ANALYTICAL INVESTIGATION OF THE QUASI TEM MODE OF THE SHIELDED MICROSTRIP AT LOW AND HIGH FREQUENCIES

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1. Introduction

Several papers have appeared on the quasi TEM mode of shielded microstrips [3]—[10]. In all of these papers the unknown phase constant is determined from a large determinant equation, where the elements of the determinant are transcendent functions of the phase constant. At zero frequency the normalized phase constant is obtained by extrapolation. High-frequency investigations have not been made yet. Though high-frequency results are of practical interest by permitting to conclude on the qualitative variation of the dispersion curve.

In this paper an analytical solution is given for the phase constant and for the field at zero frequency. The high-frequency behaviour also is investigated. The method of the author published recently [1] is used in the paper.

2. Zero-frequency case

Let us examine the shielded microstrip in Fig. 1. The unknown aperture electric field can be determined (according to the (14) formula of [1]) from the following system of equations

$$\sum_{i=1}^{2}\left[\widetilde{\mathbf{P}}\left\langle (k_0^2\varepsilon_i-\beta^2)\,\frac{\operatorname{ctg}\sqrt{k_0^2\varepsilon_i-\beta^2-\left(\frac{n\pi}{b}\right)^2}\,a_i}{k_0\sqrt{k_0^2\varepsilon_i-\beta^2-\left(\frac{n\pi}{b}\right)^2}}\right\rangle \mathbf{P}\mathbf{V}^0 - \right.$$

$$\left. -\beta\widetilde{\mathbf{P}}\left\langle \frac{n\pi}{b}\cdot\frac{\operatorname{ctg}\sqrt{k_0^2\varepsilon_i-\beta^2-\left(\frac{n\pi}{b}\right)^2}\,a_i}{k_0\sqrt{k_0^2\varepsilon_i-\beta^2-\left(\frac{n\pi}{b}\right)^2}}\right\rangle \mathbf{Q}\mathbf{W}\right]=\mathbf{0}\,, \tag{1}$$

$$\sum_{i=1}^{2}\left[\beta\tilde{\mathbf{Q}}\left\langle\frac{n\pi}{b}\;\frac{\operatorname{ctg}\sqrt{k_0^2\varepsilon_i-\beta^2-\left(\frac{n\pi}{b}\right)^2}\,a_i}{k_0\sqrt{k_0^2\varepsilon_i-\beta^2-\left(\frac{n\pi}{b}\right)^2}}\right\rangle\mathbf{P}\mathbf{V}^0+\right.$$
$$\left.+\tilde{\mathbf{Q}}\left\langle\left(\frac{n^2\pi^2}{b^2}-k_0^2\varepsilon_i\right)\frac{\operatorname{ctg}\sqrt{k_0^2\varepsilon_i-\beta^2-\left(\frac{n\pi}{b}\right)^2}\,a_i}{k_0\sqrt{k_0^2\varepsilon_i-\beta^2-\left(\frac{n\pi}{b}\right)^2}}\right\rangle\mathbf{Q}\mathbf{W}\right]=\mathbf{0}, \tag{2}$$

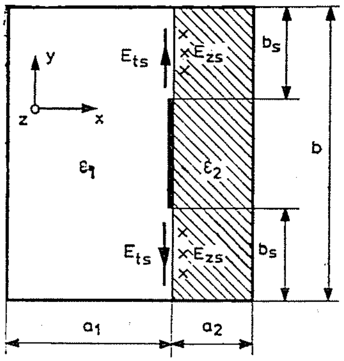

Fig. 1. The geometry of the problem

where β is the phase constant to be determined, k_0 is the free-space phase constant, $\mathbf{V}^0$ and $\mathbf{W}$ are the expanding coefficients of the unknown aperture field components:

$$E_{ts}=\sum_{p=0}^{P}(-1)^p V_p \cos\frac{p\pi}{b_s}y, \tag{3}$$

$$E_{zs}=\sum_{p=1}^{P}(-1)^p W_p \sin\frac{p\pi}{b_s}y. \tag{4}$$

The elements of matrices $\mathbf{P}$ and $\mathbf{Q}$ are:

$$P_{np}=\frac{n}{b_s b}\cdot\frac{\sin n\pi b_s/b}{(n/b)^2-(p/b_s)^2},\quad n=1,3,5,\ldots \quad p=0,1,2,\ldots P. \tag{5}$$

$$Q_{np}=\frac{p}{n}\cdot\frac{b}{b_s}P_{np},\quad n=1,3,5,\ldots \quad p=0,1,2,\ldots P. \tag{6}$$

Column vector $\mathbf{V}^0$ includes the coefficient V_0.

The physical meaning of Eqs (1) and (2) is to demand continuity of the tangential magnetic field generated by the aperture electric field in the aperture.

Let us divide (1) by k_0, (2) by $\pi p/b_s$, introduce the notation $Y = \beta^2/k_0^2$ and eliminate $\mathbf{Q}$ using (6). Finally replace W by introducing a new variable such as:

$$\frac{1}{k_0} \cdot \frac{\pi}{b_s} \langle p \rangle \mathbf{W} = \mathbf{W}' . \tag{7}$$

Then (1) and (2) become

$$\sum_{i=1}^{2} \left[\left\langle \tilde{\mathbf{P}} (\varepsilon_i - Y) \frac{\operatorname{ctg} \sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\frac{n\pi}{b}\right)^2} a_i}{\sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\frac{n\pi}{b}\right)^2}} \right\rangle \mathbf{P}\mathbf{V}^0 - \right.$$
$$\left. - \sqrt{Y} \tilde{\mathbf{P}} \left\langle \frac{\operatorname{ctg} \sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\frac{n\pi}{b}\right)^2} a_i}{\sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\frac{n\pi}{b}\right)}} \right\rangle \mathbf{P}' \mathbf{W}' \right] = \mathbf{0}, \tag{8}$$

$$\sum_{i=1}^{2} \left[\sqrt{Y} \tilde{\mathbf{P}}' \left\langle \frac{\operatorname{ctg} \left| \sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\frac{n\pi}{b}\right)^2} a_i \right.}{\sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\frac{n\pi}{b}\right)^2}} \right\rangle \mathbf{P}\mathbf{V}^0 + \right.$$
$$\left. + \tilde{\mathbf{P}}' \left\langle \left(1 - k_0^2 \frac{\varepsilon_i b^2}{n^2 \pi^2}\right) \frac{\operatorname{ctg} \sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\frac{n\pi}{b}\right)^2} a_i}{\sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\frac{n\pi}{b}\right)^2}} \right\rangle \mathbf{P}' \mathbf{W}' \right] = \mathbf{0}. \tag{9}$$

$\mathbf{P}'$ means that $p \neq 0$ in the elements P'_{np}.

Now the zero frequency limit is examined. Then the second term in the brackets of (9) is zero, so all matrix elements in (8) and (9) contain the sum (relating e.g., for the left-side subwaveguide of subscript 1):

$$T^1_{pr} = \tilde{\mathbf{P}} \left\langle \frac{\operatorname{cth} n\pi a_1/b}{n\pi/b} \right\rangle \mathbf{P} \Big|_{pr} = \sum_{n=1,3,\dots}^{\infty} P_{np} P_{nr} \frac{\operatorname{cth} n\pi a_1/b}{n\pi/b} =$$
$$= \sum_{n=1,3,\dots}^{\infty} \frac{1}{\pi b b_s^2} \cdot \frac{n \sin^2 (n\pi b_s/b) \cdot \operatorname{cth} n\pi a_1/b}{[(n/b)^2 - (p/b_s)^2]\,[(n/b)^2 - (r/b_s)^2]}, \tag{10}$$
$$p, r = 0, 1, 2, \dots, P.$$

Because of the oscillatory nature of the sine function and of the singularity caused by the denominator, series (10) is little fit for practical applications. The result of a series transformation can be found in the Appendix. This transformation is an *essential* step for the furthers.

