SOME PROBLEMS IN THE THEORY AND APPLICATION OF GENERALIZED INTEGRALS AND DERIVATIVES OF REAL ORDER

PART I

By

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1. Introduction

Starting from the well-known definition of the generalized integrals of real (not only integer) order, the first part of the present study will discuss the properties and computation aspects essential for interpretation and application, then an example of application in theoretical electricity will be given.

In the second part the concept of real order integral and derivative will be extended to the domain of generalized functions, then differential equations containing also non-integer real-order derivatives will be analysed, and conditions of validity and form of the mean value theorems and the Bernoulli-L'Hospital rule for the generalized derivatives will be interpreted below, as well as application of approximate functions with properties similar to Taylor's or Hermite's polynomials.

A member of the qth generalized integral family — for $q > 0$ — is known to be defined by the convolution integral as [1]:

$$J^{(q)}[f(t)] = \frac{1}{\Gamma(q)} \int_0^t \tau^{q-1} f(t-\tau)\, d\tau \tag{1a}$$

or by its equivalent as:

$$J^{(q)}[f(t)] = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau)\, d\tau \tag{1b}$$

where $f(t)$ is the function to be integrated in the generalized sense.

This definition is obvious for functions that can be transformed using Laplace's rule; namely for $t \geq 0$,

$$\mathfrak{L}[f(t)] = F(s),$$

then

$$J^{(q)}[f(t)] = \frac{F(s)}{s^q}$$

since formally

$$\mathfrak{L}\left[\frac{t^{q-1}}{\Gamma(q)}\right] = \frac{1}{s^q};$$

the convergency conditions will not be discussed here. The properties of the operator $J^{(q)}$, discussed in the subsequent chapter, will demonstrate in direct content relations that $J^{(q)}$ is really the generalization of the operator of integration.

2. Properties of the operator $J^{(q)}$

By definition:

$$J^{(1)}[f(t)] = \int_0^t f(\tau)d\tau \tag{2}$$

further $J^{(q)}$ is a homogeneous linear operator:

$$J^{(q)}[c_1 f_1(t) + c_2 f_2(t)] = c_1 J^{(q)}[f_1(t)] + c_2 J^{(q)}[f_2(t)] \tag{3}$$

Replacing $\tau = ut$ in (1*b*):

$$J^{(q)}[f(t)] = \frac{t^q}{\Gamma(q)} \int_0^1 (1-u)^{q-1}(tu)\, du. \tag{4}$$

Hence, for $a > 0$:

$$J^{(q)}[f(at)] = a^{-q}\{J^{(q)}[f(v)]\}_{v=at} \tag{5}$$

An essential relationship:

$$J^{(p)}\{J^{(q)}[f(t)]\} = J^{(p+q)}[f(t)] \tag{6}$$

This is proven by writing, using (4):

$$F(t) = J^{(q)}[f(t)] \frac{t^q}{\Gamma(q)} \int_0^1 (1-v)^{q-1} f(tv)\, dv,$$

and

$$J^{(p)}[F(t)] = \frac{t^p}{\Gamma(p)} \int_0^1 (1-u)^{p-1} F(tu) du,$$

hence

$$J^{(p)}\{J^{(q)}[f(t)]\} = \frac{t^{p+q}}{\Gamma(p)\Gamma(q)} I, \qquad (*)$$

where

$$I = \int_0^1 (1-u)^{p-1} u^q \left[\int_0^1 (1-v)^{q-1} f(tuv)dv\right] du =$$
$$= \int_0^1 \int_0^1 (1-u)^{p-1} (u-uv)^{q-1} f(tuv)\, ududv\,.$$

Transformations

$$\left.\begin{array}{l} z = 1-u \\ w = uv \end{array}\right\} \text{i. e.} \quad \left.\begin{array}{l} v = \dfrac{w}{1-z} \\ u = 1-z \end{array}\right\}$$

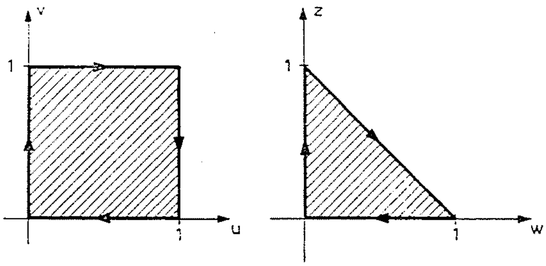

Fig. 1.

