A SYSTEMATIZATION OF CANONICAL TRANSFORMATIONS

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1. Introduction

This paper concerns time-invariant, lumped parameter, single-variable linear systems.

Canonical transformations are understood as those which change some kind of a phase-variable form into a variant of Jordan canonical form. As there are at least four Jordan forms and four phase-variable forms, we have sixteen possibilities. In this paper, the solution of the sixteen problems will be dealt with, not restricted to the case of distinct eigenvalues but allowing many repeated eigenvalues, too.

In the treatise of the above mentioned problems some new conversions will be introduced, for example, besides the transposes of matrices also the so-called anti-transposes will be defined. Systematizing the matter, first it will be shown how the various phase-variable forms can be transformed into one another, then some remarks will be made about Jordan canonical forms, finally transformation from phase-variable forms to Jordan canonical forms will be presented.

In the Appendix the results of a previous paper will be summarized for convenience to enlighten the essence of the canonical transformations. The canonical transformations, especially in the case of repeated eigenvalues are somewhat complicated. This is the reason why their systematization will bring some new results.

2. Variants of the Phase-variable Forms

The first case, that is, the *principal variant* of phase-variable forms is

$$\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x} + \mathbf{b}_0 u \qquad (1)$$

$$y = \mathbf{c}_0^T\mathbf{x} + du$$

where $\mathbf{x}$ is an $n \times 1$ matrix (a vector), u and y are scalars and

$$\mathbf{A}_0 \triangleq \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}; \quad \mathbf{b}_0 \triangleq \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{c}_0^{\mathrm{T}} \triangleq [\, b_0, \; b_1, \; b_2, \dots, \; b_{n-2}, \; b_{n-1} \,]; \quad d \triangleq b_n \tag{2}$$

For the sake of simplicity, $a_n = 1$ was assumed here and in the continuation and the transfer function was assumed in the form

$$G(s) = \frac{b_{n-1} s^{n-1} + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} + b_n$$

The second case, that is, *the first particular variant* of phase variable forms is defined as

$$\begin{aligned} \dot{\mathbf{x}}' &= \mathbf{A}_1 \mathbf{x}' + \mathbf{b}_1 u \\ y &= \mathbf{c}_1^{\mathrm{T}} \mathbf{x}' + du \end{aligned} \tag{3}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ 0 & 0 & \dots & 0 & -a_{n-2} \\ 0 & 0 & \dots & 1 & -a_{n-2} \end{bmatrix}; \quad \mathbf{b}_1 \triangleq \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{n-2} \\ b_{n-1} \end{bmatrix}$$

$$\mathbf{c}_1^{\mathrm{T}} \triangleq [\, 0, \; 0, \dots, \; 0, \; 1 \,]; \quad d \triangleq b_n \tag{4}$$

The third case, that is, *the second particular variant* of phase-variable forms is defined as

$$\begin{aligned} \dot{\mathbf{x}}'' &= \mathbf{A}_2 \mathbf{x}'' + \mathbf{b}_2 u \\ y &= \mathbf{c}_2^{\mathrm{T}} \mathbf{x}'' + du \end{aligned} \tag{5}$$

where

$$\mathbf{A}_2 \triangleq \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}; \quad b_2 \triangleq \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{c}_2^{\mathrm{T}} \triangleq [\ b_{n-1},\ b_{n-2},\ \ldots\ b_2,\ b_1,\ b_0\ ]; \quad d \triangleq b_n \tag{6}$$

Finally, the fourth case, that is, *the third particular variant* of phase-variable form is defined as

$$\begin{aligned} \dot{\mathbf{x}}''' &= \mathbf{A}_3\mathbf{x}''' + \mathbf{b}_3 u \\ y &= \mathbf{c}_3^{\mathrm{T}}\mathbf{x}''' + du \end{aligned} \tag{7}$$

where

$$\mathbf{A}_3 \triangleq \begin{bmatrix} -a_{n-1} & 1 & \cdots & 0 & 0 \\ -a_{n-2} & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ -a_2 & 0 & \cdots & 1 & 0 \\ -a_1 & 0 & \cdots & 0 & 1 \\ -a_0 & 0 & \cdots & 0 & 0 \end{bmatrix}; \quad \mathbf{b}_3 \triangleq \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \cdot \\ \cdot \\ \cdot \\ b_2 \\ b_1 \\ b_0 \end{bmatrix} \tag{8}$$

$$\mathbf{c}_3^{\mathrm{T}} \triangleq [\ 1,\ 0,\ \ldots,\ 0,\ 0\ ]; \quad d \triangleq b_n$$

Let us remark that in all the four canonical forms both the input vectors **b** and the output vectors **c** are of special form. One vector has components which are all zero but one, the latter being unity. The components of the other vector are equal to the coefficients of the numerator polynomial in the transfer function.

Sometimes the forms in Eqs. (1) and (5) are called controllability forms because the controllability matrix

$$\mathbf{Q}_c = [\mathbf{b}, \mathbf{A}\,\mathbf{b}, \ldots, \mathbf{A}^{n-1}\,\mathbf{b}]$$

is relatively simple, it is a triangular matrix with unity diagonal elements, so its determinant is unity. The forms in Eqs. (3) and (7) are called, on the hand, observability forms, because in this case the observability matrix

$$\mathbf{Q}_o = [\mathbf{c}, \mathbf{A}^{\mathrm{T}}\,\mathbf{c}, \ldots, [\mathbf{A}^{\mathrm{T}}]^{n-1}\,\mathbf{c}]$$

is of triangular form with unity diagonal elements.

For the sake of systematization let us introduce — besides the *transpose* of a matrix — its *anti-transpose*. While the former is obtained by reflecting the entries of a certain matrix about the main diagonal, the latter is found by reflecting the entries about the subsidiary diagonal. If

$$\mathbf{A} = [a_{ij}] \tag{9}$$

then the definition of the transpose is

$$\mathbf{A}^{\mathrm{T}} = [a_{ji}] \tag{10}$$

whereas the definition of the anti-transpose is

$$\mathbf{A}^{\perp} \triangleq [a_{n-j+1,n-i+1}], \tag{11}$$

Clearly,

$$\begin{gathered}[\mathbf{A}^{\mathrm{T}}]^{\mathrm{T}} = \mathbf{A};\quad [\mathbf{A}^{\perp}]^{\perp} = \mathbf{A};\quad \mathbf{A}^{\mathrm{T}\perp} = \mathbf{A}^{\perp\mathrm{T}} \\ [\mathbf{A}^{\mathrm{T}\perp}]^{\mathrm{T}} = \mathbf{A}^{\perp};\quad [\mathbf{A}^{\perp\mathrm{T}}]^{\perp} = \mathbf{A}^{\mathrm{T}};\quad [\mathbf{A}^{\mathrm{T}\perp}]^{\mathrm{T}\perp} = \mathbf{A}\,.\end{gathered} \tag{12}$$

Now, orthogonal (involutary) $n \times n$ matrices will be introduced:

$$\mathbf{E} \triangleq \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{13}$$

Obviously,

$$\mathbf{E} = \mathbf{E}^{-1} = \mathbf{E}^{\mathrm{T}} = \mathbf{E}^{\perp} \tag{14}$$

and

$$\mathbf{E}\mathbf{E}^{-1} = \mathbf{E}\mathbf{E}^{\mathrm{T}} = \mathbf{E}\mathbf{E}^{\perp} = \mathbf{E}\mathbf{E} = \mathbf{I}\,. \tag{15}$$

In a similar manner augmented $(n+1)\times(n+1)$ involutary (orthogonal) matrices $\tilde{\mathbf{E}}$ may be defined with the same properties as expressed in Eqs (14) and (15), that is,

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}^{-1} = \tilde{\mathbf{E}}^{\mathrm{T}} = \tilde{\mathbf{E}}^{\perp} \tag{16}$$

and

$$\tilde{\mathbf{E}}\tilde{\mathbf{E}}^{-1} = \tilde{\mathbf{E}}\tilde{\mathbf{E}}^{\mathrm{T}} = \tilde{\mathbf{E}}\tilde{\mathbf{E}}^{\perp} = \tilde{\mathbf{E}}\tilde{\mathbf{E}} = \tilde{\mathbf{I}} \tag{17}$$

The interchangibility (commutativity) of $\mathbf{E}$ and $\mathbf{E}^{-1}$ and $\mathbf{E}^{\mathrm{T}}$ and $\mathbf{E}^{\perp}$ or $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{E}}^{-1}$ and $\tilde{\mathbf{E}}^{\mathrm{T}}$ and $\tilde{\mathbf{E}}^{\perp}$ is also evident.