Let us introduce notations:

$$A_1 = T^1_{00}; \qquad A_2 = T^2_{00}, \tag{11}$$

$$\mathbf{B}_1 = [T^1_{p0}]; \qquad \mathbf{B}_2 = [T^2_{p0}], \qquad p = 1, 2, \ldots, P. \tag{12}$$

$$\mathbf{S}_1 = [T^1_{pr}]; \qquad \mathbf{S}_2 = [T^2_{pr}], \qquad p, r = 1, 2, \ldots, P. \tag{13}$$

With these notations, introducing $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon$ (8) and (9) can be written as:

$$\begin{bmatrix} (1-Y)A_1+(\varepsilon-Y)A_2 & (1-Y)\tilde{\mathbf{B}}_1+(\varepsilon-Y)\tilde{\mathbf{B}}_2 & -\sqrt{Y}(\tilde{\mathbf{B}}_1+\tilde{\mathbf{B}}_2) \\ (1-Y)\mathbf{B}_1+(\varepsilon-Y)\mathbf{B}_2 & (1-Y)\mathbf{S}_1+(\varepsilon-Y)\mathbf{S}_2 & -\sqrt{Y}(\mathbf{S}_1+\mathbf{S}_2) \\ \sqrt{Y}(\mathbf{B}_1+\mathbf{B}_2) & \sqrt{Y}(\mathbf{S}_1+\mathbf{S}_2) & \mathbf{S}_1+\mathbf{S}_2 \end{bmatrix} \begin{bmatrix} V_0 \\ \mathbf{V} \\ \mathbf{W}' \end{bmatrix} = \mathbf{0}. \tag{14}$$

$\mathbf{V}$ in (14) does not contain V_0. Separating the constant term of the transverse aperture field permits to obtain explicit results.

The determinant of (14) can be transformed into:

$$\begin{vmatrix} (1-Y)A_1+(\varepsilon-Y)A_2 & \tilde{\mathbf{B}}_1+\varepsilon\tilde{\mathbf{B}}_2 & -(\tilde{\mathbf{B}}_1+\tilde{\mathbf{B}}_2) \\ \mathbf{B}_1+\varepsilon\mathbf{B}_2 & \mathbf{S}_1+\varepsilon\mathbf{S}_2 & \mathbf{0} \\ Y(\mathbf{B}_1+\mathbf{B}_2) & \mathbf{0} & \mathbf{S}_1+\mathbf{S}_2 \end{vmatrix} = 0. \tag{15}$$

Using the rule of computation of determinant partitioned into blocks, (15) becomes:

$$(1-Y)A_1+(\varepsilon-Y)A_2-(\tilde{\mathbf{B}}_1+\varepsilon\tilde{\mathbf{B}}_2)(\mathbf{S}_1+\varepsilon\mathbf{S}_2)^{-1}(\mathbf{B}_1+\varepsilon\mathbf{B}_2)+ \\ + Y(\tilde{\mathbf{B}}_1+\tilde{\mathbf{B}}_2)(\mathbf{S}_1+\mathbf{S}_2)(\mathbf{B}_1+\mathbf{B}_2) = 0. \tag{16}$$

From (16) we get

$$Y = \frac{A_1+\varepsilon A_2-(\tilde{\mathbf{B}}_1+\varepsilon\tilde{\mathbf{B}}_2)(\mathbf{S}_1+\varepsilon\mathbf{S}_2)^{-1}(\mathbf{B}_1+\varepsilon\mathbf{B}_2)}{A_1+A_2-(\tilde{\mathbf{B}}_1+\tilde{\mathbf{B}}_2)(\mathbf{S}_1+\mathbf{S}_2)^{-1}(\mathbf{B}_1+\mathbf{B}_2)}. \tag{17}$$

(17) is the fundamental result of this paper. An explicit formula expressed by the matrix elements (11) to (13) was obtained for the normalized phase constant at zero frequency. For $\varepsilon = 1$ (17) yields $Y = 1$ and for $a_1 = a_2$, $Y = (1+\varepsilon)/2$ as a correct result.

