REMARK TO THE ASYMPTOTIC PROPERTIES OF THE ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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This note has been dealt with the asymptotic representation of the orthogonal polynomials on the unit circle corresponding to the Jacobi polynomials. This problem has already been approached in a previous paper [4]. The results detailed there will be developed further.

First of all, let us refer to a misprint in [4]. Formula [4] (1.23) reads correctly as follows:

$$\left(\sin\frac{\theta}{2}\right)^{a}\left(\cos\frac{\theta}{2}\right)^{b} P_n^{(a,b)}(\cos\theta) = \left(n+\frac{a+b+1}{2}\right)^{-a} \times$$
$$\times \frac{\Gamma(n+a+1)}{n!}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_a\left(\left[n+\frac{a+b+1}{2}\right]\theta\right)+ \tag{1}$$
$$+\begin{cases}\theta^{1/2}O(n^{-3/2}) & \text{if} \quad cn^{-1}\leqq\theta\leqq\pi-\varepsilon\\ \theta^{a+2}O(n^{a}) & \text{if} \quad 0<\theta\leqq cn^{-1},\end{cases}$$

where $J_a(z)$ stands for the Bessel function of the first kind with index a. Thus using the correct formula (1), in [4] (1.26) and (1.27) are the following:

$$p_n(d\alpha,\cos\theta) = \frac{n^{1/2}}{2^{\frac{a+b+1}{2}}}\,\theta^{1/2}\left(\sin\frac{\theta}{2}\right)^{-a-1/2}\left(\cos\frac{\theta}{2}\right)^{-b-1/2} \times$$
$$\times J_a\left(\left[n+\frac{a+b+1}{2}\right]\theta\right)\left[1+O\left(\frac{1}{n}\right)\right]+ \tag{2}$$
$$+\left(\sin\frac{\theta}{2}\right)^{-a}\left(\cos\frac{\theta}{2}\right)^{-b}\begin{cases}\theta^{1/2}\,O(n^{-1}) & \text{if} \quad cn^{-1}\leqq\theta\leqq\pi-\varepsilon\\ \theta^{a+2}\,O(n^{a+1/2}) & \text{if} \quad 0<\theta\leqq cn^{-1}\end{cases},$$

$$p_n(d\beta,\cos\theta) = \frac{n^{1/2}}{2^{\frac{a+b+3}{2}}}\,\theta^{1/2}\left(\sin\frac{\theta}{2}\right)^{-a-3/2}\left(\cos\frac{\theta}{2}\right)^{-b-3/2} \times$$
$$\times J_a\left(\left[n+\frac{a+b+3}{2}\right]\theta\right)\left[1+O\left(\frac{1}{n}\right)\right]+ \tag{3}$$
$$+\left(\sin\frac{\theta}{2}\right)^{-a-1}\left(\cos\frac{\theta}{2}\right)^{-b-1}\begin{cases}\theta^{1/2}\,O(n^{-1}) & \text{if} \quad cn^{-1}\leqq\theta\leqq\pi-\varepsilon\\ \theta^{a+3}\,O(n^{a+3/2}) & \text{if} \quad 0<\theta\leqq cn^{-1}\end{cases}.$$

Finally resulting in:

$$\Phi_{2n}(d\mu_1;e^{i\theta}) = \frac{e^{in\theta}}{2}\sqrt{2\pi}\Bigg[\frac{n^{1/2}}{2^{\frac{a+b+1}{2}}}\,\theta^{1/2}\left(\sin\frac{\theta}{2}\right)^{-a-1/2}\left(\cos\frac{\theta}{2}\right)^{-b-1/2}\times$$

$$\times J_a\left(\left[n+\frac{a+b+1}{2}\right]\theta\right)(1+i)+\theta^{1/2}\left(\sin\frac{\theta}{2}\right)^{-a-1/2}\left(\cos\frac{\theta}{2}\right)^{-b-1/2}\times$$

$$\times J_a\left(\left[n+\frac{a+b+1}{2}\right]\theta\right)\left(O(n^{-1/2})+iO(n^{-1/2})\right)+ \tag{4}$$

$$+\left(\sin\frac{\theta}{2}\right)^{-a}\left(\cos\frac{\theta}{2}\right)^{-b}\begin{cases}\theta^{1/2}[O(n^{-1})+iO(n^{-1})] & \text{if } cn^{-1}\leqq\theta\leqq\pi-\varepsilon\\ \theta^{a+2}O(n^{a+1/2})(1+i\theta O(n)) & \text{if } 0<\theta\leq cn^{-1}\end{cases}\Bigg],$$

$$\Phi_{2n-1}(d\mu_1;e^{i\theta}) = \frac{e^{i(n-1)\theta}}{2}\sqrt{2\pi}\Bigg[\frac{n^{1/2}}{2^{\frac{a+b+1}{2}}}\,\theta^{1/2}\left(\sin\frac{\theta}{2}\right)^{-a-1/2}\left(\cos\frac{\theta}{2}\right)^{-b-1/2}\times$$

$$\times J_a\left(\left[n+\frac{a+b+1}{2}\right]\theta\right)(1+i)+\theta^{1/2}\left(\sin\frac{\theta}{2}\right)^{-a-1/2}\left(\cos\frac{\theta}{2}\right)^{-b-1/2}\times$$

$$\times J_a\left(\left[n+\frac{a+b+1}{2}\right]\theta\right)\left(O(n^{-1/2})+iO(n^{-1/2})\right)+ \tag{5}$$

$$+\left(\sin\frac{\theta}{2}\right)^{-a}\left(\cos\frac{\theta}{2}\right)^{-b}\begin{cases}\theta^{1/2}[O(n^{-1})+iO(n^{-1})] & \text{if } cn^{-1}\leqq\theta\leqq\pi-\varepsilon\\ \theta^{a+2}O(n^{a+1/2})[O(1)+i\theta O(n)] & \text{if } 0<\theta\leq cn^{-1}\end{cases}\Bigg].$$

With notations in [4], now it will be shown that results of [4] can be improved, more precisely refined.

