MATRIX ANALYSIS OF TRANSIENT PHENOMENA IN PIPELINE FLOW

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Introduction

Possibilities of the analytical solution of dynamic problems related to the flow of elastic liquids of assumed constant density in a pipeline are examined in the following. For the sake of simplicity, the flow pipe is assumed to be horizontal, of constant cross section and material characteristics, frictional losses in flow are proportional to velocity. Under such conditions, relationships of a general validity can be obtained in a closed form for flow characteristics, such as pressure, velocity, at various points of the flow pipe. Thereby, in view of determination, dynamic response functions are obtained for the given input parameter arbitrarily varying in time. Approximating input parameters and response function e.g. by a broken line function composed of straight sections, the general relatonship takes the form of a matrix equation. Matrices do not depend on flow velocity and pressure, thus they represent the linear operator characteristic of the given flow pipe.

1. Stating the problem
Survey of known solutions. Objectives

1.1. *The problem*

In the case of practical problems, — e.g. oil pipelines, — pressure and velocity values are recorded at fixed points of the pipe, where disturbances are to be expected, and where also interventions take place. Thus, let us separate a pipe of length L from the rest of the flow system, with pressure and velocity parameters at both ends representing the effect of the omitted parts (Fig. 1).

The four characteristic parameters are obtained from boundary values of pressure and velocity:

$$\begin{aligned} p_{10}(t) &= p_0(t, 0) \\ v_{10}(t) &= v_0(t, 0) \\ p_{20}(t) &= p_0(t, L) \\ v_{20}(t) &= v_0(t, L) \end{aligned} \tag{1}$$

Let us determine the relationship between the above parameters, i.e. of the function f_1:

$$f_1(p_{10}(t), v_{10}(t), p_{20}(t), v_{20}(t)) = 0 \tag{2}$$

Among parameter values (1) there are known ones, these are named the boundary conditions. Eq. (2) serves for determining the other unknown parameters.

1.2. *Known solutions*

After having separated the examined system let us survey the known solutions fitting the problem. A fairly good survey is found in [5].

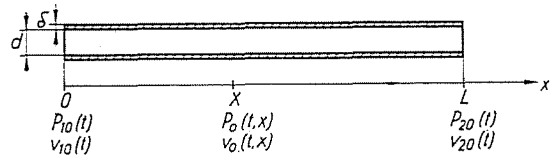

Fig. 1

a) *Numerical methods*

Numerical solution methods for the basic differential equation system of the problem give the relationship between parameters in algorithm form. Among these, the method of characteristics and the algebraic methods derived from them are of basic importance [[1] [2] [5]. After transforming the system of partial differential equations to total differential equations, these are converted to finite differences. These equations permit to determine the pressure and velocity values from initial and boundary conditions from point to point as a function of place and time. It is an advantage that actual friction losses can be taken into consideration, the method does not require an excessive sectioning, the calculation is easy to survey, and arbitrary initial and boundary conditions are easy to handle. Its disadvantage is to require some experience and inventiveness, and the use of a computer in any case.

b) *Analytical solutions*

After having linearized the friction force, the system of basic differential equations yields a linear partial differential equation of the second order, the so-called Telegraph Equation, of which various analytical solutions are known, the most important being the determination in a closed form of the transient generated by the abrupt change of pressure or velocity [6], or of response functions to the constant oscillation of disturbed variables (the so-called impedance method) [5].

The relationship f_1 between arbitrary boundary values can be determined from the above mentioned by using *Duhamel's theorem*, or by expanding to a *Fourier* series.

1.3. *Scope*

In the present work the linearized partial differential equation is solved for the generally given boundary conditions, to directly yield the functional relationship. Since the result contains arbitrary time functions of the boundary values, actual input parameters can closely be approximated, e.g. by a broken line of straight sections. The problem is reduced to the so-called Cauchy problem of the basic equation, to be solved by the method of Riemann [7].

Let us suppose at the beginning an initial condition belonging to a previous steady state to prevail:

$$t = 0, \quad p_0(0, x) = p_0(x)$$
$$v_0(0, x) = v_0(x),$$

hence, the values assumed at the boundaries are

$$p_{01} = p_0(0), \; p_{02} = p_0(L)$$
$$v_{01} = v_0(0), \; v_{02} = v_0(L)$$

This problem can be reduced to homogeneous initial conditions, on account of the linearity of the partial differential equation, by superposition.

Be the new initial conditions

$$p(0, x) = 0$$
$$v(0, x) = 0$$

replacing boundary values (1) by the following

$$\begin{aligned} p_1(t) &= p_{10} - p_{01} \\ v_1(t) &= v_{10} - v_{01} \\ p_2(t) &= p_{20} - p_{02} \\ v_2(t) &= v_{20} - v_{02} \end{aligned} \tag{3}$$

The new variables at an arbitrary place of the space part are:

$$p(t, x) = p_0(t, x) - p_0(x)$$
$$v(t, x) = v_0(t, x) - v_0(x)$$

Since (3) can be calculated from (1) in a mutually unambiguous way, in the following the solution of the last mentioned boundary value problem will be discussed.