Multiplying from the right side, that is, post-multiplication by $\tilde{\mathbf{E}}$ interchanges the columns whereas multiplying from the left side, that is, premultiplication interchanges the rows of the multiplicand matrix.

The final step in our systematization is made by introducing $(n+1)\times(n+1)$ hyper-matrices as follows:

$$\tilde{\mathbf{A}}_0 \triangleq \begin{bmatrix} \mathbf{A}_0 & \mathbf{b}_0 \\ \mathbf{c}_0^{\mathrm{T}} & d_1 \end{bmatrix}; \tag{18}$$

$$\tilde{\mathbf{A}}_1 \triangleq \begin{bmatrix} \mathbf{A}_1 & \mathbf{b}_1 \\ \mathbf{c}_1^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_0^{\mathrm{T}} & \mathbf{c}_0 \\ \mathbf{b}_0^{\mathrm{T}} & d \end{bmatrix} = \tilde{\mathbf{A}}_0^{\mathrm{T}} \tag{19}$$

$$\tilde{\mathbf{A}}_2 \triangleq \begin{bmatrix} d & \mathbf{c}_2^{\mathrm{T}} \\ \mathbf{b}_2 & \mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} d & \mathbf{c}_0^{\perp} \\ \mathbf{b}_0^{\mathrm{T}\perp} & \mathbf{A}_0^{\mathrm{T}\perp} \end{bmatrix} = \tilde{\mathbf{A}}_0^{\mathrm{T}\perp} \tag{20}$$

$$\tilde{\mathbf{A}}_3 \triangleq \begin{bmatrix} d & \mathbf{c}_3^{\mathrm{T}} \\ \mathbf{b}_3 & \mathbf{A}_3 \end{bmatrix} = \begin{bmatrix} d & \mathbf{b}_0^{\perp} \\ \mathbf{c}_0^{\mathrm{T}\perp} & \mathbf{A}_0^{\perp} \end{bmatrix} = \tilde{\mathbf{A}}_0^{\perp}. \tag{21}$$

By introducing the transposition operator T and anti-transposition operator $\perp$ the complete transformation cycle can be visualized as follows:

$$\begin{aligned}
\tilde{\mathbf{A}}_1 &= \mathrm{T}(\tilde{\mathbf{A}}_0) = \mathbf{A}_0^{\mathrm{T}} \\
\tilde{\mathbf{A}}_2 &= \perp(\tilde{\mathbf{A}}_1) = \perp\mathrm{T}(\tilde{\mathbf{A}}_0) = \tilde{\mathbf{A}}_0^{\mathrm{T}\perp} \\
\tilde{\mathbf{A}}_3 &= \mathrm{T}(\tilde{\mathbf{A}}_2) = \mathrm{T}\perp(\tilde{\mathbf{A}}_1) = \mathrm{T}\perp\mathrm{T}(\tilde{\mathbf{A}}_0) = \perp(\tilde{\mathbf{A}}_0) = \mathbf{A}_0^{\perp} \\
\tilde{\mathbf{A}}_0 &= \perp(\tilde{\mathbf{A}}_3) = \perp\mathrm{T}(\mathbf{A}_2) = \perp\mathrm{T}\perp(\tilde{\mathbf{A}}_1) = \perp\mathrm{T}\perp\mathrm{T}(\tilde{\mathbf{A}}_0) = \tilde{\mathbf{A}}_0
\end{aligned} \tag{22}$$

Thus, the cycle is claused.

Eqs (18) to (22) expressed the particular variants by the principal variant. This is, however, not the only possibility. Table 1 summarizes all the transformations among phase-variable forms.

Table 1

Relationships of phase-variable forms

	$\tilde{\mathbf{A}}_0$	$\tilde{\mathbf{A}}_1$	$\tilde{\mathbf{A}}_2$	$\tilde{\mathbf{A}}_3$
$\tilde{\mathbf{A}}_0 =$	$\tilde{\mathbf{A}}_0$	$\tilde{\mathbf{A}}_1^{\mathrm{T}}$	$\tilde{\mathbf{A}}_2^{\mathrm{T}\perp}$	$\tilde{\mathbf{A}}_3^{\perp}$
$\tilde{\mathbf{A}}_1 =$	$\tilde{\mathbf{A}}_0^{\mathrm{T}}$	$\tilde{\mathbf{A}}_1$	$\tilde{\mathbf{A}}_2^{\perp}$	$\tilde{\mathbf{A}}_3^{\mathrm{T}\perp}$
$\tilde{\mathbf{A}}_2 =$	$\tilde{\mathbf{A}}_0^{\mathrm{T}\perp}$	$\tilde{\mathbf{A}}_1^{\perp}$	$\tilde{\mathbf{A}}_2$	$\tilde{\mathbf{A}}_3^{\mathrm{T}}$
$\tilde{\mathbf{A}}_3 =$	$\tilde{\mathbf{A}}_0^{\perp}$	$\tilde{\mathbf{A}}_1^{\mathrm{T}\perp}$	$\tilde{\mathbf{A}}_2^{\mathrm{T}}$	$\tilde{\mathbf{A}}_3$

Introducing the augmented involutary matrix in Eq. (16), Table 1 may somehow be changed into Table 2.

Table 2

Relationships of phase-variable forms

	$\tilde{\mathbf{A}}_0$	$\tilde{\mathbf{A}}_1$	$\tilde{\mathbf{A}}_2$	$\tilde{\mathbf{A}}_3$
$\tilde{\mathbf{A}}_0 =$	$\tilde{\mathbf{A}}_0$	$\tilde{\mathbf{A}}_1^{\mathrm{T}} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_1^{\perp}\tilde{\mathbf{E}}$	$\tilde{\mathbf{A}}_2^{\mathrm{T}\perp} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_2\tilde{\mathbf{E}}$	$\tilde{\mathbf{A}}_3^{\perp} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_3^{\mathrm{T}}\tilde{\mathbf{E}}$
$\tilde{\mathbf{A}}_1 =$	$\tilde{\mathbf{A}}_0^{\mathrm{T}} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_0^{\perp}\tilde{\mathbf{E}}$	$\tilde{\mathbf{A}}_1$	$\tilde{\mathbf{A}}_2^{\perp} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}^{\mathrm{T}}\tilde{\mathbf{E}}$	$\tilde{\mathbf{A}}_3^{\mathrm{T}\perp} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_3\tilde{\mathbf{E}}$
$\tilde{\mathbf{A}}_2 =$	$\tilde{\mathbf{A}}_0^{\mathrm{T}\perp} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_0\tilde{\mathbf{E}}$	$\tilde{\mathbf{A}}_1^{\perp} = \tilde{\mathbf{E}}\mathbf{A}_1^{\mathrm{T}}\tilde{\mathbf{E}}$	$\tilde{\mathbf{A}}_2$	$\tilde{\mathbf{A}}_3^{\mathrm{T}} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_3^{\perp}\tilde{\mathbf{E}}$
$\tilde{\mathbf{A}}_3 =$	$\tilde{\mathbf{A}}_0^{\perp} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_0^{\mathrm{T}}\tilde{\mathbf{E}}$	$\tilde{\mathbf{A}}_1^{\mathrm{T}\perp} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_1\tilde{\mathbf{E}}$	$\tilde{\mathbf{A}}_2^{\mathrm{T}} = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_2^{\perp}\tilde{\mathbf{E}}$	$\tilde{\mathbf{A}}_3$

The hyper-matrices can easily be partitioned yielding similar results as given in Eqs (18) to (21),

$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{b}_0 \\ \mathbf{c}_0^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}} & \mathbf{c}_1 \\ \mathbf{b}_1^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2^{\mathrm{T}\perp} & \mathbf{b}_2^{\mathrm{T}\perp} \\ \mathbf{c}_2^{\perp} & d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_3^{\perp} & \mathbf{c}_3^{\mathrm{T}\perp} \\ \mathbf{b}_3^{\perp} & d \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} \mathbf{A}_0^{\mathrm{T}} & \mathbf{c}_0 \\ \mathbf{b}_0^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{b}_1 \\ \mathbf{c}_1^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_2^{\perp} & \mathbf{c}_2^{\mathrm{T}\perp} \\ \mathbf{b}_2^{\perp} & d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_3^{\mathrm{T}\perp} & \mathbf{b}_3^{\mathrm{T}\perp} \\ \mathbf{c}_3^{\perp} & d \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} d & \mathbf{c}_0^{\perp} \\ \mathbf{b}_0^{\mathrm{T}\perp} & \mathbf{A}^{\mathrm{T}\perp} \end{bmatrix} = \begin{bmatrix} d & \mathbf{b}_1^{\perp} \\ \mathbf{c}_1^{\mathrm{T}\perp} & \mathbf{A}_1^{\perp} \end{bmatrix} = \begin{bmatrix} d & \mathbf{c}_2^{\mathrm{T}} \\ \mathbf{b}_2 & \mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} d & \mathbf{b}_3^{\mathrm{T}} \\ \mathbf{c}_3 & \mathbf{A}_3^{\mathrm{T}} \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} d & \mathbf{b}_0^{\perp} \\ \mathbf{c}_0^{\mathrm{T}\perp} & \mathbf{A}_0^{\perp} \end{bmatrix} = \begin{bmatrix} d & \mathbf{c}_1^{\perp} \\ \mathbf{b}_1^{\mathrm{T}\perp} & \mathbf{A}_1^{\mathrm{T}\perp} \end{bmatrix} = \begin{bmatrix} d & \mathbf{b}_2^{\mathrm{T}} \\ \mathbf{c}_2 & \mathbf{A}_2^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} d & \mathbf{c}_3^{\mathrm{T}} \\ \mathbf{b}_3 & \mathbf{A}_3 \end{bmatrix}. \tag{26}$$