Using the relations obtained in [3]:

$$p_n(d\alpha,x) = \frac{1}{\sqrt{2\pi}}\left[1+\frac{\Phi_{2n}(d\mu_1,0)}{\varkappa_{2n}(d\mu_1)}\right]^{-1/2}[z^{-n}\Phi_{2n}(d\mu_1,z)+z^n\Phi_{2n}(d\mu_1,z^{-1})] =$$

$$= \frac{1}{\sqrt{2\pi}}\left[1-\frac{\Phi_{2n}(d\mu_1,0)}{\varkappa_{2n}(d\mu_1)}\right]^{-1/2}[z^{-n+1}\Phi_{2n-1}(d\mu_1,z)+z^{n-1}\Phi_{2n-1}(d\mu_1,z^{-1})], \tag{6}$$

$$p_{n-1}(d\beta,x) = \sqrt{\frac{2}{\pi}}\left[1-\frac{\Phi_{2n}(d\mu_1,0)}{\varkappa_{2n}(d\mu_1)}\right]^{-1/2}\frac{z^{-n+1}\Phi_{2n}(d\mu_1,z)-z^{n-1}\Phi_{2n}(d\mu_1,z^{-1})}{z-z^{-1}} =$$

$$= \sqrt{\frac{2}{\pi}}\left[1+\frac{\Phi_{2n}(d\mu_1,0)}{\varkappa_{2n}(d\mu_1)}\right]^{-1/2}\frac{z^{-n+1}\Phi_{2n-1}(d\mu_1,z)-z^{n-1}\Phi_{2n-1}(d\mu_1,z^{-1})}{z-z^{-1}} \tag{7}$$

we have

$$\Phi_{2n}(d\mu_1,z) = \frac{z^n}{2}\left[\sqrt{2\pi}\, p_n(d\alpha,x)\left(1+\frac{\Phi_{2n}(d\mu_1,0)}{\varkappa_{2n}(d\mu_1)}\right)^{1/2} + \right.$$
$$\left. + \sqrt{\frac{\pi}{2}}\, p_{n-1}(d\beta,x)\left(1-\frac{\Phi_{2n}(d\mu_1,0)}{\varkappa_{2n}(d\mu_1)}\right)^{1/2}(z-z^{-1})\right], \tag{8}$$

$$\Phi_{2n-1}(d\mu_1,z) = \frac{z^{n-1}}{2}\left[\sqrt{2\pi}\, p_n(d\alpha,x)\left(1-\frac{\Phi_{2n}(d\mu_1,0)}{\varkappa_{2n}(d\mu_1)}\right)^{1/2} + \right.$$
$$\left. + \sqrt{\frac{\pi}{2}}\, p_{n-1}(d\beta,x)\left(1+\frac{\Phi_{2n}(d\mu_1,0)}{\varkappa_{2n}(d\mu_1)}\right)^{1/2}(z-z^{-1})\right]. \tag{9}$$

It is easy to see that for the asymptotic behaviour of functions $\Phi_{2n}(d\mu_1, z)$ and $\Phi_{2n-1}(d\mu_1, z)$ only the evaluation of quotient $\Phi_{2n}(d\mu_1, 0)/\varkappa_{2n}(d\mu_1)$ is needed. Since $\Phi_{2n}(d\mu_1, 0)/\varkappa_{2n}(d\mu_1)$ can be exactly determined, thus, instead of (4) and (5), arbitrary exact formulas can be obtained. The asymptotic examination of $\Phi_n(d\mu_1, z)$ can be reduced to the asymptotic behaviour of $p_n(d\alpha, x)$. (This examination for $p_n(d\alpha, x)$ is a problem already solved.)

Indeed, by using formulas [4] (1.4), (1.5), (1.11) and (1.12) we obtain

$$\frac{\Phi_{2n}(d\mu_1,0)}{\varkappa_{2n}(d\mu_1)} = \frac{a+b+1}{a+b+2n+1}.$$

Finally

$$\Phi_{2n}(d\mu_1,z) = \frac{z^n}{2}\left[\sqrt{2\pi}\left(\frac{2(n+a+b+1)}{2n+a+b+1}\right)^{1/2} p_n(d\alpha,x) + \right.$$
$$\left. + \sqrt{\frac{\pi}{2}}\left(\frac{2n}{2n+a+b+1}\right)^{1/2}(z-z^{-1})p_{n-1}(d\beta,x)\right],$$

$$\Phi_{2n-1}(d\mu_1,z) = \frac{z^{n-1}}{2}\left[\sqrt{2\pi}\left(\frac{2n}{2n+a+b+1}\right)^{1/2} p_n(d\alpha,x) + \right.$$
$$\left. + \sqrt{\frac{\pi}{2}}\left(\frac{2(n+a+b+1)}{2n+a+b+1}\right)^{1/2}(z-z^{-1})p_{n-1}(d\beta,x)\right].$$

Summary

This note is closely connected with a previous paper [4], where it has been shown that for the study of the asymptotic behaviour of the orthogonal polynomials on the unit circle corresponding to the Jacobi polynomials only the asymptotics of the Jacobi polynomials orthogonal on [−1, 1] are needed.

References

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