SOLUTION OF MAXWELL'S EQUATIONS BY RELATIVISTIC ELECTRIC AND MAGNETIC POTENTIALS

By

I. BÁRDI

Department of Theoretical Electricity, Technical University Budapest

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Presented by Prof. Dr. I. VÁGÓ

Introduction

Maxwell's equations are known to be covariant with respect to Lorentz's transformation. Formally, Maxwell's equations can be described by four-dimensional matrices formed of the respective quantities and by the operations performed with these. This four dimensional presentation of Maxwell's equations has been described in the literature.

In certain cases it is usual to complete Maxwell's equations by the fictitious magnetic charge and current. The first part of the present paper gives the four-dimension form of Maxwell's equations for this case.

In the further parts of the paper the solution of four-dimensional Maxwell's equations by means of four-dimensional potentials is discussed. References [1], [2] present only the four-dimensional solution by means of the electric scalar and vector potential for the case of known electric charge and current density.

If charge and current density are unknown, and there is also a magnetic charge and current density, there exists no published four-dimension electric solution, or one based on the four-dimensional magnetic potentials.

The present paper gives a solution for this general case.

In the knowledge of the relevant four-dimension potential, changing over to co-ordinate systems performing uniform, rectilinear movement relative to each other, Lorentz's transformation has to be applied on the four-dimensional potential vector.

In the case of motional induction problems, this can be used advantageously. Looking for the solution in one of the co-ordinate systems in the form of TM or TE mode, the electric or magnetic four-dimension potential should be transformed into the other co-ordinate system. If the two co-ordinate systems are fixed to field parts with different material constants, the boundary condition can be written for the potentials.

The method was practically applied for determining the electromagnetic field in linear induction motors. Because of space shortage however, the example cannot be published.

The four-dimension form of Maxwell's equation

Introducing a new co-ordinate, time, as the fourth dimension, Maxwell's equations can be written in four-dimension form.

First let us recapitulate the three-dimension form of Maxwell's equations completed by magnetic charge and current density.

$$\mathrm{rot}\, H = J^E + \frac{\partial D}{\partial t} \tag{1}$$

$$\mathrm{rot}\, E = -J^M - \frac{\partial B}{\partial t} \tag{2}$$

$$\mathit{div}\, D = \varrho^E \tag{3}$$

$$\mathit{div}\, B = \varrho^M \tag{4}$$

$$D = \varepsilon E;\ B = \mu H;\ J^E = \sigma^E E;\ J^M = \sigma^M H;\ \varepsilon = \varepsilon_0 \varepsilon_r;\ \mu = \mu_0 \mu_r \tag{5}$$

(the inserted electric and magnetic field strength values E_b, H_b, being zero).
Here:
ϱ^E and ϱ^M, the electric and magnetic charge densities in unit volume, respectively,
J^E and J^M, the electric and magnetic current densities, respectively,
σ^E and σ^M, electric and magnetic conductivity, respectively.
(Italicized bold face type denotes three-dimension vectors.) The meaning of other denominations is as usual.

By introducing fictitious magnetic charge and current density, the system of Maxwell's equations shows a more better complete symmetry.

Two members in each of equation groups (1), (2), further of (3), (4), and (5) are to be formally identical.

The structural identity of equations describing various physical processes points to a regularity between corresponding quantities of the two different physical phenomena, called duality in the literature. The corresponding quantities are dual quantities.

On the basis of the equations the dual quantities are:

$$B - D;\ H - E;\ \varrho^M - \varrho^E;\ J^M - J^E;\ \mu - \varepsilon;\ \sigma^M - \sigma^E .$$

Accordingly, the magnetic and electric quantities are dual ones.

Whether magnetic charge and current density exist in reality or not, is irrelevant for the discussion, since the duality principle means physical quantities to correspond each other on formal basis. Thereby different phenomena can be discussed by identical mathematical methods involving a simplification. It should be noted that magnetic currents and dipoles can be realized by elementary current loops.