The same result can be expressed in a somewhat modified form:

$$\begin{bmatrix} \mathbf{A}_0 & \mathbf{b}_0 \\ \mathbf{c}_0^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1^{\mathrm{T}} & \mathbf{c}_1 \\ \mathbf{b}_1^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{E}\mathbf{A}_2\mathbf{E} & \mathbf{E}\mathbf{b}_2 \\ \mathbf{c}_2^{\mathrm{T}}\mathbf{E} & d \end{bmatrix} = \begin{bmatrix} \mathbf{E}\mathbf{A}_3^{\mathrm{T}}\mathbf{E} & \mathbf{E}\mathbf{c}_3 \\ \mathbf{b}_3\mathbf{E} & d \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} \mathbf{A}_0^{\mathrm{T}} & \mathbf{c}_0 \\ \mathbf{b}_0^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{b}_1 \\ \mathbf{c}_1^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{E}\mathbf{A}_2^{\mathrm{T}}\mathbf{E} & \mathbf{E}\mathbf{c}_2 \\ \mathbf{b}_2\mathbf{E} & d \end{bmatrix} = \begin{bmatrix} \mathbf{E}\mathbf{A}_3\mathbf{E} & \mathbf{E}\mathbf{b}_3 \\ \mathbf{c}_3\mathbf{E} & d \end{bmatrix} \tag{28}$$

$$\begin{bmatrix} d & \mathbf{c}_0\mathbf{E} \\ \mathbf{E}\mathbf{b}_0 & \mathbf{E}\mathbf{A}_0\mathbf{E} \end{bmatrix} = \begin{bmatrix} d & \mathbf{b}_1\mathbf{E} \\ \mathbf{E}\mathbf{c}_1 & \mathbf{E}\mathbf{A}_1^{\mathrm{T}}\mathbf{E} \end{bmatrix} = \begin{bmatrix} d & \mathbf{c}_2^{\mathrm{T}} \\ \mathbf{b}_2 & \mathbf{A}_2 \end{bmatrix} = \begin{bmatrix} d & \mathbf{b}_3^{\mathrm{T}} \\ \mathbf{c}_3 & \mathbf{A}_3^{\mathrm{T}} \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} d & \mathbf{b}_0\mathbf{E} \\ \mathbf{E}\mathbf{c}_0 & \mathbf{E}\mathbf{A}_0^{\mathrm{T}}\mathbf{E} \end{bmatrix} = \begin{bmatrix} d & \mathbf{c}_1\mathbf{E} \\ \mathbf{E}\mathbf{b}_1 & \mathbf{E}\mathbf{A}_1\mathbf{E} \end{bmatrix} = \begin{bmatrix} d & \mathbf{b}_2^{\mathrm{T}} \\ \mathbf{c}_2 & \mathbf{A}_2^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} d & \mathbf{c}_3^{\mathrm{T}} \\ \mathbf{b}_3 & \mathbf{A}_3 \end{bmatrix}. \tag{30}$$

Owing to the fact that in general there is no similarity transformation by which a transpose $\mathbf{A}^T$ or anti-transpose $\mathbf{A}^{\perp}$ could be obtained from a matrix $\mathbf{A}$ or vice versa, the system of relationships in Table 2 and Eqs (27) to (30) clearly splits into two subsystems. In Table 2, on the one hand, matrix $\tilde{\mathbf{A}}_0$ can be obtained by a similarity transform from $\tilde{\mathbf{A}}_2$ and so can be $\tilde{\mathbf{A}}_1$ from $\tilde{\mathbf{A}}_3$ and vice versa. The identities $\tilde{\mathbf{A}}_0 = \tilde{\mathbf{A}}_0, \ldots, \tilde{\mathbf{A}}_3 = \tilde{\mathbf{A}}_3$ do also belong to this subsystem. On the other hand, $\tilde{\mathbf{A}}_0$ could not be obtained by a similarity transform from $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_3$, and $\tilde{\mathbf{A}}_1$ could not be obtained from $\tilde{\mathbf{A}}_0$ and $\tilde{\mathbf{A}}_2$, and so on, but also transpositions or anti-transposition are to be resorted to. In Table 2, the first and second subsystems are situated like the black and white fields of a chess-board, respectively. Therefore these circumstances will be referred to as the chess-board rule.

3. Variants of JORDAN Canonical Forms

The *principal variant* of the JORDAN canonical form may be given in the form

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{J}\mathbf{z} + \mathbf{b}u \\ y &= \mathbf{c}^T\mathbf{z} + d\,u \end{aligned} \tag{31}$$

where the JORDAN matrix is given in pseudodiagonal hyper-matrix form

$$\mathbf{J} = \mathbf{diag}\,[\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_m] \tag{32}$$

where the $k_i \times k_i$ JORDAN block

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & & & \\ & \lambda_i & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \lambda_i & 1 \\ & & & & & \lambda_i \end{bmatrix} \qquad (i = 1, 2, \ldots, m) \tag{33}$$

belongs to a repeated eigenvalue of multiplicity k_i, where $k_1 + k_2 + \ldots + k_m = n$. If some of the eigevalues is distinct ($k_i = 1$, $\exists i$) then the corresponding JORDAN block reduces to a scalar quantity which is nothing else but the eigenvalue itself $\mathbf{J}_i = \lambda_i$, $\exists i$. Thus, the distinct eigenvalue can be considered as the special case of the repeated eigenvalue.

The first particular variant of the JORDAN form can be expressed as

$$\begin{aligned} \dot{\mathbf{z}}' &= \mathbf{J}^{\mathrm{T}}\mathbf{z}' + \mathbf{c}u \\ y &= \mathbf{b}^{\mathrm{T}}\mathbf{z}' + du \end{aligned} \tag{34}$$

where

$$\mathbf{J}^{\mathrm{T}} = \mathbf{diag}\ [\mathbf{J}_1^{\mathrm{T}}, \mathbf{J}_2^{\mathrm{T}}, \ldots, \mathbf{J}_m^{\mathrm{T}}] \tag{35}$$

with

$$\mathbf{J}_i^{\mathrm{T}} = \begin{bmatrix} \lambda_i & & & & & \\ 1 & \lambda_i & & & & \\ & \cdot & \cdot & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 1 & \lambda_i & \\ & & & & & 1 & \lambda_i \end{bmatrix}. \tag{36}$$

The second particular variant is

$$\begin{aligned} \dot{\mathbf{z}}'' &= \mathbf{J}^{\mathrm{T}\perp}\, \mathbf{z}'' + \mathbf{b}^{\mathrm{T}\perp}u \\ y &= \mathbf{c}^{\perp}\, \mathbf{z}'' + du \end{aligned} \tag{37}$$

where

$$\mathbf{J}^{\mathrm{T}\perp} = \mathbf{diag}\ [\mathbf{J}_m^{\mathrm{T}}, \ldots, \mathbf{J}_2^{\mathrm{T}}, \mathbf{J}_1^{\mathrm{T}}] \tag{38}$$

and

$$\mathbf{b}^{\mathrm{T}\perp} = \mathbf{E}\mathbf{b},\ \mathbf{c}^{\perp} = \mathbf{c}^{\mathrm{T}}\mathbf{E} \tag{39}$$