affect the domain of integration according to Fig. 1 and

$$\begin{vmatrix} \dfrac{\partial u}{\partial w} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial w} & \dfrac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ \dfrac{1}{1-z} & \dfrac{w}{(1-z)^2} \end{vmatrix} = \frac{1}{1-z} = \frac{1}{u}$$

$$ududv = u\frac{1}{u} dzdw = dzdw,$$

consequently:

$$I = \int_0^1 \left[\int_0^{1-w} z^{p-1}(1-z-w)^{q-1} dz\right] f(tw)\, dw.$$

Substituting $z = (1-w)r$ and $dz = (1-w)dr$ the integral in the brackets becomes:

$$(1-w)^{p+q-1} \int_0^1 r^{p-1}(1-r)^{q-1} dr = (1-w)^{p+q-1} B(p,q)$$

where $B(p,q)$ is the beta function:

$$B(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)},$$

and thus

$$I = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \int_0^1 (1-w)^{p+q-1} f(tw)\, dw .$$

From Eq. (4):

$$t^{p+q}\, I = \Gamma(p)\,\Gamma(q)\; J^{(p+q}\, [f(t)],$$

that, substituted in formula (*) leads to (6).

Differentiating (1*b*) *k*-times where $k < q$ yields:

$$\frac{d^k}{dt^k} \{J^{(q)}\, [f(t)]\} = J^{(q-k)}\, [f(t)] \tag{7}$$

(derivative with respect to the upper integration limit being zero).

On the other hand, if $f(t)$ can be differentiated *k*-times, then — in the case of $k < q$ — *k*-fold partial integration of the defining formula (1*b*) results in:

$$\begin{aligned} J^{(q)}\, [f^{(k)}(t)] = J^{(q-k)}\, [f(t)] - \Bigg[& \frac{t^{q-1}}{\Gamma(q)} f^{(k-1)}(0) + \\ & + \frac{t^{q-2}}{\Gamma(q-1)} f^{(k-2)}(0) + \ldots + \frac{t^{q-k}}{\Gamma(q-k+1)} f(0) \Bigg] . \end{aligned} \tag{8}$$

Confined to functions $f(t)$ differentiable *k*-times, and $f(0) = f'(0) = = f''(0) = \ldots f^{(k-2)}(0) = f^{(k-1)}(0) = 0$, and the integrals (1*a*) and (1*b*) can be interpreted for $f^{(k)}(t)$, the derivative of non-integer order $q < k$ can be defined as:

$$\frac{d^{(q)}[f(t)]}{dt^q} = f^{(q)}(t) = J^{(k-q)}\, [f^{(k)}(t)] . \tag{9}$$

Definition (9) is justified by

$$J^{(q)}\, [f^{(q)}(t)] = J^{(q)}\, \{J^{(k-q)}\, [f^{(q)}(t)]\},$$

and thus, from (6) and (8):

$$\begin{aligned} J^{(q)}\, [f^{(q)}(t)] = J^{(k)}\, [f^{(k)}(t)] = J^{(k)} \left\{ \frac{d^{k-1}}{dt^{k-1}} [f'(t)] \right\} = \\ = J^{(1)} f'(t) = f(t). \end{aligned}$$

For $q < k$ and $p < m$, provided function $f(t)$ can be differentiated $(k + m)$-times (k and m being integer numbers), and $f^{(\nu)}(0) = 0$, $(\nu = 0, 1, 2, \ldots, k + m - 1)$:

$$\frac{d^{(p)}}{dt^p}[f^{(q)}(t)] = f^{(p+q)}(t). \tag{10}$$

This is proven by considering that with the conditions made for $f(t)$:

$$f^{(k)}(t) = J^{(m)}[f^{(m+k)}(t)],$$

and thus, using relationship (6), by definition (9):

$$f^{(q)}(t) = J^{(m+k-q)} f^{(m+k)}(t).$$

and

$$\frac{d^{(p)}}{dt^p}[f^{(q)}(t)] = J^{(m-p)}\left(\frac{d^m}{dt^m}\{J^{(m+k-q)}[f^{(m+k)}(t)]\}\right).$$