Rewriting the equations to four dimensions, using the method usual in the literature, we obtain:

$$\mathbf{div}\ \mathsf{G} = \mathbf{s}^E \tag{6}$$

$$\mathbf{rot}\ \mathsf{F} = -\mathbf{div}\ \mathsf{F}^* = \mathbf{s}^{M'} \tag{7}$$

$$\mathbf{s}^E = \sigma^E \mathsf{F}\mathbf{u};\quad \mathbf{s}^{M'} = \sigma^M \mathsf{G}^*\mathbf{u};\quad \mathsf{G} = \frac{1}{\mu}\mathsf{F} \tag{8}$$

where

$$\mathsf{F} = \begin{bmatrix} 0 & B_z & -B_y & -j/c\,E_x \\ -B_z & 0 & B_x & -j/c\,E_y \\ B_y & -B_x & 0 & -j/c\,E_z \\ j/c\,E_x & j/c\,E_y & j/c\,E_z & 0 \end{bmatrix};\quad \mathbf{s}^{M'} = \begin{bmatrix} j/c\,J^M \\ \varrho^M \end{bmatrix};\quad \mathbf{s}^E = \begin{bmatrix} J^E \\ j/c\ \ \varrho^E \end{bmatrix}. \tag{9}$$

(Italicized heavy types designate four-dimension vectors, while standing heavy types four-dimension tensors. The meaning of asterisk(*) and description of operations on four-dimension tensors are found in the Appendix.)

Four-dimension equations in this form are not dual formally. With a view to make the equations formally dual, let us introduce the tensor

$$\mathsf{K} = -jc\,\mathsf{F}^*. \tag{10}$$

Eq. (7) becomes:

$$\mathbf{div}\ \mathsf{K} = \mathbf{s}^M \tag{11}$$

resulting formally in a form corresponding to Eq. (6), where

$$\mathbf{s}^M = \begin{bmatrix} J^M \\ jc\ \ \varrho^M \end{bmatrix}.$$

These yield the four-dimension Maxwell's equations:

$$\mathbf{div}\ \mathsf{G} = \mathbf{s}^E \tag{12}$$

$$\mathbf{div}\ \mathsf{K} = \mathbf{s}^M \tag{13}$$

$$\mathbf{s}^E = \frac{j}{c}\,\sigma^E \mathsf{K}^*\mathbf{u} \tag{14}$$

$$\mathbf{s}^M = \frac{1}{jc}\,\sigma^M\,\mathsf{G}^*\mathbf{u} \tag{15}$$

$$\mathsf{K} = -j \sqrt{\frac{\mu}{\varepsilon}}\, \mathsf{G}^* \tag{16}$$

where

$$\mathbf{u} = \begin{bmatrix} \varkappa \mathsf{v} \\ j \varkappa c \end{bmatrix}; \quad \varkappa = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Duality is evident also in the four-dimension presentation. The dual of the electric current density $\mathbf{s}^E$ is the magnetic current density $\mathbf{s}^M$, that of tensor G is tensor K, while the dual of σ^E is σ^M.

Material characteristics are lacking from Eqs (12) and (13). The correlation between tensors K and G in Eq. (16) is determined by the material characteristics. Eqs (14) and (15) are the differential electric and magnetic Ohm's law, respectively.

If ε and μ are constant in each field part, the equations can be written in the following form:

$$\mathbf{div}\ \mathsf{F} = \mu \mathbf{s}^E$$

$$\mathbf{div}\ \mathsf{P} = \varepsilon \mathbf{s}^M$$

$$\mathbf{s}^E = \sigma^E\, \mathsf{F}\mathbf{u}; \quad \mathbf{s}^M = \sigma^M\, \mathsf{P}\mathbf{u}; \quad \mathsf{P} = -j \sqrt{\frac{\varepsilon}{\mu}}\, \mathsf{F}^*$$

where

$$\mathsf{P} = \frac{1}{\varepsilon} \mathsf{K}.$$

In the following, the solution of these equations will be discussed.

Solution by introducing the electric vector potential

Maxwell's equations represent a system of partial differential equations, to be solved with respect to the boundary conditions corresponding to the given problem. For the case of the three-dimension way of writing, the solution method with electric and magnetic scalar and vector potentials has been developed. The method of electric and magnetic potentials is a dual one.

For the case of the four-dimension way of writing, no corresponding four-dimension methods have been developed. In the literature only a solution with electric potentials for the case $\bar{\mathbf{s}}^M = \mathbf{O}$ can be found for known electric charge and current distribution.

The equations to be solved are

$$\mathbf{div}\ \mathsf{F} = \mu \mathbf{s}^E; \ \mathbf{div}\ \mathsf{P} = \mathbf{O}; \ \mathbf{s}^E = \sigma^E \mathsf{F}\mathbf{u}; \ \mathsf{P} = -j \sqrt{\frac{\varepsilon}{\mu}}\, \mathsf{F}^*.$$

The logic of the solution is identical with that for the three-dimension case.