The third particular variant of the JORDAN form is

$$\begin{aligned} \dot{\mathbf{z}}''' &= \mathbf{J}^{\perp}\mathbf{z}''' + \mathbf{c}^{\mathrm{T}\perp}u \\ y &= \mathbf{b}^{\perp}\mathbf{z}''' + du \end{aligned} \tag{40}$$

where

$$\mathbf{J}^{\perp} = \mathbf{diag}\ [\mathbf{J}_m, \ldots, \mathbf{J}_2, \mathbf{J}_1] \tag{41}$$

and

$$\mathbf{c}^{\mathrm{T}\perp} = \mathbf{E}\mathbf{c}, \qquad \mathbf{b}^{\perp} = \mathbf{b}^{\mathrm{T}}\mathbf{E}\,, \tag{42}$$

We remark that in particular:

$$\mathbf{J}_i = \mathbf{J}_i^{\perp},\ \ \mathbf{J}_i^{\mathrm{T}} = \mathbf{J}_i^{\mathrm{T}\perp}, \tag{43}$$

Let us define a $(n+1)\times(n+1)$ augmented canonical matrix,

$$\tilde{\mathbf{J}} \triangleq \begin{bmatrix} \mathbf{J} & \mathbf{b} \\ \mathbf{c}^{\mathrm{T}} & d \end{bmatrix}. \tag{44}$$

Then,

$$\tilde{\mathbf{J}}^{\mathrm{T}} = \begin{bmatrix} \mathbf{J}^{\mathrm{T}} & \mathbf{c} \\ \mathbf{b}^{\mathrm{T}} & d \end{bmatrix} \tag{45}$$

$$\tilde{\mathbf{J}}^{\mathrm{T}\perp} = \begin{bmatrix} d & \mathbf{b}^{\mathrm{T}\perp} \\ \mathbf{c}^{\perp} & \mathbf{J}^{\mathrm{T}\perp} \end{bmatrix} \tag{46}$$

and

$$\tilde{\mathbf{J}}^{\perp} = \begin{bmatrix} d & \mathbf{c}^{\mathrm{T}\perp} \\ \mathbf{b}^{\perp} & \mathbf{J}^{\perp} \end{bmatrix}. \tag{47}$$

4. Canonical Transformations

It can be shown (see Appendix) that there exist a transformation

$$\mathbf{x} = \mathbf{L}\mathbf{z}, \quad \mathbf{z} = \mathbf{L}^{-1}\mathbf{x} \tag{48}$$

which converts the principal phase variable form into the principal JORDAN form yielding

$$\mathbf{J} = \mathbf{L}^{-1}\mathbf{A}_0\mathbf{L}, \quad \mathbf{b} = \mathbf{L}^{-1}\mathbf{b}_0, \quad \mathbf{c}^{\mathrm{T}} = \mathbf{c}_0^{\mathrm{T}}\mathbf{L} \tag{49}$$

where

$$\mathbf{b} = \begin{bmatrix} \mathbf{j}_1 \\ \mathbf{j}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{j}_{m-1} \\ \mathbf{j}_m \end{bmatrix} \triangleq \mathbf{j}; \quad \mathbf{j}_i \triangleq \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \qquad (i = 1, 2, \ldots, m) \tag{50}$$

Introducing the augmented transformation matrix and its inverse:

$$\tilde{\mathbf{L}} \triangleq \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix}; \quad \tilde{\mathbf{L}}^{-1} \triangleq \begin{bmatrix} \mathbf{L}^{-1} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{51}$$

$$\begin{bmatrix} \mathbf{J} & \mathbf{b} \\ \mathbf{c}^{\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{-1} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_0 & \mathbf{b}_0 \\ \mathbf{c}_0^{\mathrm{T}} & \mathbf{d} \end{bmatrix} \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{52}$$

or

$$\tilde{\mathbf{J}} = \tilde{\mathbf{L}}^{-1}\tilde{\mathbf{A}}_0\tilde{\mathbf{L}} \tag{53}$$

Taking Eq. (19) into account a transposition yields

$$\tilde{\mathbf{J}}^{\mathrm{T}} = \tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{A}}_1\tilde{\mathbf{L}}^{-1\mathrm{T}} \tag{54}$$

or

$$\begin{bmatrix}\mathbf{J}^{\mathrm{T}} & \mathbf{c}\\ \mathbf{b}^{\mathrm{T}} & d\end{bmatrix} = \begin{bmatrix}\mathbf{L}^{\mathrm{T}} & \mathbf{0}\\ \mathbf{0}^{\mathrm{T}} & 1\end{bmatrix}\begin{bmatrix}\mathbf{A}_1 & \mathbf{b}_1\\ \mathbf{c}_1^{\mathrm{T}} & d\end{bmatrix}\begin{bmatrix}\mathbf{L}^{-1\mathrm{T}} & \mathbf{0}\\ \mathbf{0}^{\mathrm{T}} & 1\end{bmatrix} = \begin{bmatrix}\mathbf{L}^{\mathrm{T}}\mathbf{A}_1\mathbf{L}^{-1\mathrm{T}} & \mathbf{L}^{\mathrm{T}}\mathbf{b}_1\\ \mathbf{c}_1^{\mathrm{T}}\mathbf{L}^{-1\mathrm{T}} & d\end{bmatrix} \tag{55}$$

From Eq. (54), and taking Eq. (20) into consideration an anti-transposition yields

$$\tilde{\mathbf{J}}^{\mathrm{T}\perp} = \tilde{\mathbf{L}}^{-1\mathrm{T}\perp}\tilde{\mathbf{A}}_2\tilde{\mathbf{L}}^{\mathrm{T}\perp} \tag{56}$$

or

$$\begin{bmatrix}d & \mathbf{c}^{\perp}\\ \mathbf{b}^{\mathrm{T}\perp} & \mathbf{J}^{\mathrm{T}\perp}\end{bmatrix} = \begin{bmatrix}1 & \mathbf{0}^{\mathrm{T}}\\ \mathbf{0} & \mathbf{L}^{-1\mathrm{T}\perp}\end{bmatrix}\begin{bmatrix}d & \mathbf{c}_2^{\mathrm{T}}\\ \mathbf{b}_2 & \mathbf{A}_2\end{bmatrix}\begin{bmatrix}1 & \mathbf{0}^{\mathrm{T}}\\ \mathbf{0} & \mathbf{L}^{\mathrm{T}\perp}\end{bmatrix} = \begin{bmatrix}d & \mathbf{c}_2^{\mathrm{T}}\mathbf{L}^{\mathrm{T}\perp}\\ \mathbf{L}^{-1\mathrm{T}\perp}\mathbf{b}_2 & \mathbf{L}^{-1\mathrm{T}\perp}\mathbf{A}_2\mathbf{L}^{\mathrm{T}\perp}\end{bmatrix}. \tag{57}$$

Here $\mathbf{0}^{\perp} = \mathbf{0}^{\mathrm{T}}$ and $\mathbf{0}^{\mathrm{T}\perp} = \mathbf{0}$ were also considered. Finally, from Eq. (53) and taking Eq. (21) into account, an anti-transposition yields

$$\tilde{\mathbf{J}}^{\perp} = \tilde{\mathbf{L}}^{\perp}\tilde{\mathbf{A}}_3\tilde{\mathbf{L}}^{-1\perp} \tag{58}$$

or

$$\begin{bmatrix}d & \mathbf{c}^{\mathrm{T}\perp}\\ \mathbf{b}^{\perp} & \mathbf{J}^{\perp}\end{bmatrix} = \begin{bmatrix}1 & \mathbf{0}^{\mathrm{T}}\\ \mathbf{0} & \mathbf{L}^{\perp}\end{bmatrix}\begin{bmatrix}d & \mathbf{c}_3^{\mathrm{T}}\\ \mathbf{b}_3 & \mathbf{A}_3\end{bmatrix}\begin{bmatrix}1 & \mathbf{0}^{\mathrm{T}}\\ \mathbf{0} & \mathbf{L}^{-1\perp}\end{bmatrix} = \begin{bmatrix}d & \mathbf{c}_3\mathbf{L}^{-1\perp}\\ \mathbf{L}^{\perp}\mathbf{b}_3 & \mathbf{L}^{\perp}\mathbf{A}_3\mathbf{L}^{-1\perp}\end{bmatrix}. \tag{59}$$

Substituting $\tilde{\mathbf{A}}_0 = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_2\tilde{\mathbf{E}}$ into Eq. (53) we obtain

$$\tilde{\mathbf{J}} = [\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{E}}]\tilde{\mathbf{A}}_2\,[\tilde{\mathbf{E}}\tilde{\mathbf{L}}] \tag{60}$$

Substituting $\tilde{\mathbf{A}}_1 = \tilde{\mathbf{E}}\tilde{\mathbf{A}}_3\tilde{\mathbf{E}}$ into Eq. (54) we obtain

$$\tilde{\mathbf{J}}^{\mathrm{T}} = [\tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{E}}]\tilde{\mathbf{A}}_3\,[\tilde{\mathbf{E}}\,\tilde{\mathbf{L}}^{-1\mathrm{T}}] = [\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{\perp}]\,\tilde{\mathbf{A}}_3\,[\tilde{\mathbf{L}}^{-1\perp}\,\tilde{\mathbf{E}}] \tag{61}$$

where $\tilde{\mathbf{L}}^{\mathrm{T}} = \tilde{\mathbf{E}}\tilde{\mathbf{L}}^{\perp}\tilde{\mathbf{E}}$ and $\tilde{\mathbf{E}}\tilde{\mathbf{E}} = \tilde{\mathbf{I}}$ were also considered.