Finally, according to (7)

$$\frac{d^{(p)}}{dt^p}[f^{(q)}(t)] = J^{(m-p)}\{J^{(k-q)}[f^{(m+k)}(t)]\} =$$
$$= J^{[m+k-(p+q)]}[f^{(m+k)}(t)] = f^{(p+q)}(t).$$

3. Methods of computing the generalized integral

a) For power functions a solution in explicit form will be obtained, namely for $f(t) = t^r$ the integral in (4) becomes:

$$\int_0^1 (1-u)^{q-1} t^r u^r \, du = t^r B(q, r+1) = t^r \frac{\Gamma(q)\,\Gamma(r+1)}{\Gamma(q+r+1)}$$

Consequently

$$J^{(q)}(t^r) = \frac{\Gamma(r+1)}{\Gamma(q+r+1)} t^{q+r}; \tag{11}$$

this equation is valid for $q + r \geq 0$ except for $r = -1, -2, -3, -4 \ldots$

It must be noted that, if k is the greatest integer number contained in r, conditions for (9) are met and the generalized derivate of qth order becomes:

$$f^{(q)}(t) = J^{(k-q)}[f^{(k)}(t)] = J^{(k-q)}[r(r-1)(r-2)\ldots(r-k+1)t^r]$$

thus, using (11):

$$f^{(q)}(t) = \frac{\Gamma(r-k+1)}{\Gamma(r-q+1)} r(r-1)(r-2)\ldots(r-k+1)\, t^{r-q},$$

hence

$$\frac{d^{(q)}(t^r)}{dt^q} = \frac{\Gamma(r+1)}{\Gamma(r+1-q)} t^{r-q} \tag{12}$$

valid for $0 < q \leq k$.

b) The value of $J^{(q)}\,[f(t)]$ can be found for any t by numerically evaluating integral in (4).

c) If $f(t)$ can be differentiated $(n+1)$-times and to the latter the formulas (1a) and (1b) can be applied (n being an integer number), then the $(n+1)$-fold partial integration of (4) will give the following series:

$$\begin{aligned} J^{(q)}[f(t)] = {} & \frac{f(0)}{\Gamma(q+1)} t^q + \frac{f'(0)}{\Gamma(q+2)} t^{q+1} + \frac{f''(0)}{\Gamma(q+3)} t^{q+2} + \\ & + \ldots + \frac{f^{(n)}(0)}{\Gamma(q+n+1)} t^{q+n} + \frac{t^{q+n+1}}{\Gamma(q+n+1)} \int_0^1 (1-u)^{q+n} f^{(n+1)}(tu)\,du \end{aligned} \tag{13}$$

d) In some cases $J^{(q)}[f(t)]$ can be written in differential equation form. If for instance $f(t)$ meets differential equation

$$af''(t) + bf'(t) + cf(t) = 0, \tag{14a}$$

then written according to (4)

$$J^{(q)}[f(t)] = \frac{t^q}{\Gamma(q)} g(t), \tag{14b}$$

$g(t)$ meets differential equation

$$\begin{aligned} at^2g''(t) + (2aqt + bt^2)g'(t) + [aq(q-1) + btq + ct^2]g(t) = \\ = atf'(0) + [a(q-1) + bt]f(0) \end{aligned} \tag{14c}$$

and

$$g(0) = \frac{f(0)}{q}\,;\; g'(0) = \frac{f'(0)}{2q}\,. \tag{14d}$$

For $\lim\limits_{t=\infty} g(t) = 0$, $g(t)$ tends asymptotically to

$$g_a(t) = \frac{af'(0) + bf(0)}{ct}\,. \tag{14e}$$

To prove the previous statements, a comparison of (14b) and (1a) will give:

$$t^q g(t) = \int_0^t \tau^{q-1} f(t-\tau)\, d\tau ,$$

multiplied by t^2

$$t^{q+2} g(t) = t^2 \int_0^t \tau^{q-1} f(t-\tau) d\tau \qquad (*)$$

Differentiating:

$$(q+2)t^{q+1} g(t) + t^{q+2} g'(t) = 2t\,[t^q g(t)] + t^{q+1} f(0) + t^2 \int_0^t \tau^{q-1} f'(t-\tau) d\tau,$$

after reducing:

$$t^{q+2} g'(t) + q t^{q+1} g(t) - t^{q+1} f(0) = t^2 \int_0^t \tau^{q-1} f'(t-\tau)\, d\tau \qquad (**)$$

