ON SOME QUESTIONS OF LINEAR SYSTEM IDENTIFICATION BY THE LAGUERRE ORTHONORMAL SYSTEM

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(Received: November 8, 1974.)

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Introduction

The aim of identification is to determine the mathematical model of the system in the knowledge of its measurable input and output signals. Assuming a given model structure the task is to assess the model parameters in a way that some functional of the difference between the output signals should reach a minimum (Fig. 1).

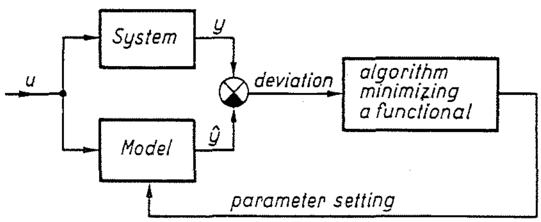

Fig. 1. Scheme of identification

The model is linear in its parameters, if

$$\hat{w} = \mathbf{c}^T \varphi \tag{1}$$

where $\hat{w}$ — the approximate weighting function of the system,
$\mathbf{c}$ — the vector of the parameters with
T — designating the transposition, and
φ — the vector of the functions regarded as weighting functions.

The structure is chosen by accepting the system φ; $\hat{w}$ might be approximated successfully by some infinite series, "cut-off" at a given value N. If the series is converging fast enough, i.e. if a low value of N can be chosen, then the approximation is efficient. An orthogonal system φ is advantageous for computation purposes.

The possibility of obtaining such a structure is offered by the system of the Laguerre functions as orthonormed in $(0, \infty)$.

The Laguerre functions may be given by the following relationship:

$$\varphi_n(t) = l_n(t) = \sqrt{2\alpha}\frac{e^{\alpha t}}{n!}\frac{d^n}{dt^n}\{t^n e^{-2\alpha t}\} \qquad n = 0, 1, 2, \ldots \tag{2}$$

or

$$l_n(t) = \sqrt{2\alpha}\, e^{-\alpha t}\sum_{k=0}^{n}\frac{n!\,(-2\alpha)^k}{(n-k)!\,(k!)^2}t^k \qquad n = 0, 1, 2, \ldots \tag{3}$$

It is advised to select the parameter α on the basis of a-priori knowledge concerning the system. The selection of α may be conceived as a scaling essentially affecting the rate of the convergence.

The Laguerre functions satisfy the following differential equation:

$$-t\ddot{\Phi}_n - \dot{\Phi}_n + t\alpha^2\Phi_n = 2\alpha\left(n + \frac{1}{2}\right)\Phi_n \tag{4}$$

so they may be regarded as weighting functions indeed. Their favourable property is that their Laplace-transforms can be derived from each other and can be formed easily:

$$L_n(s) = \frac{\sqrt{2\alpha}}{s+\alpha}\left(\frac{s-\alpha}{s+\alpha}\right)^n \qquad n = 0,1,2, \ldots \tag{5}$$

The coefficients c_n of the Laguerre expansion of the weighting function w may be formed by the relationship below based on the condition of orthogonality:

$$c_n = \frac{\int_0^\infty w(t)\, l_n(t)\, dt}{\int_0^\infty l_n^2(t)\, dt} \tag{6}$$

As the Laguerre system is orthonormal:

$$c_n = \int_0^\infty w(t)\, l_n(t)\, dt\,. \tag{7}$$

The error of the approximation may be evaluated by the following functional:

$$e_1 = \frac{\int_0^\infty \left(w - \sum_{k=0}^{N} c_k l_k\right)^2 dt}{\int_0^\infty w^2\, dt}, \tag{8}$$

The error depends on the number of the Laguerre terms N and the value of the parameter α.

On the selection of the parameter α of the Laguerre-functions

If the transfer function of the studied system is

$$Y(s) = \frac{A}{(1 + sT)^m},$$

then with $\alpha = \frac{1}{T}$ the system function is accurately reproduced by the Laguerre expansion to a number $N \geq m$ of the terms. The expansion is convergent also with other values of α, but then a greater number of the terms must be considered for attaining the specified error.

A practical method for selecting the parameter α is given by T. W. Parks [1] for the case where the following quantities can be determined by a test measurement:

$$M_1 = \int_0^\infty t w^2 \, dt ; \qquad M_2 = \int_0^\infty t \dot{w}^2 \, dt . \tag{9}$$

In this case the value of

$$\alpha = \sqrt{\frac{M_2}{M_1}} \tag{10}$$

may be assumed.