Similarly, substituting $\tilde{\mathbf{A}}_2 = \tilde{\mathbf{E}}\,\tilde{\mathbf{A}}_0\,\tilde{\mathbf{E}}$ into Eq. (56) and taking $\tilde{\mathbf{L}}^{\mathrm{TL}} = \tilde{\mathbf{E}}\,\tilde{\mathbf{L}}\,\tilde{\mathbf{E}}$, $\tilde{\mathbf{L}}^{-1\mathrm{T}\perp} = \tilde{\mathbf{E}}\,\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{E}}$ and $\tilde{\mathbf{E}}\,\tilde{\mathbf{E}} = \tilde{\mathbf{I}}$ into consideration yields:

$$\tilde{\mathbf{J}}^{\mathrm{T}\perp} = [\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{-1}]\,\tilde{\mathbf{A}}_0\,[\tilde{\mathbf{L}}\tilde{\mathbf{E}}] \tag{62}$$

Finally, substituting $\tilde{\mathbf{A}}_3 = \tilde{\mathbf{E}}\,\tilde{\mathbf{A}}_1\tilde{\mathbf{E}}$ into Eq. (58) yields

$$\tilde{\mathbf{J}}^{\perp} = [\tilde{\mathbf{L}}^{\perp}\,\tilde{\mathbf{E}}]\,\tilde{\mathbf{A}}_1\,[\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{-1\perp}] = [\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{\mathrm{T}}]\,\tilde{\mathbf{A}}_1\,[\tilde{\mathbf{L}}^{-1\mathrm{T}}\tilde{\mathbf{E}}] \tag{63}$$

where $\tilde{\mathbf{L}}^{\perp} = \tilde{\mathbf{E}}\tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{E}}$ and $\tilde{\mathbf{E}}\,\tilde{\mathbf{E}} = \tilde{\mathbf{I}}$ were also considered.

Eqs. (60) to (63) could be obtained also in another way, for example Eq. (54) and $\tilde{\mathbf{J}}^{\perp} = \tilde{\mathbf{E}}\,\tilde{\mathbf{J}}^{\mathrm{T}}\tilde{\mathbf{E}}$ would immediately yield the second expression of Eq. (63).

The canonical transformations are summarized in Table 3. This configuration shows clearly the chessboard rule.

Table 3

Canonical transformations

	$\mathbf{A}_0$	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{A}_3$
$\tilde{\mathbf{J}}$	$\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{A}}_0\tilde{\mathbf{L}}$		$[\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{E}}]\tilde{\mathbf{A}}_2[\tilde{\mathbf{E}}\tilde{\mathbf{L}}]$	
$\tilde{\mathbf{J}}^{\mathrm{T}}$		$\tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{A}}_1\tilde{\mathbf{L}}^{-1\mathrm{T}}$		$\begin{cases}[\tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{E}}]\tilde{\mathbf{A}}_3[\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{-1\mathrm{T}}]\\ [\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{\perp}]\tilde{\mathbf{A}}_3[\tilde{\mathbf{L}}^{-1\perp}\tilde{\mathbf{E}}]\end{cases}$
$\tilde{\mathbf{J}}^{\mathrm{T}\perp}$	$[\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{-1}]\tilde{\mathbf{A}}_0[\tilde{\mathbf{L}}\tilde{\mathbf{E}}]$		$\tilde{\mathbf{L}}^{-1\mathrm{T}\perp}\tilde{\mathbf{A}}_2\tilde{\mathbf{L}}^{\mathrm{T}\perp}$	
$\tilde{\mathbf{J}}^{\perp}$		$\begin{cases}[\tilde{\mathbf{L}}^{\perp}\tilde{\mathbf{E}}]\tilde{\mathbf{A}}_1[\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{-1\perp}]\\ [\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{\perp}]\tilde{\mathbf{A}}_1[\tilde{\mathbf{L}}^{-1\mathrm{T}}\tilde{\mathbf{E}}]\end{cases}$		$\tilde{\mathbf{L}}^{\perp}\tilde{\mathbf{A}}_3\tilde{\mathbf{L}}^{-1\perp}$

5. An Illustrative Example

For the sake of simplicity we solve only a very simple example. Let us start with the improper transfer function:

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{(s+1)^2(s+2)} + d = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + 4s^2 + 5s + 2} + d\,.$$

(for $d = 0$, $G(s)$ becomes a proper transfer function.)

The roots of $b_2 s^2 + b_1 s + b_0 = 0$, that is, the zeros of the transfer function are assumed not to coincide with -1 or -2. Such a system is completely controllable and observable.

The principal case of the phase variable form is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [\; b_0, \quad b_1, \quad b_2 \;]\; \mathbf{x} + \quad du\,.$$

According to the Appendix, and with

$$N_1'(-1) = 2s + 4\,\big|_{s=-1} = 2; \qquad N_2'(-1) = 1\,\big|_{s=-1} = 1; \qquad N_3'(-1) = 0$$

$$N_1(-1) = s^2 + 4s + 5\,\big|_{s=-1} = 2; \quad N_2(-1) = s + 4\,\big|_{s=-1} = 3; \quad N_3(-1) = 1$$

$$N_1(-2) = s^2 + 4s + 5\,\big|_{s=-2} = 1; \quad N_2(-2) = s + 4\,\big|_{s=-2} = 2; \quad N_2(-1) = 1$$

$$t_1^{(1)} = \delta_1^{(1)} = \frac{1}{s+2}\bigg|_{s=-1} = 1; \qquad t_2^{(1)} = \delta_2^{(1)} = -\frac{1}{(s+2)^2}\bigg|_{s=-1} = -1$$

$$t_1^{(2)} = \delta_1^{(2)} = \frac{1}{(s+1)^2}\bigg|_{s=-2} = 1 .$$

Thus, the transformation matrix is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -3 & 4 \end{bmatrix}$$

whereas its inverse is

$$\mathbf{L}^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

The entries of $\mathbf{J}$ and $\mathbf{b}$, $\mathbf{c}^T$ are

$$\mathbf{J} = \mathbf{L}^{-1}\mathbf{A}_0\mathbf{L} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad \mathbf{b} = \mathbf{L}^{-1}\mathbf{b}_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{c}^T = \mathbf{c}_0\mathbf{L} = [b_0 - b_1 + b_2, \ -b_0 + 2b_1 - 3b_2, \ b_0 - 2b_1 + 4b_2] .$$

The second particular variant of the phase variable form is

$$\mathbf{x}'' = \begin{bmatrix} -4 & -5 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}'' + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [\ b_2, \ b_1, \ b_0] \mathbf{x}'' + du .$$

The appropriate transformation matrix is now

$$\mathbf{EL} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ -1 & 2 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

and its inverse is

$$\mathbf{L}^{-1}\mathbf{E} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Really,

$$\mathbf{J} = [\mathbf{L}^{-1}\mathbf{E}]\mathbf{A}_2[\mathbf{EL}] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \ \mathbf{b} = [\mathbf{L}^{-1}\mathbf{E}]\,\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{c}^T = \mathbf{c}_2^T[\mathbf{EL}] = [b_2 - b_1 + b_0, \ -3b_2 + 2b_1 - b_0, 4b_2 - 2b_1 + b_0] .$$

Thus, again the principal variant of the JORDAN canonical form is obtained. The validity of the other transformations summarized in Table 3 can similarly be illustrated.

6. Supplementary Variants of Canonical Transformations

Table 3 shows some holes which cause a feeling of want. Allowing transpositions or anti-transpositions could fill in these empty spaces but this would be only an illusory solution yielding the already obtained results. (For example, $\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{A}}_1^{\mathrm{T}}\ \tilde{\mathbf{L}}$ is essentially the same as $\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{A}}_0\tilde{\mathbf{L}}$.) The question arises whether there are still other possibilities.