2. The system of differential equations

2.1. In a horizontal pipe of constant cross-section the flow equation [1] is given by

$$\frac{1}{\varrho}\frac{\partial p}{\partial x}+\frac{\partial v}{\partial t}=-S \tag{4}$$

$$\text{where} \quad S=\frac{\lambda\cdot v^2}{2d} \quad \text{sing}\, v$$

In the case of a linearized friction resistance S will be expressed as:

$$S=k\cdot v+S_0. \tag{5}$$

In laminar flow, for Re $<$ 2300, the pipe friction coefficient can be calculated by the following known formula [3].

$$\lambda=\frac{64}{Re}, \quad \text{where} \quad Re=\frac{v\cdot d}{\nu}.$$

Substituting transforms the expression of S:

$$S=\frac{32\cdot\nu\cdot v}{d^2}$$

Similarly, for *turbulent flow* in a smooth surface pipe, for $2300 < \text{Re} < 10^5$, from

$$\lambda=\frac{0{,}316}{\sqrt[4]{Re}},$$

$$S=\frac{0{,}316\cdot\nu^{1/4}\cdot|v|^{3/4}}{2d^{5/4}}\cdot v \tag{6}$$

k and S_0, values to be considered in the following as constant, can be determined on the basis of the expectable set of values for v, depending on the character of the problem (Fig. 2). In the case of $v \leqslant v_m$, relationship (6) can be linearized by line a. Now, comparing (5) and (6)

$$S_0=0$$

$$k=\frac{0{,}316\cdot\nu^{1/4}\cdot|v_m|^{3/4}}{2d^{5/4}} \tag{6/a}$$

The approximation by a line b is advisable if v does not change its sign during the examinations, and it is changing in a narrow range,

$$v\in\Delta v$$

From the equation of the straight line b, k and S_0 can be determined.

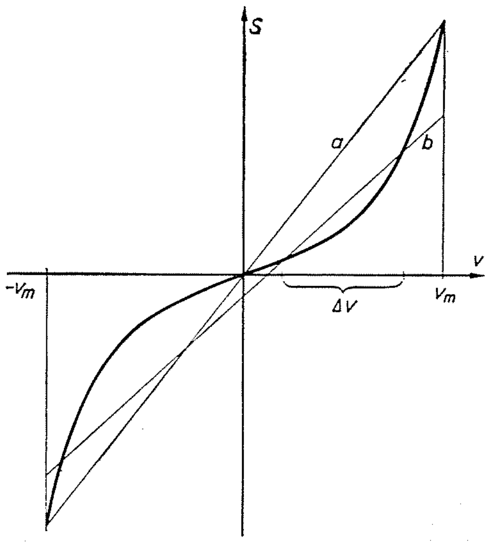

Fig. 2

2.2. The equation expressing the continuity of flow is the following:

$$\varrho w^2 \frac{\partial v}{\partial x} + \frac{\partial p}{\partial t} = 0 \tag{7}$$

where

$$w = \sqrt{\frac{E_r}{\varrho}}$$

$$E_r = \frac{1}{\dfrac{1}{E_f} + \dfrac{\delta}{d}\dfrac{1}{E_{cs}}}$$

From the system of differential equations (4) and (7), substituting Eq. (5), a linear partial differential equation of second order is obtained. To this end Eqs (4) and (7) will be partially differentiated with respect to t and x, respectively. Supposing that the mixed derivatives are identical, arranging yields:

$$\frac{\partial^2 v}{\partial t^2} + k \frac{\partial v}{\partial t} - w^2 \frac{\partial^2 v}{\partial x^2} = 0. \tag{8}$$

Similarly, by differentiating Eqs (4) and (7) with respect to x and t, respectively and arranging:

$$\frac{\partial^2 p}{\partial t^2} + k \frac{\partial p}{\partial t} - w^2 \frac{\partial^2 p}{\partial x^2} = 0. \tag{9}$$

Eqs (8) and (9) are formally identical with the known basic equation for electric transmission lines.

3. Reduction of the general relationship f_1 by introducing new boundary values

The differential operators for Eqs (8) and (9) are identical, hence:

$$F \cdot v(t, x) = 0$$

$$F \cdot p(t, x) = 0$$

where

$$F = \frac{\partial^2}{\partial t^2} + k \frac{\partial}{\partial t} - w^2 \frac{\partial^2}{\partial x^2}$$

The symmetry of pressure and velocity with respect to the above differential equation will find an important application, permitting to replace relationship (2) by a function of three variables without impairing generality.