With this choice of α the e_1 error is given by

$$e_1 = \frac{1}{N+1} \left(\sqrt{m_1 m_2} - \frac{1}{2} \right) \quad \text{for} \quad \frac{1}{2} \leq \sqrt{m_1 m_2} \leq N + 1 + \frac{1}{2} \tag{11}$$

where

$$m_1 = \frac{M_1}{\int_0^\infty w^2 \, dt} \qquad \text{and} \qquad m_2 = \frac{M_2}{\int_0^\infty w^2 \, dt}$$

We have to remark that the α factor selected in this way generally doesn't coincide with its actual optimum value because of the inaccuracy of the given method of estimation. E.g., for the system $1/(1 + s)^2$ the α value given on the basis of (10) is 0.576 instead of the optimum value $\alpha = 1$ for which the Laguerre expansion of this transfer function is finite, containing only two nonvanishing terms L_0 and L_1. Nevertheless, the estimation provides quite good results considering the orders of magnitude and so it may give an initial α value for the identification.

A practical problem is that the signal y on the output of the system fails to agree with the weighting function. The method may be applied with the following modification:

Be the input signal u of the system the solution of the differential equation $Du = \delta$, where δ designates the Dirac-delta and D the differential operator.

Then the weighting function may be given from the y output signal by the following relationship:

$$w = Dy \tag{12}$$

This can be seen easily:
It is well known that

$$u * w = y \tag{13}$$

where $*$ denotes the convolution. Applying now the D differential operator to (13) we obtain

$$D(u * w) = Dy. \tag{14}$$

As $D(u * w) = (Du) * w = \delta * w = w$ (14) may be rewritten in the form

$$w = Dy$$

so equation (12) holds indeed.

Taking into account (12), relationship (10) can be given also in a form more suitable for the practical measurements:

$$\alpha = \frac{\int_0^\infty t(D\dot{y})^2\,dt}{\int_0^\infty t(Dy)^2\,dt} \tag{15}$$

Let the input signal be $u = 10 \cdot e^{-10t} \cdot 1(t)$, which is the solution of the differential equation $0.1\,\frac{du}{dt} + u = \delta(t)$, and so the differential operator is $D = 0.1\,\frac{d}{dt} + 1.$

Table 1. contains the values obtained for the parameter α for several systems with this input signal using equation (15).

Table 1

Transfer function	$\frac{\sqrt{2}}{s+1}\;\frac{s-1}{s+1}$	$\frac{1}{(1+s)^2}$	$\frac{1}{1+0.8s+s^2}$	$\frac{1}{(1+s)(1+5s)}$	$\frac{1}{(1+0.1s)(1+0.5s)}$
α	1.049	0.576	0.87	0.208	2.08

The error of the Laguerre-approximation

The measure of the error in terms of the weighting function deviation is given by relationship (8). During the experimentation acceptable values (below 5%) of the error interpreted in this way were obtained and a considerable deviation between the steady-state values of the approximate and the effective unit step responses appeared (Fig. 2). This fact is supported by the consideration that as the weighting function is the derivate of the unit step response, the error expressed by (8) can be no adequate measuring number of the steady-state deviation.

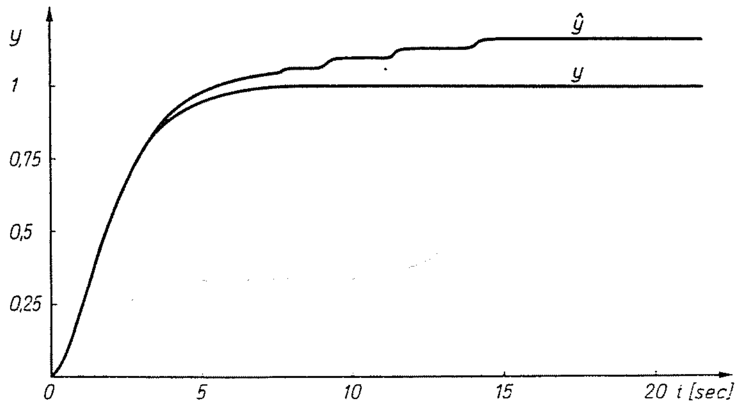

Fig. 2. In spite of the small weighting function error the deviation in the gain factor may be significant

For the Laguerre-approximation of proportional lag-elements in the steady-state the following relationship may be written up:

$$\lim_{s\to 0} \sum_{k=0}^{\infty} c_k \frac{\sqrt{2\alpha}}{s+\alpha} \left(\frac{s-\alpha}{s+\alpha}\right)^k = A \tag{16}$$

where A is the gain factor of the lag-element, i.e.

$$\sqrt{\frac{2}{\alpha}} \sum_{k=0}^{\infty} (-1)^k c_k = A \tag{17}$$

The measure of the deviation is seen to depend on the number of the approximation terms and on α.