Owing to the special structure of $\mathbf{J}$, matrices $\mathbf{J}^{\mathrm{T}}$ and $\mathbf{J}^{\perp}$ may be expressed by special similarity transformations. For any JORDAN block $\mathbf{J}_i$, namely,

$$\mathbf{J}_i^{\mathrm{T}} = \mathbf{E}_i\mathbf{J}_i\mathbf{E}_i \quad \text{and} \quad \mathbf{J}_i^{\perp} = \mathbf{I}_i\mathbf{J}_i\mathbf{I}_i = \mathbf{J}_i \tag{64}$$

where $\mathbf{E}_i$ and $\mathbf{I}_i$ are $k_i \times k_i$ matrices both reducing to the scalar 1 in the case of $k_i = 1$, that is, for distinct eigenvalues.

Let us now define a transformation matrix and its inverse by

$$\mathbf{E}_J = \mathbf{E}_J^{-1} = \mathbf{E}_J^{\mathrm{T}} = \mathbf{E}_J^{-1\mathrm{T}} \triangleq \mathbf{diag}\,[\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_m]\,. \tag{65}$$

It can be shown that

$$\mathbf{J}^{\mathrm{T}} = \mathbf{E}_J\mathbf{J}\mathbf{E}_J \quad \text{and} \quad \mathbf{J} = \mathbf{E}_J\mathbf{J}^{\mathrm{T}}\mathbf{E}_J \tag{66}$$

Furthermore,

$$\mathbf{J}^{\perp} = \mathbf{E}\mathbf{E}_J\mathbf{J}\mathbf{E}_J\mathbf{E} \quad \text{and} \quad \mathbf{J} = \mathbf{E}_J\mathbf{E}\mathbf{J}^{\perp}\mathbf{E}\mathbf{E}_J \tag{67}$$

Eqs (66) and (67) express the desired similarity transformations.

Let us start now with the reduced Eq. (54). Taking Eq. (66) into account we obtain

$$\mathbf{J} = \mathbf{E}_J\mathbf{L}^{\mathrm{T}}\mathbf{A}_1\mathbf{L}^{-1\mathrm{T}}\mathbf{E}_J \tag{68}$$

expressing $\mathbf{J}$ in terms of $\mathbf{A}_1$. Thus, the hole in the first row and second column of Table 3 seems to be filled in. Transpositions and Eq. (19) yield

$$\mathbf{J}^{\mathrm{T}} = \mathbf{E}_J\mathbf{L}^{-1}\mathbf{A}_0\mathbf{L}\mathbf{E}_J \tag{69}$$

expressing $\mathbf{J}^{\mathrm{T}}$ in terms of $\mathbf{A}_0$. Eq. (68) seems to fill in the hole in the second row and first column of Table 3.

Let us now define an augmented transformation matrix:

$$\tilde{\mathbf{E}}_J = \tilde{\mathbf{E}}_J^{-1} = \tilde{\mathbf{E}}_J^{\mathrm{T}} = \tilde{\mathbf{E}}_J^{-1\mathrm{T}} = \mathbf{diag}\,[\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_m, 1]\,. \tag{70}$$

An analogue expression to that of Eq. (68) would be

$$\tilde{\mathbf{E}}_J\tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{A}}_1\tilde{\mathbf{L}}^{-1\mathrm{T}}\mathbf{E}_J = \begin{bmatrix} \mathbf{E}_J\mathbf{L}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{b}_1 \\ \mathbf{c}_1^{\mathrm{T}} & d \end{bmatrix} \begin{bmatrix} \mathbf{L}^{-1\mathrm{T}}\mathbf{E}_J & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} = \\ = \begin{bmatrix} \mathbf{E}_J\mathbf{L}^{\mathrm{T}}\mathbf{A}_1\mathbf{L}^{-1\mathrm{T}}\mathbf{E}_J & \mathbf{E}_J\mathbf{L}^{\mathrm{T}}\mathbf{b}_1 \\ \mathbf{c}_1^{\mathrm{T}}\mathbf{L}^{-1\mathrm{T}}\ \mathbf{E}_J & d \end{bmatrix}. \tag{71}$$

Although there is $\mathbf{J}$ again but $\mathbf{E}_J\mathbf{L}^{\mathrm{T}}\mathbf{b}_1 = \mathbf{E}_J\mathbf{L}^{\mathrm{T}}\mathbf{c}_0 \neq \mathbf{b}$ and $\mathbf{c}_1^{\mathrm{T}}\mathbf{L}^{-1\mathrm{T}}\mathbf{E}_J = \mathbf{b}_0\mathbf{L}^{-1\mathrm{T}}\mathbf{E}_J \neq \neq \mathbf{c}^{\mathrm{T}}$, thus, $\tilde{\mathbf{J}}$ is not obtained. Remark that $\mathbf{E}_J\mathbf{L}^{\mathrm{T}}\mathbf{b}_1$ gives $\mathbf{c}$ and $\mathbf{c}_1^{\mathrm{T}}\mathbf{L}^{-1\mathrm{T}}\mathbf{E}_J$ gives $\mathbf{b}^{\mathrm{T}}$ however, both with interchanged entries.

Table 4

Supplementary canonical transformations

	$\tilde{\mathbf{A}}_0$	$\tilde{\mathbf{A}}_1$	$\tilde{\mathbf{A}}_2$	$\tilde{\mathbf{A}}_3$
$[\tilde{\mathbf{J}}]$		$\tilde{\mathbf{E}}_J\tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{A}}_1\tilde{\mathbf{L}}^{-1\mathrm{T}}\tilde{\mathbf{E}}_J$		$\begin{cases}\tilde{\mathbf{E}}_J\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{\perp}\tilde{\mathbf{A}}_3\tilde{\mathbf{L}}^{-1\perp}\tilde{\mathbf{E}}\tilde{\mathbf{E}}_J \\ \tilde{\mathbf{E}}_J\tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{E}}\tilde{\mathbf{A}}_3\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{-1\mathrm{T}}\tilde{\mathbf{E}}_J\end{cases}$
$[\tilde{\mathbf{J}}^{\mathrm{T}}]$	$\tilde{\mathbf{E}}_J\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{A}}_0\tilde{\mathbf{L}}\tilde{\mathbf{E}}_J$		$\tilde{\mathbf{E}}_J\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{E}}\tilde{\mathbf{A}}_2\tilde{\mathbf{E}}\tilde{\mathbf{L}}\tilde{\mathbf{E}}_J$	
$[\tilde{\mathbf{J}}^{\mathrm{T}\perp}]$		$\tilde{\mathbf{E}}\tilde{\mathbf{E}}_J\tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{A}}_1\tilde{\mathbf{L}}^{-1\mathrm{T}}\tilde{\mathbf{E}}_J\tilde{\mathbf{E}}$		$\tilde{\mathbf{E}}\tilde{\mathbf{E}}_J\tilde{\mathbf{L}}^{\mathrm{T}}\tilde{\mathbf{E}}\tilde{\mathbf{A}}_3\tilde{\mathbf{E}}\tilde{\mathbf{L}}^{-1\mathrm{T}}\tilde{\mathbf{E}}_J\tilde{\mathbf{E}}$
$[\tilde{\mathbf{J}}^{\perp}]$	$\tilde{\mathbf{E}}\tilde{\mathbf{E}}_J\mathbf{L}^{-1}\tilde{\mathbf{A}}_1\tilde{\mathbf{L}}\tilde{\mathbf{E}}_J\tilde{\mathbf{E}}$		$\tilde{\mathbf{E}}\tilde{\mathbf{E}}_J\tilde{\mathbf{L}}^{-1}\tilde{\mathbf{E}}\tilde{\mathbf{A}}_2\tilde{\mathbf{E}}\tilde{\mathbf{L}}\tilde{\mathbf{E}}_J\tilde{\mathbf{E}}$	

Table 4 summarizes the supplementary transformations. The relationships are given for multiple eigenvalues, whereas the brackets applied to the augmented JORDAN forms will indicate that $\mathbf{b}$ and $\mathbf{c}^{\mathrm{T}}$ or $\mathbf{c}$ and $\mathbf{b}^{\mathrm{T}}$ are not on their prescribed places and they are not obtained in the desired form but only with interchanged entries whereas the $n \times n$ JORDAN forms are correct. For distinct eigenvalues the canonical form is obtained with rightly ordered $\mathbf{b}$ and $\mathbf{c}^{\mathrm{T}}$ but their places are interchanged.