Taking into consideration that in the numerator and denominator of (17) the first two terms are dominant, the phase constant is seen to be determined by the upper left element of matrix (14) as a first approximation. It is strange enough to see that the phase constant is determined by the requirement of continuity of the longitudinal magnetic field of zero value as a first approximation. In [2] the phase constant was computed in this first approximation.

Now let us examine the solution of (14). The system of equations (14) cannot be solved without V_0 because $V_0 b_s$ means the static voltage between the strip and the shield. So the matrix of (14) has a rank $2P$, i.e. all of the unknowns in (14) can be expressed by V_0.$\mathbf{V}$ and $\mathbf{W}'$ are nonzero hence $\mathbf{W} = k_0 \mathbf{W}'$ is zero. The longitudinal electric field is zero in the aperture, but in this case the field is of type TE [1]. Also the longitudinal magnetic field containing the factor $k_0^2 \varepsilon_i - \beta^2$ is zero. So it has been shown how the hybrid field of the strip line becomes TEM field for $k_0 \to 0$.

Omitting the first row of (14), after some arrangement the aperture field coefficients are obtained in terms of V_0:

$$(\mathbf{S}_1 + \varepsilon \mathbf{S}_2)\mathbf{V} = -(\mathbf{B}_1 + \varepsilon \mathbf{B}_2) V_0. \tag{18}$$

Knowing the aperture field, the waveguide electromagnetic field can be found from the TE type equations (3) and (5) in [1]. So the static field is the limiting case of the TE field, there is no double root at zero frequency as DALY [5] and RHODES [12] supposed.

3. High-frequency behaviour

For $k_0 \to \infty$ and $\beta \to \infty$, the system of equations (1) and (2) becomes:

$$\begin{bmatrix} \sum_{i=1}^{2} \tilde{\mathbf{P}} \left\langle (\varepsilon_i - Y) \dfrac{\operatorname{ctg} \sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\dfrac{n\pi}{b}\right)^2} \, a_i}{\sqrt{\varepsilon_i - Y - \left(\dfrac{n\pi}{k_0 b}\right)^2}} \right\rangle \mathbf{P V}^0 & \mathbf{0} \\ \mathbf{0} & -\sum_{i=1}^{2} \tilde{\mathbf{Q}} \left\langle \varepsilon_i \dfrac{\operatorname{ctg} \sqrt{k_0^2 \varepsilon_i - \beta^2 - \left(\dfrac{n\pi}{b}\right)^2} \, a_i}{\sqrt{\varepsilon_i - Y - \left(\dfrac{n\pi}{k_0 b}\right)^2}} \right\rangle \mathbf{Q W} \end{bmatrix} = \mathbf{0}. \tag{19}$$

In the shaded region A of Fig 2, the TE_{01}-mode of the dielectric-filled subwaveguide is dominant. The left edge of the region A lies at a distance of a few k_{0h} from the origin. Choosing only the term $n = 1$ and using only one term from (3) in the upper left block of (19), the dispersion equation will be

$$(\varepsilon_1 - Y)\frac{\operatorname{cth}\sqrt{\left(\frac{\pi}{b}\right)^2 + \beta^2 - k_0^2\,\varepsilon_1\,a_1}}{\sqrt{\left(\frac{\pi}{k_0\,b}\right)^2 + Y - \varepsilon_1}} + \\ + (\varepsilon_2 - Y)\frac{\operatorname{cth}\sqrt{\left(\frac{\pi}{b}\right)^2 + \beta^2 - k_0^2\,\varepsilon_2\,a_2}}{\sqrt{\left(\frac{\pi}{k_0\,b}\right)^2 + Y - \varepsilon_2}} = 0\,. \tag{20}$$