For measuring the steady-state deviation we introduce the following measuring number:

$$e_2 = \frac{\left| A - \sqrt{\frac{2}{\alpha}} \sum_{k=0}^{N} (-1)^k c_k \right|}{A} . \tag{18}$$

The course of the errors e_1 and e_2 for the transfer function $Y(s) = \frac{1}{(1+s)^2}$ versus the number of the approximation terms is given by Fig. 3 for $\alpha = 0.5$ (here the course of e_2 is shown without absolute value) and by Figs 4, 5, 6 versus α with $N = 4, 6, 10$.

The number of the terms necessary for the prescribed error may be read off the diagram for a given α and in the case of a preset number of terms the permissible range for α, respectively, taking into account the more pessimistic e_2 curve. A resultant error may be defined as $\mu e_1 + \nu e_2$, μ and ν being weighting factors and $\mu + \nu = 1$.

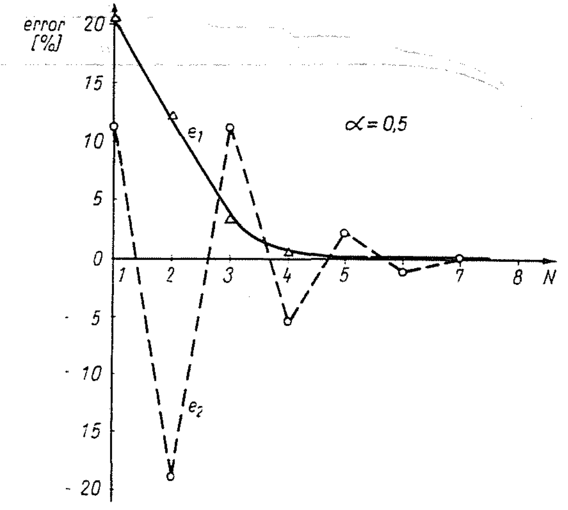

Fig. 3. Weighting function and gain factor deviations versus the number of the Laguerre terms