Let

$$\mathsf{F} = \mathbf{rot}\ \mathbf{A} \tag{17}$$

where $\mathbf{A}$ denotes four-dimension electric potential.

By this choice the equation $\mathbf{div}\ \mathsf{P} = \mathbf{O}$ becomes an identity, namely

$$\mathbf{rot}\ \mathbf{rot}\ \mathbf{A} \equiv \mathbf{O}\,.$$

Substituting into Eq. (6):

$$\mathbf{div}\ \mathbf{rot}\ \mathbf{A} = \mu\ \mathbf{s}^E. \tag{18}$$

Since

$$\mathbf{div}\ \mathbf{rot}\ \mathbf{A} = \mathbf{grad}\ \mathbf{div}\ \mathbf{A} - \Box\ \mathbf{A}\,, \tag{19}$$

choosing

$$\mathbf{div}\ \mathbf{A} = 0, \tag{20}$$

we obtain the inhomogeneous differential equation:

$$\Box\,\mathbf{A} = -\mu\,\mathbf{s}^E. \tag{21}$$

(For the designations see Appendix.)

The first three components of the equation are identical with the inhomogeneous wave equation for the electric vector potential, while the fourth component is the equation relating to the electric scalar potential. Thus in the four-dimension vector potential $\mathbf{A}$ the first three elements represent the three-dimension vector potential, while the fourth element the scalar potential.

Also the homogeneous wave equation can be obtained by choosing (17), in the case of $\mathbf{s}^M = \mathbf{O}$. In the case of $\mathbf{s}^M \neq \mathbf{O}$, however, this is not true of the four-dimension formalism. The same problem arises in the magnetic potential-type the solution in the case of $\mathbf{s}^E \neq \mathbf{O}$.

To solve the problems needs in the general case, both the magnetic and electric potentials. In their knowledge, the linearly independent TE and TM mode solutions can be determined.

In the following the solution relating to vector potentials will be examined for the case where $\mathbf{s}^E \neq \mathbf{O}$ and $\mathbf{s}^M \neq \mathbf{O}$, and the distribution of current densities is not known.

For determining the electric potential let

$$\mathsf{F} = \mathbf{rot}\ \mathbf{A} + \mathsf{V}. \tag{22}$$

This choice can be justified also by formal logical considerations. Namely in solving Maxwell's equations, completed by the magnetic charge and current densities, by means of the three-dimension wave equation relating to electric potentials, the electric field is seen to have a component proportional to the vector potential. The elements of tensor V accordingly consist of the elements of the vector potential. These help to make statements on the structure of tensor V. Namely, the first three rows and columns of tensor F contain the vector of magnetic induction. In the three-dimension solution, vector **B** is in an other than proportional relationship with the electric vector potential, thus these elements of tensor V are zero.

F being an alternating tensor,

$$\mathsf{V} = \begin{bmatrix} 0 & 0 & 0 & V_{14} \\ 0 & 0 & 0 & V_{24} \\ 0 & 0 & 0 & V_{34} \\ -V_{14} & -V_{24} & -V_{34} & 0 \end{bmatrix} \tag{23}$$

V_{14}, V_{24}, V_{34} can be formed from the elements of vector **A**. To determine these, choose the elements of tensor V in such a way that substituting (22) Eq. (11) becoming an identity:

$$-j\sqrt{\frac{\varepsilon}{\mu}}\,\mathbf{div}\,(\mathbf{rot}\,\mathbf{A} + \mathsf{V})^* = \varepsilon \mathbf{s}^M , \tag{24}$$

that is

$$j\sqrt{\frac{\varepsilon}{\mu}}\,\mathbf{rot}\,(\mathbf{rot}\,\mathbf{A} + \mathsf{V}) = \varepsilon \mathbf{s}^M . \tag{25}$$

Since

$$\mathbf{rot}\,\mathbf{rot}\,\mathbf{A} \equiv \mathbf{O} , \tag{26}$$

thus

$$-j\sqrt{\frac{\varepsilon}{\mu}}\,\mathbf{div}\,\mathsf{V}^* = \varepsilon \mathbf{s}^M . \tag{27}$$

Using Eq. (14):

$$\mathbf{div}\,\mathsf{V}^* = \varepsilon\sigma^M\left[(\mathbf{rot}\,\mathbf{A})^* + \mathsf{V}^*\right]\mathbf{u} . \tag{28}$$