Differentiated once again:

$$t^{q+2} g''(t) + (2q+2)t^{q+1} g'(t) + q(q+1)t^q g(t) - (q+1)t^q f(0) =$$
$$= 2[t^{q+1} g'(t) + q t^q g(t) - t^q f(0)] + t^{q+1} f'(0) + t^2 \int_0^t \tau^{q-1} f''(t-\tau) d\tau,$$

hence

$$t^{q+2} g''(t) + 2q t^{q+1} g'(t) + q(q-1)t^q g(t) - (q-1)t^q f(0) - t^{q+1} f'(0) =$$
$$= t^2 \int_0^t \tau^{q-1} f''(t-\tau)\, d\tau \qquad (***)$$

Adding b-times equation (**) and a-times equation (***) to c-times equation (*), yields, according to (14a):

$$t^q [c t^2 g(t) + b t^2 g'(t) + q b t g(t) - b t f(0) + a t^2 g''(t) +$$
$$+ 2aqt g'(t) + aq(q-1) g(t) - a(q-1) f(0) - a t f'(0)] = 0,$$

resulting after reduction in (14c).

(14d) directly follows from (14c). For $\lim_{t\to\infty} g(t) = 0$, left-hand side of the differential equation is dominated by $ct^2 g(t)$ and its right-hand side by $atf'(0) + btf(0)$ for high t-values, leading to Eq. (14e)

In the following, asymptotic behaviour of generalized integrals of functions meeting differential equation (14a) will be discussed.

The general form of these functions is $Im(e^{At})$, where A is a complex number and Im designates the imaginary part.

This latter operation, being interchangeable with the integrals, can be left to the end of the computation and thus it will not be indicated. Let first be

$$q = k + p, \quad k = \text{integer}; \quad 0 < p < 1; \quad q > 0.$$

Since

$$e^{At} = A^{-k} \frac{d^k(e^{At})}{dt^k},$$

from Eq. (8):

$$J^{(p)}(e^{At}) = A^{-k} J^{(p)}(e^{At}) - \left[\frac{t^{q-1}}{A\Gamma(q)} + \frac{t^{q-2}}{A^2 \Gamma(q-1)} + \ldots + \right. \\ \left. + \frac{t^{q-k+1}}{A^{k-1}\Gamma(q-k+1)}\right] \tag{15}$$

This formula demonstrates to be sufficient to deal only with the generalized integrals of e^{At} of an order below unit, and with their asymptotic behaviour, namely the integrals of higher order differ from them by a value determined by an explicit formula.

Thus, in the following:

$$0 < q < 1.$$

For $Re(A) < 0$, comparison of formulae (14b) and (4) gives:

$$g(t) = \int_0^1 (1-u)^{q-1} e^{Atu} \, du ,$$

thus $\lim_{t \to \infty} g(t) = 0$, hence asymptotic approximation (14e) is valid. (Remind that $g(t)$ tends to zero even for $q > 1$, and thus Eq. (14e) holds in this case, too; this, however, under consideration of (14b), gives only a term of the form t^{q-1}, in contrary to power functions in (15) giving the asymptotic function more exactly.)

For $Re(A) \geq 0$, the asymptotic solution can be defined as follows: According to formula (1a)

$$J^{(q)}(e^{At}) = \frac{1}{\Gamma(q)} \int_0^t \tau^{q-1} e^{A(t-\tau)} \, d\tau = \frac{e^{At}}{\Gamma(q)} \int_0^t \tau^{q-1} e^{-A\tau} \, d\tau.$$

Thus $J^{(q)}[e^{At}]$ tends asymptotically to

$$\frac{e^{At}}{\Gamma(q)} \int_0^\infty \tau^{q-1} e^{-A\tau} \, d\tau ,$$

if the latter integral has a finite value.