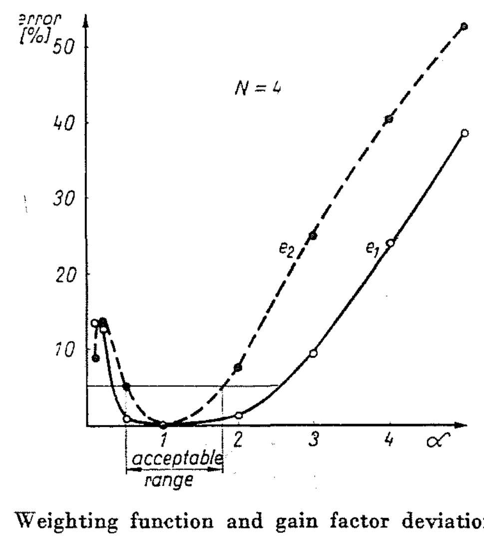

Fig. 4. Weighting function and gain factor deviations versus α

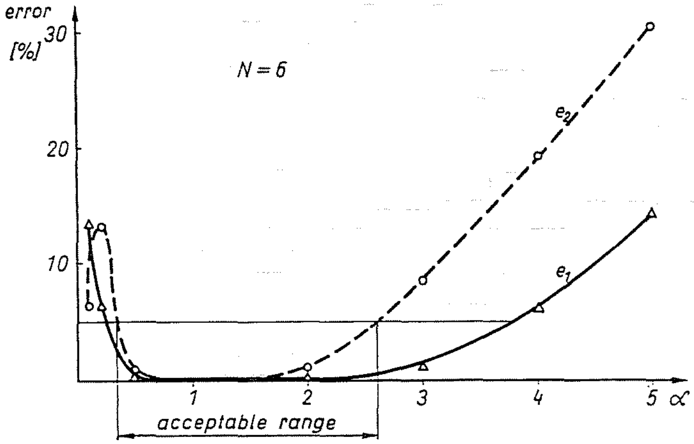

Fig. 5. Weighting function and gain factor deviations versus α

$N = 10$

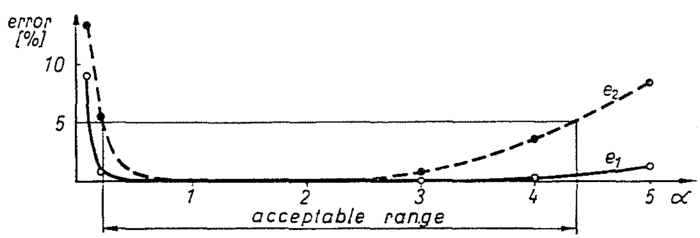

Fig. 6. Weighting function and gain factor deviations versus α

Identification by the Laguerre-expansion

Fig. 7 shows the off-line identification scheme suggested by N. Wiener (continuous line) utilizing relationship (6) to determine the coefficients of the orthogonal system. Here the following question arises:

Which are the conditions where the following equation is true?

$$c_n = \frac{\int_0^\infty w \cdot l_n dt}{\int_0^\infty l_n^2 dt} = \frac{M(y \cdot v_n)}{M(v_n^2)} \tag{19}$$

where M is the symbol of the expected value.

This relationship is only applicable when the output signals v_n of the Laguerre terms are orthogonal and this condition is met with when the input signal u is a white noise.

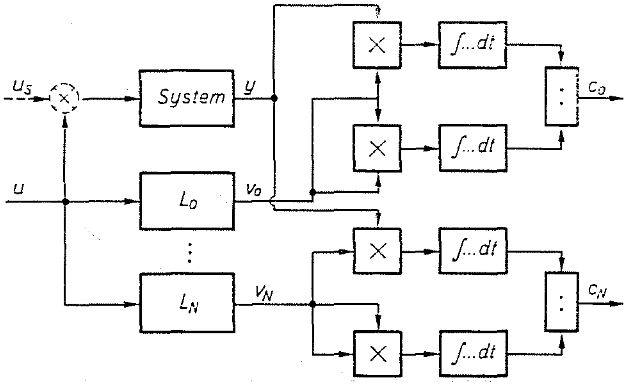

Fig. 7. Off-line identification scheme

YEGOROV [2] completed the scheme of Fig. 7 by the superposition of the signal u_s applied to the system (dashed part). If

$$M(u \cdot u_s) = 0\,, \tag{20}$$

then the values of the coefficients are not distorted by the effective input signal u_s of the system. The amplitude of the white test signal u may be selected low.

An on-line identification algorithm may be given for minimizing according to c the functional

$$I(\mathbf{c}) = M\left\{\frac{1}{2}(y - \mathbf{c}^T\mathbf{v})^2\right\} \tag{21}$$

According to the stochastic approximation method the value of the coefficient c_n may be derived in step $i+1$ from its value in step i with the help of the following algorithm [3]:

$$c_n[i] = c_n[i-1] + r_n[i]\left(y[i] - \sum_{k=0}^{N} c_k[i-1]\,v_k[i]\right) v_n[i] \tag{22}$$

r_n is here the coefficient of the convergence, which must satisfy the conditions of convergence given in [3]. By minimizing the quadratic deviation according to the coefficient of the convergence, the matrix of the optimum convergence coefficients may be obtained, which may be calculated also recursively [4, 5] as:

$$\boldsymbol{R}[i] = \boldsymbol{R}[i-1] - \frac{(\boldsymbol{R}[i-1]\,\mathbf{v}[i])(\boldsymbol{R}[i-1]\,\mathbf{v}[i])^T}{1 + \mathbf{v}^T[i]\,\boldsymbol{R}[i-1]\,\mathbf{v}[i]} \tag{23}$$

where

$$R[0] = \left[\sum_{m=-j}^{0} \mathbf{v}[m]\mathbf{v}^T[m] \right]^{-1} \tag{24}$$

is the result of a preceding off-line estimation by utilizing the right-hand side of relationship (19) to determine the values of the vector **v**.

