\xi(L_{aa}L_{bb}L_q - L_{aa}L_{bm}^2) + \sin^2\xi(L_{aa}L_{bb}L_d - L_{am}^2L_{bb}) \\ \sin\xi\cos\xi[L_{aa}L_{bb}(L_q - L_d) - L_{aa}L_{bm}^2 + L_{bb}L_{am}^2) \end{bmatrix};$$

$$l_4 = \begin{bmatrix} \sin\xi(-L_{bb}L_{am}L_q + L_{bm}^2L_{am}) \\ \cos\xi(-L_{aa}L_{bm}L_d + L_{am}^2L_{bm}) \\ \sin\xi\cos\xi[L_{aa}L_{bb}(L_q - L_d) - L_{aa}L_{bm}^2 + L_{bb}L_{am}^2] \\ \cos^2\xi(L_{aa}L_{bb}L_d - L_{am}^2L_{bb}) + \sin^2\xi(L_{aa}L_{bb}L_d - L_{bm}^2L_{aa}) \end{bmatrix}.$$

Appendix 2

The submatrices of A(t):
$\mathbf{E}_2$ is an identity matrix of second order,
$\mathbf{O}_2$ is a zero matrix of second order,
$\mathbf{F}$ is a time-dependent matrix defined as

$$\mathbf{F} = \begin{bmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{bmatrix}$$

matrices $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$, $\mathbf{D}_1$ are constant diagonal matrices;

$$\mathbf{A}_1 = \begin{bmatrix} aR_s & 0 \\ 0 & cR_s \end{bmatrix}; \quad \mathbf{B}_1 = \begin{bmatrix} bR_s & 0 \\ 0 & dR_s \end{bmatrix}; \quad \mathbf{C}_1 = \begin{bmatrix} bR_r & 0 \\ 0 & dR_r \end{bmatrix}; \quad \mathbf{D}_1 = \begin{bmatrix} eR_r & 0 \\ 0 & fR_r \end{bmatrix};$$

where the constant elements are

$$a = (L_{bb}L_dL_q - L_{bm}^2L_d)k_{11}^{-1} \qquad b = (-L_{bb}L_{am}L_q + L_{bm}^2L_{am})k_{11}^{-1}$$

$$c = (L_{aa}L_dL_q - L_{am}^2L_q)k_{11}^{-1} \qquad d = (-L_{aa}L_{bm}L_d + L_{am}^2L_{bm})k_{11}^{-1}$$

$$e = (L_{aa}L_{bb}L_q - L_{aa}L_{bm}^2)k_{11}^{-1} \qquad f = (L_{aa}L_{bb}L_d - L_{am}^2L_{bb})k_{11}^{-1}$$

Summary

The problem was to a priori derive a transformation for transforming a time-dependent differential equation of a general electric machine to time-invariant one.

The general transformation matrix independent of the machines impedance is derived and with special assumptions it is shown how the application of this theory leads to some transformations well-known in electrical machine theory.

References

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