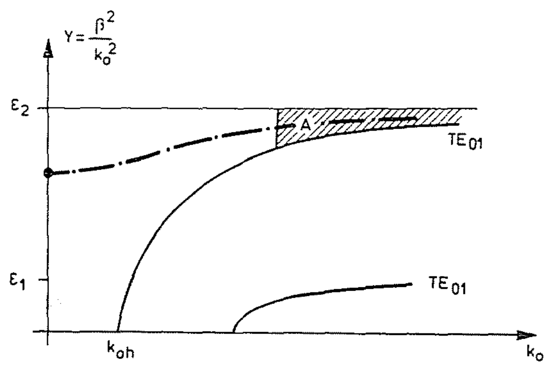

Fig. 2. Qualitative variation of the quasi TEM dispersion curve

The left-hand side of (20) differs in sign at the lines surrounding the region A. At the line $k_0^2\varepsilon_2 - \beta^2 - (\pi/b)^2 = 0$, defining the dispersion curve of the TE_{01}-mode the left-hand side of (20) is negative, while at the line $Y = \varepsilon_2$ it is positive, between the two lines it varies monotonically. So (20) has a root in the region A.

At the same time the dispersion equation defined by the bottom right block of (19) has no root in the region A.

The numerical results showed the situation to be the same for several subwaveguide modes having different n values and for several aperture expanding functions. The determinant defined by the upper left block in (19) has one root, the one defined by the bottom right block has no root in the region A. So the system of equations (19) can be solved for $\mathbf{W} = \mathbf{0}$, i.e. the quasi TEM mode behaves as a TE-mode at high frequencies.

From Fig 2 it can be seen that the characteristic frequency separating the low-frequency and the high-frequency regions is the cutoff frequency of the TE_{01}-mode of the dielectric-filled subwaveguide.

4. Numerical results

Numerical calculations involved parameters $a_1 = 3a_2$; $b = 7a_2$; $b_s/b = 5/14$; $\varepsilon_1 = 1$; $\varepsilon_2 = 9$, identical to those used by HORNSBY and GOPINATH [4]. Evaluating (10) with $P = 4$ from (17) we get $Y = 6{,}07$. With the given parameters, Y values from 5,5 to 6,6 have been published [7] but these values are quite uncertain. Choosing $V_0 = 1$, from (18) the Fourier coefficients of the aperture field are: $\tilde{V} = [1{,}40\ 0{,}927\ 0{,}702\ 0{,}565]$. The aperture field is seen in Fig 3. The estimated true aperture field is marked by dotted line.

Figs 4 and 5 show the qualitative field plot at zero and high frequencies.

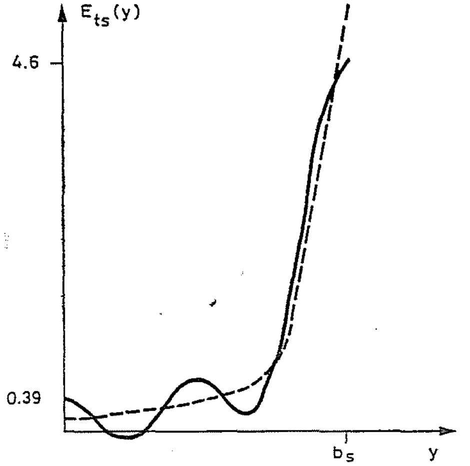

Fig. 3. The y component of the aperture electric field computed from a four term Fourier polynomial. The dotted line curve refers to the estimated exact field