A suboptimum scalar coefficient of the convergence requiring less computation, but giving a slower convergence may be obtained by the realtionship below:

$$r[i] = \frac{1}{y[i] - \mathbf{c}^T[i-1]\mathbf{v}[i]} \cdot \frac{\mathbf{v}^T[i]\left(\mathbf{k}[i] - \Phi[i]\,\mathbf{c}[i-1]\right)}{\mathbf{v}^T[i]\,\Phi[i]\,\mathbf{v}[i]} \tag{25}$$

with

$$\mathbf{k}[i] = \mathbf{k}[i-1] + y[i]\,\mathbf{v}[i]; \qquad \mathbf{k}[0] = \mathbf{0} \tag{26}$$

and

$$\Phi[i] = \Phi[i-1] + \mathbf{v}[i]\,\mathbf{v}^T[i]; \qquad \Phi[0] = 0 \tag{27}$$

Fig. 8 shows the course of the Laguerre coefficients c_0 and c_1 for the transfer function $Y(s) = 1/(1+s)^2$ during the identification. The input signal was a

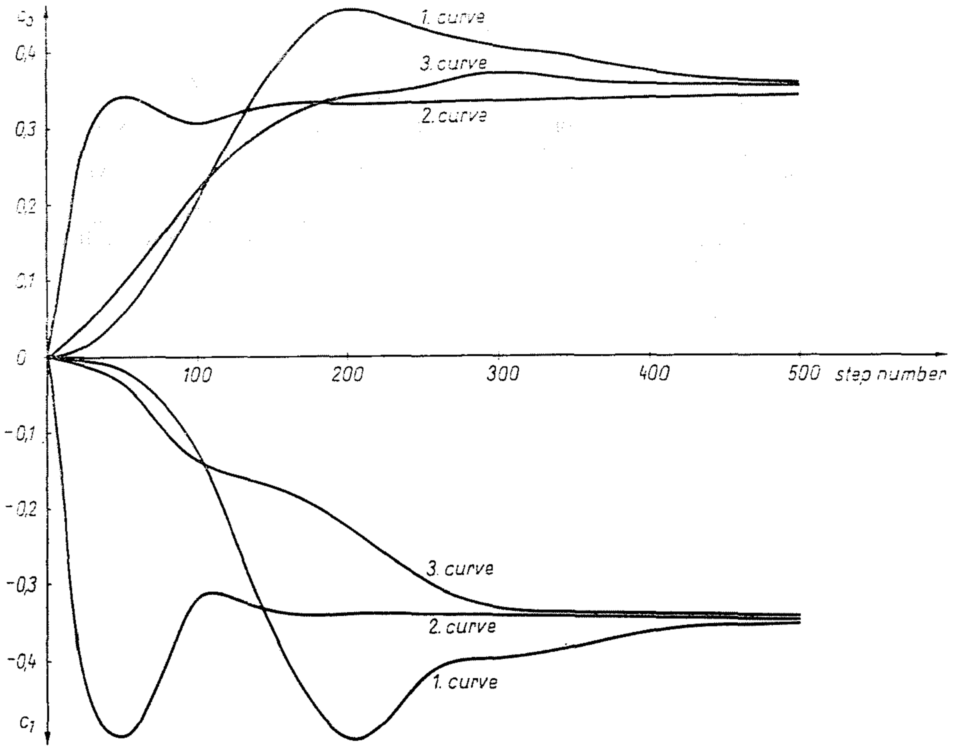

Fig. 8. The course of the Laguerre coefficients c_0 and c_1 during the identification for the system $Y(s) = 1/(1+s)^2$. Step size: 0,05 sec. 1. curve: off-line algorithm; 2. curve: stochastic approximation algorithm with optimum coefficient of convergence; 3. curve: stochastic approximation algorithm with suboptimum coefficient of convergence

sequence approximating white noise with the expected value zero. The fastest convergence is seen to be provided by the stochastic approximation algorithm (22) with the optimum convergence matrix (curves $c_0(2)$) and ($c_1(2)$). The off-line algorithm (19) utilizing the property of orthogonality gives also a quick set-up (curves $c_0(1)$, $c_1(1)$), and the results of the suboptimum stochastic approximation algorithm are also acceptable (curves $c_0(3)$, $c_1(3)$). The application of the above algorithms supplied good results in the case of other transfer functions as well, and the estimation was undistorted when condition (20) prevailed.

Summary

Some questions of the identification of linear systems by the Laguerre orthonormal system are dealt with. The choice of parameter α as a scaling factor of the Laguerre functions is touched upon. A measuring number is introduced for characterizing the correctness of the approximation of the identified gain factor. The course of the deviations of the weighting function and the gain factor, respectively, versus α and the number of the approximating terms is discussed. Some identification results obtained by applying the off-line algorithm utilizing the orthogonality property of the Laguerre system and by stochastic approximation algorithms are presented.

References

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