Substituting $z = A\tau$ the integral turns into the complex integral

$$\frac{1}{A^q}\int_{(L)} z^{q-1}\, e^{-z}\, dz$$

with path L as shown in Fig. 2.

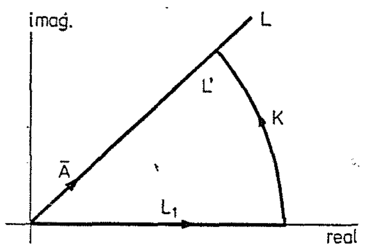

Fig. 2.

Replacing it provisorily by L' ending at the circular arc K, regularity of the integrand permits to write:

$$\int_{(L')} = \int_{(L_1)} + \int_{(K)} = \int_0^R x^{q-1}\, e^{-x}\, dx + \int_{(K)}$$

For $R \to \infty$, L' turns into L and the integral along the arc is zeroed such as:

$$B = \int_{(K)} z^{q-1}\, e^{-z}\, dz = \Big[-e^{-z}\, z^{q-1}\Big]_{z=R}^{z=R\frac{A}{|A|}} + \int_{(K)} (q-1) z^{q-2}\, e^{-z}\, dz;$$

namely the absolute value of the term in square brackets falls below

$$2R^{q-1} = \frac{2}{R^{1-q}},$$

and the absolute value of the integral (on the right-hand side) falls below:

$$(q-1)\, R^{q-2} \cdot R \,\mathrm{arc}\,(A) = \frac{(q-1)\,\mathrm{arc}\,(A)}{R^{1-q}};$$

and so, for $q < 1$ and $R \to \infty$, $B \to 0$.

Consequently

$$\int_{(L)} = \lim_{R\to\infty} \int_{(L_1)} = \int_0^\infty x^{q-1} e^{-x}\, dx = \Gamma(q)\,,$$

thus

$$[J^{(q)}(e^{At})]_a = \frac{e^{AT}}{A^q}\,, \quad \text{for} \quad 0 < q < 1\,,\ Re\,(A) \geqq 0, \dots \tag{16}$$

where subscript a indicates "asymptote".

4. An example of application

Eddy currents are induced in a semi-infinite metal block due to the variable voltage connected to the induction coil. (Fig. 3).

In the co-ordinate system the magnetic field H has only a component of direction z, the electric field a component of direction y, both depending on x and t (time) alone. Thus, on the basis of Maxwell's equations obtained by neglecting the displacement current and of Ohm's differential law [2]:

$$\frac{\partial H}{\partial x} = -\gamma E\,, \tag{17}$$

$$\frac{\partial E}{\partial x} = -\mu \frac{\partial H}{\partial t}\,. \tag{18}$$

The voltage connected to the coil and the currect intensity have to be related. $(E)_{x=0} = E_0$ being proportional to the voltage, and $(H)_{x=0} = H_0$ to the current, therefore it is only relevant to know $H_0(t)$ for a given $E_0(t)$. Using the Laplace transform of Eqs (17) and (18):

$$\frac{d\hat{H}}{dx} = -\gamma \hat{E} \tag{19}$$

$$\frac{d\hat{E}}{dx} = -\mu s \hat{H}, \tag{20}$$

hence

$$\frac{d^2\hat{H}}{dx^2} = \gamma\mu s \hat{H}\,,$$

the general solution being

$$\hat{H} = A e^{\sqrt{\gamma\mu s}\,x} + B e^{-\sqrt{\gamma\mu s}\,x}\,.$$

For $Re(\sqrt{s}) > 0$ and $x \to \infty$, $\hat{H}$ remains only finite if $A = 0$, and thus

$$\hat{H} = Be^{-\sqrt{\gamma\mu s}\,x},$$

Denoting the Laplace transform of $H_0(t)$ by $\hat{H}_0(s)$, and since $H_0(t) = (H)_{x=0}$.

$$\hat{H}_0(s) = (\hat{H})_{x=0} = B,$$

hence

$$\hat{H} = \hat{H}_0(s)e^{-\sqrt{\gamma\mu s}\,x}.$$

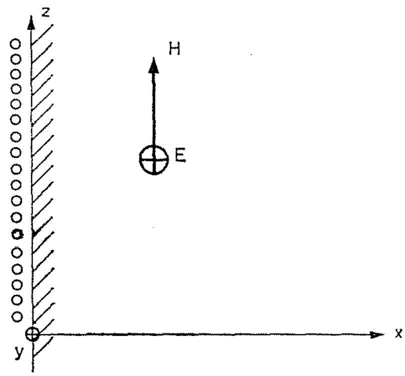

Fig. 3.