7. An Illustrative Example

Let us start once more with the problem discussed in the previous illustrative example. The phase-variable form is given by the first particular variant, that is,

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -5 \\ 0 & 1 & -4 \end{bmatrix}; \quad \mathbf{b}_1 = \begin{bmatrix} b_1 \\ b_1 \\ b_2 \end{bmatrix}$$

$$\mathbf{c}_1^{\mathrm{T}} = [0, \quad 0, \quad 1]\ ; \quad d_1 = d\ .$$

According to Table 4, the suitably reduced transformation matrix and its inverse are

$$\mathbf{E}_J\mathbf{L}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -3 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$$

$$\mathbf{L}^{-1\mathrm{T}}\mathbf{E}_J = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

These yield

$$[\tilde{\mathbf{J}}] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & -b_0 + 2b_1 - 3b_2 \\ 0 & -1 & 0 & b_0 - b_1 + b_2 \\ 0 & 0 & -2 & b_0 - 2b_1 + 4b_2 \\ \hline 1 & 0 & 1 & d \end{array}\right].$$

8. Some Special Ccases

In the case of distinct eigenvalues JORDAN canonical forms $\tilde{\mathbf{J}}$ reduce to diagonal canonical forms $\tilde{\mathbf{\Lambda}}$.

Let us see once more the first particular variant of the phase variable form.

If the eigenvalues are distinct then $\mathbf{E}_J$ and $\tilde{\mathbf{E}}_J$ become $n \times n$ and $(n+1) \times (n+1)$ unity matrices, respectively, and then, from Eq. (71) the solution is

$$\begin{bmatrix} \mathbf{L}^{\mathrm{T}}\mathbf{A}_1\mathbf{L}^{-1\mathrm{T}} & \mathbf{L}^{\mathrm{T}}\mathbf{b}_1 \\ \mathbf{c}_1^{\mathrm{T}}\mathbf{L}^{-1\mathrm{T}} & d \end{bmatrix} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{L}^{\mathrm{T}}\mathfrak{c}_0 \\ \mathfrak{b}_0\mathbf{L}^{-1\mathrm{T}} & d \end{bmatrix} = \tilde{\mathbf{\Lambda}}^{\mathrm{T}} \tag{72}$$

where $\mathbf{\Lambda}$ is the diagonal matrix of the eigenvalues. Eq. (72) supplies the transpose of the form looked for.

In this case, instead of $[\tilde{\mathbf{\Lambda}}]$ we obtain $\tilde{\mathbf{\Lambda}}^{\mathrm{T}}$. The latter result could also be obtained from the appropriate expression in Table 3.

For distinct eigenvalues Tables 3 and 4 express the same relationships.

Another interesting special case arises if the JORDAN canonical form consists of a single JORDAN block, that is, a single multiple eigenvalue exists, and moreover there is no numerator dynamics and the gain factor is unity in the proper transfer function. In such cases

$$G(s) = \frac{1}{(s - \lambda_1)^{k_1}}.$$

9. Two Illustrative Examples

First, let us start with the improper transfer function

$$G(s)=\frac{b_2 s^2+b_1 s+b_0}{s(s+1)(s+2)}+d\,.$$

According to Table 3, from the principal variant of the phase variable form we obtain:

$$\begin{bmatrix}2&3&1&0\\0&2&1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&-2&-3&1\\b_0&b_1&b_2&d\end{bmatrix}\begin{bmatrix}1/2&-1&1/2&0\\0&1&-1&0\\0&-1&2&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0&1\\0&-1&0&1\\0&0&-2&1\\c_1&c_2&c_3&d\end{bmatrix}=\tilde{\Lambda}$$

where

$$c_1=(1/2)\,b_0,\quad c_2=-b_0+b_1-b_2,\quad c_3=(1/2)\,b_0-b_1+2\,b_2\,.$$

On the other hand, according to Table 4, from the first particular variant of the phase-variable form we obtain:

$$\begin{bmatrix}2&0&0&0\\3&2&1&0\\1&1&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&b_0\\1&0&-2&b_1\\0&1&-3&b_2\\0&0&1&d\end{bmatrix}\begin{bmatrix}1/2&0&0&0\\-1&1&1&0\\1/2&-1&2&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0&c_1\\0&-1&0&c_2\\0&0&-2&c_3\\1&1&1&d\end{bmatrix}=\tilde{\Lambda}^{\mathrm{T}}.$$

Secondly, let $\lambda_1=-\alpha$ and $k_1=2$. The principal variant and the first particular variant of the phase variable form is represented as

$$\tilde{\mathbf{A}}_0=\begin{bmatrix}0&1&0\\-\alpha^2&-2\alpha&1\\1&0&0\end{bmatrix},\quad \tilde{\mathbf{A}}_1=\begin{bmatrix}0&-\alpha^2&1\\1&-2\alpha&0\\0&1&0\end{bmatrix}.$$

Now

$$\mathbf{L}=\begin{bmatrix}1&0\\-\alpha&1\end{bmatrix},\quad \mathbf{L}^{-1}=\begin{bmatrix}1&0\\\alpha&1\end{bmatrix}$$

and according to the first row first column expression in Table 3 we obtain:

$$\begin{bmatrix}1&0&0\\\alpha&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\-\alpha^2&-2\alpha&1\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\-\alpha&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-\alpha&1&0\\0&-\alpha&1\\1&0&0\end{bmatrix}=\mathbf{J}$$

whereas according to the first row second column expression in Table 4 we have:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\alpha^2 & 1 \\ 1 & -2\alpha & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\alpha & 1 & 0 \\ 0 & -\alpha & 1 \\ 1 & 0 & 0 \end{bmatrix} = \mathbf{J}.$$

In this case Tables 3 and 4 merge and we have sixteen canonical transformations.

10. Conclusions

In the foregoing treatise four variants of phase-variable forms, four variants of JORDAN canonical forms have been defined and for the sake of descriptiveness we introduced augmented matrices. Besides similarity transformations allowing also transpositions and anti-transpositions would yield sixteen possibilities of turning from one variant of phase-variable form to one variant of JORDAN canonical form, see Table 5. If, however, nothing but similarity transformations are considered then there are only eight possibilities, see Table 3. Modifying somewhat the JORDAN canonical forms results in eight other possibilities as given in Table 4.

Table 5

Relationships of phase-variable and JORDAN canonical forms

	$\tilde{A}_0$	$\tilde{A}_1 = \tilde{A}_0^T$	$\tilde{A}_2 = \tilde{A}_0^{T\perp}$	$\tilde{A}_3 = \tilde{A}_0^{\perp}$
$\tilde{J}$	$\tilde{L}^{-1}\tilde{A}_0\tilde{L}$	$[\tilde{L}^T\tilde{A}_1\tilde{L}^{-1T}]^T$	$\tilde{L}^{-1}\tilde{A}_2^{T\perp}\tilde{L}$	$[\tilde{L}^{\perp}\tilde{A}_3\tilde{L}^{-1\perp}]^{\perp}$
$\tilde{J}^T$	$[\tilde{L}^{-1}\tilde{A}_0\tilde{L}]^T$	$\tilde{L}^T\tilde{A}_1\tilde{L}^{-1T}$	$[\tilde{L}^{-1T\perp}\tilde{A}_2\tilde{L}^{T\perp}]^{\perp}$	$\tilde{L}^T\tilde{A}_3^T\tilde{L}^{-1\perp}$
$\tilde{J}^{T\perp}$	$\tilde{L}^{-1T\perp}\tilde{A}_0^{T\perp}\tilde{L}^{T\perp}$	$[\tilde{L}^T\tilde{A}_1\tilde{L}^{-1T}]^{\perp}$	$\tilde{L}^{-1T\perp}\tilde{A}_2\tilde{L}^{T\perp}$	$[\tilde{L}^{\perp}\tilde{A}_3\tilde{L}^{-1\perp}]^T$
$\tilde{J}^{\perp}$	$[\tilde{L}^{-1}\tilde{A}_0\tilde{L}]^{\perp}$	$\tilde{L}^{\perp}\tilde{A}_1^{T\perp}\tilde{L}^{-1\perp}$	$[\tilde{L}^{-1T\perp}\tilde{A}_2\tilde{L}^{T\perp}]^T$	$\tilde{L}^{\perp}\tilde{A}_3\tilde{L}^{-1\perp}$

For distinct eigenvalues, Tables 3 and 4 express essentially the same similarity transformations.

In the very special case of one multiple eigenvalue and unity gain, Tables 3 and 4 yield althogether sixteen similarity transformations.

Appendix

In previous papers [1,2,4] we have shown that when starting from the principal variant of the phase-variable form, Eq. (1), the principal variant of the JORDAN canonical form, Eqs (31) to (33), can be obtained by an appropriate similarity transformation, Eq. (48), which is called modal transformation. At the same time the requirements of Eqs (49), (50) can also be satisfied.