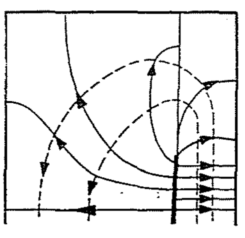

Fig. 4. Field plot at zero frequency. Solid line: E field, dotted line: H field

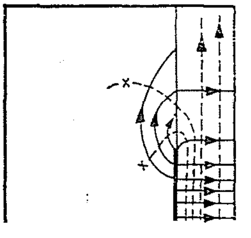

Fig. 5. Field plot at high frequency. Solid line: E field, dotted line: H field

It must be noted that the summation of the series (c. f. the Appendix) can be used to improve the convergence properties also at non-zero frequencies. The sum is added to, and substracted term by term from the matrix elements of (1) and (2). So the complete dispersion diagram of Fig. 6 was obtained.

The inflexion point [13] seems to be at the frequency k_{oh} and this is quite clear from physical point of view: the fundamental high-frequency component of the qTEM mode begins to propagate.

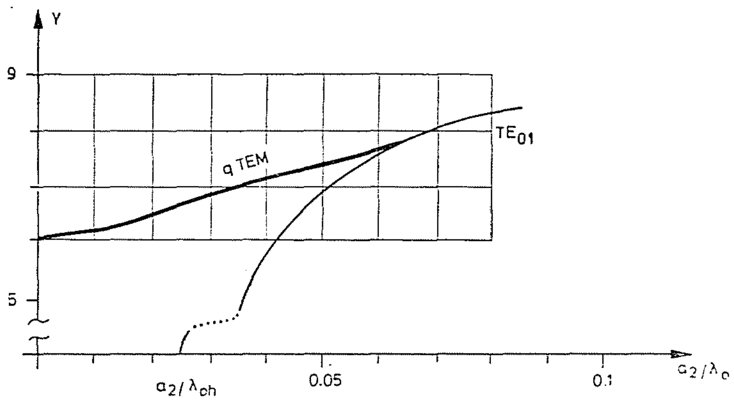

Fig. 6. Computed dispersion curve of the quasi TEM mode. The inflexion point is at a_2/λ_{oh}

5. Appendix

The contour integration method [11] leads for the sum (10) to:

$$T^1_{pr} = t^1_{pr} - \frac{b_s^3}{2\pi a_1^2} \sum_{i=1,2,\dots} \frac{i \operatorname{sh} \dfrac{i\pi b_s}{a_1} \operatorname{ch} \dfrac{i\pi b}{a_1}\left(\dfrac{b_s}{b} - \dfrac{1}{2}\right)}{\left[\left(\dfrac{ib_s}{a_1}\right)^2 + p^2\right]\left[\left(\dfrac{ib_s}{a_1}\right)^2 + r^2\right] \operatorname{ch} \dfrac{i\pi b}{2a_1}}, \tag{21}$$

here

$$t^1_{pr} = \frac{\pi b_s}{8} \frac{1}{p} \operatorname{cth} \frac{p\pi a_1}{b_s}, \quad \text{for} \quad p = r,$$

$$t^1_{pr} = -\frac{b_s^2}{4\, a_1 p^2}, \qquad \text{for} \quad r = 0,$$

$$t^1_{pr} = \frac{\pi^2 b^2}{8 a_1}\left[\frac{2}{3}\left(\frac{a_1}{b}\right)^2 + \frac{b_s}{b} - \frac{4}{3}\left(\frac{b_s}{b}\right)^2\right], \quad \text{for } p = r = 0,$$

else $\qquad t^1_{pr} = 0.$

The sum in (21) has to be summed up to I so as to have $Ib_s/a_1 \gg p_M$, where p_M means the larger among p and r. For high values the terms of the sum decrease as i^{-3}.

Summary

An analytical result based on the subwaveguide method is presented for the zero-frequency phase constant of the shielded microstrip. The Fourier coefficients of the aperture field are calculated and the series closely approximates the edge singularity. The quasi TEM field is seen to be a special TE type at zero frequency and of TE type at high frequencies.

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