From (19):

$$\hat{E} = -\frac{1}{\gamma}\frac{d\hat{H}}{dx} = \sqrt{\frac{\mu}{\gamma}}\,H_0(s)\sqrt{s}\,e^{-\sqrt{\gamma\mu s}\,x},$$

and

$$E_0(s) = \mathfrak{L}[E_0(t)] = (\hat{E})_{x=0} = \sqrt{\frac{\mu}{\gamma}}\,H_0(s)\cdot\sqrt{s}.$$

hence

$$\hat{H}_0(s) = \sqrt{\frac{\gamma}{\mu}}\,\frac{\hat{E}_0(s)}{\sqrt{s}}.$$

Since

$$\frac{1}{\sqrt{s}} = \mathfrak{L}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)\sqrt{t}}\right],$$

the convolution theorem transforms the previous equation to:

$$H_0(t) = \sqrt{\frac{\gamma}{\mu}}\,\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t \frac{E_0(\tau)}{\sqrt{t-\tau}}\,d\tau \tag{21}$$

hence, by definition (1*b*):

$$H_0(t) = \sqrt{\frac{\gamma}{\mu}} J^{(\frac{1}{2})} [E_0(t)]. \tag{22}$$

Consequently, *the current-time function is proportional to the* $\frac{1}{2}$*-th integral of the voltage-time function*. If direct voltage is connected to the coil, then $E_0(t) = = 1(t) . E_0$ where $1(t)$ is the unity step function, and thus, on the basis of (21)-using $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (Fig. 4)

$$H_0(t) = 2\sqrt{\frac{\gamma}{\mu\pi}}\sqrt{t}.$$

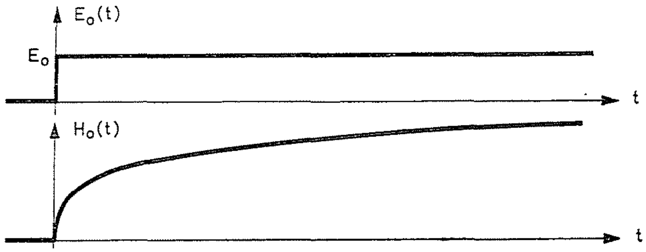

Fig. 4.

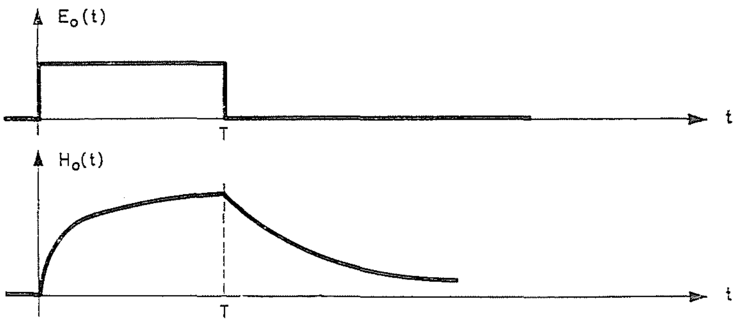

Fig. 5.

If an impulse of width T, shown in Fig. 5, is applied to the coil, then, again from (21):

$$H_0(t) = 2\sqrt{\frac{\gamma}{\mu\pi}}[\sqrt{t} - 1(t-T)\sqrt{t-T}].$$

Finally, connecting a sinusoidal voltage to the coil:

$$E_0(t) = E_0 \sin(\omega t + \alpha).$$

Substituting

$$\varphi(t) = J^{(\frac{1}{2})}[\sin(t+\alpha)]$$

leads on the basis of Eqs (22), (3) and (5) to:

$$H_0(t) = \sqrt{\frac{\gamma}{\mu\omega}}\,E_0\,\varphi(\omega t)\,.$$

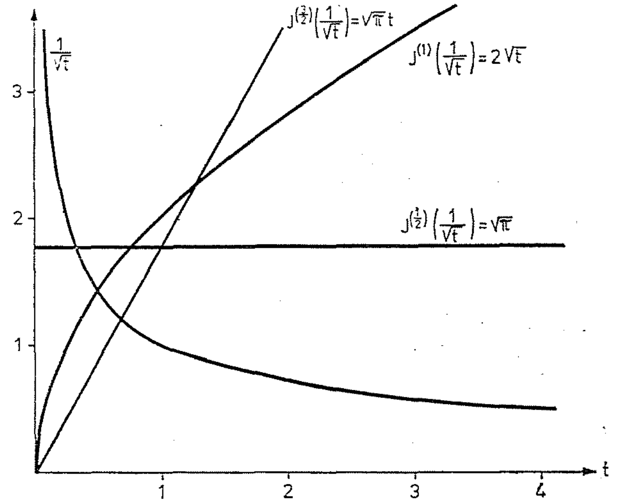

Fig. 6.

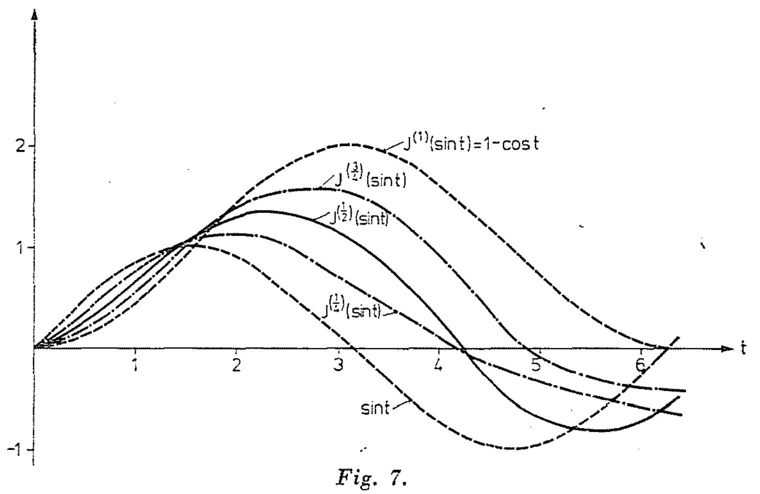

Fig. 7.

According to (14*a*), (14*b*), (14*e*) and (14*d*) — where $a = c = 1$, $b = 0$, $f(o) =$ $= \sin\alpha$ and $f'(o) = \cos\alpha$:

$$\varphi(t) = \sqrt{\frac{t}{\pi}}\,g(t)\,,$$

and the solution of the differential equation

$$t^2 g''(t) + tg'(t) + \left(t^2 - \frac{1}{4}\right) g(t) = t \cos\alpha - \frac{1}{2} \sin\alpha$$

is $g(t)$, with initial values

$$g(0) = 2 \sin\alpha, \quad g'(0) = \cos\alpha.$$

Considering that $\sin(t + \alpha) = Im(e^{j\alpha} e^{jt})$, on the basis of (16):

$$\varphi_a(t) = Im\left(e^{j\alpha} \frac{e^{jt}}{\sqrt{j}}\right) = Im\left[e^{j\left(t+\alpha-\frac{\pi}{4}\right)}\right]$$

thus:

$$\varphi_a(t) = \sin\left(t + a - \frac{\pi}{4}\right).$$

Appendix

For sake of demonstration, Fig. 6 shows the curves of the "halfth", first and "one-and halfth" integrals of $\frac{1}{\sqrt{t}}$, Fig. 7 the integral curves of $\frac{1}{4}$-th, $\frac{1}{2}$-th, $\frac{3}{4}$-th, and first order of $\sin t$.

Summary

Starting from the well-known definition of generalized real-order-integrals, their most important features (homogeneous linear operator; transition integer-order integrals to traditional integrals; addition of the orders upon sequential application of the operator) is presented. After defining the real-order derivatives, calculation methods are described. At last, as an example in the field of theoretical electricity, the current of the excitation coil of a semi-infinite conductor is shown to be proportional to the $\frac{1}{2}$-th integral of the voltage-time function.

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