VANDERMONDE matrices are known to be modal transformations. VANDERMONDE matrices, $\mathbf{V}$ do not, however, satisfy these requirement in Eqs (49), (50) in general. For this purpose a special matrix $\mathbf{T}$ has to be introduced such that

$$\mathbf{L} = \mathbf{VT}, \quad \mathbf{L}^{-1} = \mathbf{T}^{-1}\mathbf{V}^{-1} \tag{A1}$$

and

$$\mathbf{T}^{-1}\mathbf{V}^{-1}\mathbf{b}_0 = \mathbf{j} \quad \text{or} \quad \mathbf{Tj} = \mathbf{V}^{-1}\mathbf{b}_0 . \tag{A2}$$

Furthermore,

$$\mathbf{T}^{-1}\mathbf{JT} = \mathbf{J} \tag{A3}$$

or

$$\mathbf{JT} = \mathbf{TJ}, \quad \mathbf{T}^{-1}\mathbf{J} = \mathbf{JT}^{-1} \tag{A4}$$

must also hold. According to Eq. (A3) $\mathbf{T}$ will be called *commutativity matrix.*

The JORDAN matrix is given in pseudodiagonal form and so are $\mathbf{T}$ and $\mathbf{T}^{-1}$:

$$\begin{aligned} \mathbf{T} &= \mathbf{diag}\,[\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_m] \\ \mathbf{T}^{-1} &= \mathbf{diag}\,[\mathbf{T}_1^{-1}, \mathbf{T}_2^{-1}, \ldots, \mathbf{T}_m^{-1}] \end{aligned} \tag{A5}$$

with

$$\mathbf{T}_i^{-1}\mathbf{J}_i\mathbf{T}_i = \mathbf{J}_i \tag{A6}$$

and

$$\mathbf{J}_i\mathbf{T}_i = \mathbf{T}_i\mathbf{J}_i, \quad \mathbf{T}_i^{-1}\mathbf{J}_i = \mathbf{J}_i\mathbf{T}_i^{-1} . \tag{A7}$$

Evaluating the latter relationships in Eq. (A6), for the elements $t_k^{(i)}$ of $\mathbf{T}_i$ and elements $\mathbf{T}_k^{(i)}$ of $\mathbf{T}^{-1}$ it becomes obvious that both $\mathbf{T}_i$ and $\mathbf{T}_i^{-1}$ are upper right-hand triangular matrices:

$$\mathbf{T}_i = \begin{bmatrix} t_1^{(i)} & t_2^{(i)} & \cdots & t_{k_i}^{(i)} \\ & t_1^{(i)} & \ddots & \vdots \\ & & \ddots & t_2^{(i)} \\ & & & t_1^{(i)} \end{bmatrix} \tag{A8}$$

$$\mathbf{T}_i^{-1} \begin{bmatrix} \tau_1^{(i)} & \tau_2^{(i)} & \cdots & \tau_{k_i}^{(i)} \\ & \tau_1^{(i)} & \ddots & \vdots \\ & & \ddots & \tau_2^{(i)} \\ & & & \tau_1^{(i)} \end{bmatrix} . \tag{A9}$$

After some computations, based on Eq. (A2), it is shown that

$$\begin{aligned} t_1 &= w^{(i)}_{k_i,n} \\ t_2 &= w^{(i)}_{k_i-1,n} \\ &\cdot \\ &\cdot \\ &\cdot \\ t_{k_i-1} &= w^{(i)}_{2n} \\ t_{k_i} &= w^{(i)}_{1n} \end{aligned} \tag{A10}$$

where $w^{(i)}_{kl}$ are the entries of $\mathbf{W}_i$ figuring in

$$\mathbf{W} = \mathbf{V}^{-1} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \mathbf{W}_i \\ \mathbf{W}_m \end{bmatrix}$$

Elements of the inverse matrix can be obtained by recurrent relationships:

$$\begin{aligned} \tau^{(i)}_1 &= \frac{1}{t^{(i)}_1} \\ \tau^{(i)}_j &= -\frac{1}{t^{(i)}_1} \sum_{l=1}^{j-1} t^{(i)}_{l+1}\, \tau^{(i)}_{j-l} \\ & (j = 2, 3, \ldots, k_i) \end{aligned} \tag{A11}$$

or

$$\begin{aligned} \tau^{(i)}_1 &= \frac{1}{w^{(i)}_{k_i,n}} \\ \tau^{(i)}_j &= -\frac{1}{w^{(i)}_{k_i,n}} \sum_{h=1}^{j-1} w^{(i)}_{k_i-h,n}\, \tau^{(i)}_{j-h} \\ & (j = 2, 3, \ldots, k_i)\,. \end{aligned} \tag{A12}$$

Holding Eq. (A8) and (A9), from Eq. (A2) the final transformation matrix, $\mathbf{L}$ and its inverse, $\mathbf{L}^{-1}$ can be obtained. Let us start from the characteristic equation

$$D(s) = \prod_{i=1}^{m} (s - \lambda_i)^{k_i} \tag{A13}$$

and define

$$D_i(s) = \frac{D(s)}{(s - \lambda_i)^{k_i}}, \quad \Delta_i(s) = \frac{(s - \lambda_i)^{k_i}}{D(s)} \tag{A14}$$

Let us define furthermore

$$d_k^{(i)} \triangleq \frac{1}{(k-1)!} D_i^{(k-1)}(\lambda_i) \tag{A15}$$

and

$$\delta_k^{(i)} \triangleq \frac{1}{(k-1)!} \Delta_i^{(k-1)}(\lambda_i) \tag{A16}$$

where superscripts $(k-1)$ mean derivations.

Then, after some computation the following results become obvious:

$$\begin{aligned} \tau_k^{(i)} &= d_k^{(i)} \\ t_k^{(i)} &= \delta_k^{(i)} . \end{aligned} \tag{A17}$$

Because of these properties

$$\mathbf{T}_i^{-1}\mathbf{D}_i^{-1} = \begin{bmatrix} \tau_1^{(i)} & \tau_2^{(i)} & \dots & \tau_{k_i}^{(i)} \\ & \tau_1^{(i)} & \ddots & \vdots \\ & & \ddots & \tau_2^{(i)} \\ & & & \tau_1^{(i)} \end{bmatrix} \begin{bmatrix} \delta_{k_i}^{(1)} & & \delta_2^{(i)} & \delta_1^{(i)} \\ \vdots & \cdot & \delta_1^{(i)} & \\ \delta_2^{(i)} & \cdot & & \\ \delta_1^{(i)} & & & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ & & \cdot & \\ & 1 & & \\ 1 & & & \end{bmatrix} \triangleq \mathbf{E}_i \tag{A18}$$

the latter being a row changing matrix, and

$$\mathbf{T}^{-1}\mathbf{D}^{-1} = \mathbf{diag}\,[\mathbf{E}_1, \mathbf{E}_2, \dots, \mathbf{E}_m] . \tag{A19}$$

The inverse Vandermonde matrix can be expressed as (see [1] or [2]):

$$\mathbf{W} = \mathbf{V}^{-1} = \mathbf{D}^{-1}\mathbf{Q} \tag{A20}$$

and so

$$\mathbf{L}^{-1} = \mathbf{T}^{-1}\mathbf{W} = \mathbf{T}^{-1}\mathbf{D}^{-1}\mathbf{Q} \triangleq \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_m \end{bmatrix} \tag{A21}$$

where

$$\mathbf{N}_i = \begin{bmatrix} \frac{1}{(k_i-1)!} N_1^{(k_i-1)}(\lambda_i), & \frac{1}{(k_i-1)!} N_2^{(k_i-1)}(\lambda_i), & \dots, & \frac{1}{(k_i-1)!} N_n^{(k_i-1)}(\lambda_i) \\ \vdots & \vdots & & \\ N_1'(\lambda_i), & N_2'(\lambda_i) & , \dots, & N_n'(\lambda_i) \\ N_1(\lambda_i), & N_2(\lambda_i) & , \dots, & N_n(\lambda_i) \end{bmatrix} . \tag{A22}$$

On the other hand, from Eqs (A2) and (A6)

$$\mathbf{L} = \mathbf{VT} = \mathbf{V}\ \mathbf{diag}\ [\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_m]\,. \tag{A23}$$

Although L in Eq. (A23) is more complicated than the original VANDERMONDE matrix V, its inverse, as given in Eqs (A21) and (A22), is much easier to obtain.

Summary

The possibilities of canonical transformations are systematized. Defining four JORDAN canonical forms and four phase-variable forms permit eight similarity transformations as summarized in Table 3. Some modifications of the JORDAN canonical forms enables eight other similarity transformations as shown in Table 4. Illustrative examples are also given.

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