The dual of tensor V is found to be

$$\mathsf{V}^* = \begin{bmatrix} 0 & V_{34} & -V_{24} & 0 \\ -V_{34} & 0 & V_{14} & 0 \\ V_{24} & -V_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{29}$$

Looking for the solution in the system of co-ordinates fixed to the charges:

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ u_4 \end{bmatrix}; \qquad u_4 = j\varkappa c\,. \tag{30}$$

With this condition:

$$\mathsf{V}^*\mathbf{u} \equiv 0\,. \tag{31}$$

Thus

$$\mathbf{div}\ \mathsf{V}^* = \varepsilon\sigma^M (\mathbf{rot}\ \mathsf{A})^* \mathbf{u}\,. \tag{32}$$

This is the equation for determining the elements of tensor V. Upon performng the operations we obtain:

$$V_{14} = \varepsilon\sigma^M u_4 A_1, \tag{33}$$

$$V_{24} = \varepsilon\sigma^M u_4 A_2, \tag{34}$$

$$V_{34} = \varepsilon\sigma^M u_4 A_3. \tag{35}$$

If Eq. (35) is satisfied, Eq. (11) becomes an identity. Hereafter Eq. (6) has to be solved.

$$\mathbf{div}\ (\mathbf{rot}\ \mathbf{A} + \mathsf{V}) = (\sigma^E \mu\ \mathbf{rot}\ \mathbf{A} + \mathsf{V})\ \mathbf{u}\,. \tag{36}$$

Using relationship (19):

$$\diamond\ \mathbf{A}^+ \diamond - \Box\ \mathbf{A} + \mathbf{div}\ \mathsf{V} = \sigma^E \mu\ [\diamond\ \mathbf{A}^+ - \mathbf{A} \diamond^+ + \mathsf{V}]\ \mathbf{u}\,. \tag{37}$$

(see Appendix).

Since only the condition for the rotation of vector **A** has been given, let

$$\mathbf{A}^+ \diamond = \sigma^E \mu \mathbf{A}^+ \mathbf{u}\,. \tag{38}$$

Using condition (38) and performing the operations, Eq. (37) becomes:

$$\Box\ \mathbf{A} - (\varepsilon\sigma^M + \mu\sigma^E)\ \mathbf{A} \diamond^+ \mathbf{u} - \sigma^E \sigma^M\ \mathbf{A} = \mathbf{O}\,. \tag{39}$$

The wave equation relating to electric potentials has thus been written by means of the four-dimension mathematical formalism. Eq. (38) is Lorentz's condition.

Solution by introducing the magnetic vector potential

In the following a method dual to that in the preceding chapter is presented. The magnetic potential $\mathbf{A}^M$ supplying the solution of four-dimension Maxwell's equations is looked for in the following form:

$$\mathsf{P} = \mathbf{rot}\ \mathbf{A}^M + \mathsf{V}^M\,. \tag{40}$$

Tensor V^M has to be chosen so that Eq. (6) be an identity. Similarly as in the considerations above,

$$\mathsf{V}^M = \begin{bmatrix} 0 & 0 & 0 & V_{14}^M \\ 0 & 0 & 0 & V_{24}^M \\ 0 & 0 & 0 & V_{34}^M \\ -V_{14}^M & -V_{24}^M & -V_{34}^M & 0 \end{bmatrix}. \tag{41}$$

Substituting into Eq. (6):

$$\textbf{div}\ \mathsf{V}^M = \mu\sigma^E \left[(\textbf{rot}\ \mathbf{A}^M)^* + \mathsf{V}^{M*}\right] \mathbf{u}\,. \tag{42}$$

This equation is the dual of Eq. (28).
V^M, $\mathbf{A}^M$, μ and σ^M correspond to tensors V, $\mathbf{A}$, ε and σ^E respectively. Thus:

$$\begin{aligned} V_{14}^M &= \mu\sigma^E\, u_4 A_1^M, \\ V_{24}^M &= \mu\sigma^E\, u_4 A_2^M, \\ V_{34}^M &= \mu\sigma^E\, u_4 A_3^M. \end{aligned} \tag{43}$$

Substituting into Eq. (11):

$$\textbf{div}\ (\textbf{rot}\ \mathbf{A}^M + \mathsf{V}^M) = \varepsilon\sigma^M (\textbf{rot}\ \mathbf{A}^M + \mathsf{V}^M)\, \mathbf{u}\,. \tag{44}$$

This equation, in turn, is the dual of Eq. (28).

These are used for the magnetic potentials, to obtain the wave equation

$$\Box\, \mathbf{A}^M - (\varepsilon\sigma^M + \mu\sigma^E)\, \mathbf{A}^M \diamond^{+} \mathbf{u} - \sigma^E \sigma^M \mathbf{A}^M = \mathbf{O}, \tag{45}$$

and Lorentz's condition

$$\mathbf{A}^{M+} \diamond = \sigma^M \varepsilon\, \mathbf{A}^M\, \mathbf{u}\,. \tag{46}$$

Accordingly, the four-dimension mathematical formalism helps to write not only Maxwell's equations, but also the three-dimension solution methods. Thus the solution method applying the potentials is covariant with respect to Lorentz's transformation, electric and magnetic potentials can be transformed as four-dimension vectors, in systems of co-ordinates performing uniform, rectilinear movement relative to each other.

Appendix

The most important definitions and operations relating to four-dimension tensors are summarized in the following.

In an alternating or antimetric tensor, elements symmetrically arranged with respect to the main diagonal differ only by sign.

$$V_{ik} = -\,V_{ki}\,.$$

Hence,

$$V_{kk} = 0.$$

A tensor V* can be assigned to alternating tensor V for the elements of which

$$V^*_{kl} = V_{mn}.$$

k, l, m, n represent the even number permutation of the series 1, 2, 3, 4. V* constructed in this way is the dual tensor of tensor V.

$$\mathsf{V} = \begin{bmatrix} 0 & V_{12} & V_{13} & V_{14} \\ -V_{12} & 0 & V_{23} & V_{24} \\ -V_{13} & -V_{23} & 0 & V_{34} \\ -V_{14} & -V_{24} & -V_{34} & 0 \end{bmatrix};$$

$$\mathsf{V}^* = \begin{bmatrix} 0 & V_{34} & -V_{24} & V_{23} \\ -V_{34} & 0 & V_{14} & -V_{13} \\ V_{24} & -V_{14} & 0 & V_{12} \\ -V_{23} & V_{13} & -V_{12} & 0 \end{bmatrix}.$$

The four-dimension Hamilton's operator is given by

$$\diamond = \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_4} \end{bmatrix}.$$

The four-dimension Laplace operator:

$$\Box = \diamond^* \diamond = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}.$$

$\diamond^*$ denotes the transpose of $\diamond$.

The operations on the operators are controlled by the formal rules of matrix algebra.

$$\mathrm{grad}\, s = \diamond s \qquad (s \ldots \text{scalar})$$

$$\mathrm{div}\, \mathbf{v} = \diamond^{+} \mathbf{v} = \mathbf{v}^{+} \diamond$$

$$\mathrm{rot}\, \mathbf{v} = \diamond \mathbf{v}^{+} - \mathbf{v} \diamond^{+}$$

$$\mathrm{grad}\, \mathbf{v} = \mathbf{v} \diamond^{+}$$

$$\mathrm{div}\, \mathsf{T} = \mathsf{T} \diamond = [\diamond^{+} \mathsf{T}^{+}]^{+}$$

$$\mathrm{div}\, \mathrm{grad}\, s = \diamond^{+} \diamond s = \square s$$

$$\mathrm{div}\, \mathrm{grad}\, \mathbf{v} = \mathbf{v} \diamond^{+} \diamond = \mathbf{v} \square$$

$$\mathrm{div}\, \mathrm{rot}\, \mathbf{v} = [\diamond \mathbf{v}^{+} - \mathbf{v} \diamond^{+}] \diamond = \diamond \mathbf{v}^{+} \diamond - \mathbf{v} \diamond^{+} \diamond = \mathbf{grad}\, \mathbf{div}\, \mathbf{v} - \mathbf{v} \square$$

$$\mathrm{rot}\, \mathsf{V} = -\mathbf{div}\, \mathsf{V}^{*} = -\mathsf{V}^{*} \diamond \qquad (\text{if } \mathsf{V} \text{ is alternating tensor.})$$

Summary

In certain cases Maxwell's equations are completed by a fictitious magnetic charge and current. The paper gives the way of writing the completed relativistic Maxwell's equations. A general method is elaborated for solving the electromagnetic field by the relativistic electric and magnetic vector potential. The method can be advantageously used in cases of determining the electromagnetic field in two co-ordinate systems performing rectilinear, uniform movement relative to each other.

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István BÁRDI, H-1521 Budapest