This is obvious from writing relationship (9) with variable $\varphi(t, x)$ representing p or v.

$$F \cdot \varphi(t, x) = 0 \tag{10}$$

To the above equation the boundary values should also be given in terms of the new variable.

$$\begin{aligned} \varphi_1(t) &= \varphi(t, 0) \\ \psi_1(t) &= \left.\frac{\partial \varphi(t,x)}{\partial x}\right|_{x=0} \\ \varphi_2(t) &= \varphi(t, L) \end{aligned} \tag{11}$$

Initial conditions

$$\varphi(0, x) = 0$$

$$\psi(0, x) = 0.$$

It is evident that equation

$$f_2\big(\varphi_1(t),\ \varphi_2(t),\ \psi_1(t)\big) = 0 \tag{12}$$

relating boundary conditions (11) can produce solution of the original Eq. (2), with any boundary condition.

As the first step we shall prove that differential equations (4) and (7) can bring boundary values (3) and (11) into a mutually unambiguous relationship.

To prove this, first consider Eq. (4). Upon considering Eq. (11) too, this can be written in the following form.

For $\varphi_i = p_i$; $i = 1, 2$

$$\frac{1}{\varrho}\,\psi_1(t) + \frac{dv_1(t)}{dt} = -\,S\big(v_1(t)\big). \tag{13}$$

That is, $\psi_1(t)$ can be calculated from $v_1(t)$, and *vice versa*. Similarly Eq. (7) yields:
For $\varphi_i = v_i$; $i = 1, 2$

$$\varrho w^2 \psi_1(t) + \frac{dp_1(t)}{dt} = 0\,. \tag{14}$$

that is, $\psi_1(t)$ can be calculated from $p_1(t)$ and vice versa. In the above Eqs (13) and (14) time is the only variable, therefore differentiation is indicated with respect to one variable. Now it is obvious that suitably interpreting φ, boundary values (3) and (11) can be calculated one from another by means of Eqs (13) and (14).

Let us now suppose relationship (12) to exist and to be known, and also that it can produce solutions of Eq. (2) for all boundary conditions possible.

The given boundary conditions and the indices of the relationships for the unknown boundary values are given in Table I.

Table I

Given boundary conditions (input characteristics)	Course of determining the unknown parameter (response function)
$p_1(t)$ $p_2(t)$	1. $p_1 = \varphi_1$ $p_2 = \varphi_2$ (12) $\rightarrow \psi_1$, (13) $\rightarrow v_1(t)$ 2. $v_1 = \varphi_1$ On the basis of p_1, (14) $\rightarrow \psi_1$ (12) $\rightarrow \varphi_2 = v_2(t)$
$p_1(t)$ $v_1(t)$	1. $p_1 = \varphi_1$ On the basis of v_1, (13) $\rightarrow \psi_1$, (12) $\rightarrow \varphi_2 = p_2(t)$ 2. $v_1 = \varphi_1$ On the basis of p_1, (14) $\rightarrow \psi_1$, (12) $\rightarrow \varphi_2 = v_2(t)$
$p_2(t)$ $v_1(t)$	1. $v_1 = \psi_1$ $p_2 = \varphi_2$ (12) $\rightarrow \varphi_1 = p_1(t)$ 2. $v_1 = \varphi_1$ On the basis of p_1, (14) $\rightarrow \psi_1$, (12) $\rightarrow \varphi_2 = v_2(t)$
$v_1(t)$ $v_2(t)$	1. $v_1 = \varphi_1$ $v_2 = \varphi_2$, (12) $\rightarrow \psi_1$, (14) $\rightarrow p_1(t)$ 2. $p_1 = \varphi_1$ On the basis of v_1, (13) $\rightarrow \psi_1$, (12) $\rightarrow \varphi_2 = p_2(t)$

The number of given values is always two, to be explained in solving differential equation (10).

Combinations obtained from data by index change were not regarded as different, therefore these are omitted.

Table I serves as directive in an actual problem, on the other hand it proves the generality of Eq. (12).

4. Solution of the differential equation

4.1. *Setting the Cauchy problem*

Considering differential equation (10) and the first two parameters from Eq. (11) as given, we obtain the so-called Cauchy problem for (10). This typical initial value problem is applied in our case for a typical boundary value problem.

For the sake of solution $\varphi_1(t)$ is supposed to be continuously differentiable in sections, and $\psi_1(t)$ to be continuous. At the same time, these are important sufficient conditions for the mutual unambiguity problems discussed in connection with Eqs (13) and (14).

The problem will be solved by the method of Riemann. The theory of the solution is discussed in [7]. As the first step, Eq. (10) is to be transformed to the characteristic canonic form.

4.2. *Transformations*

The characteristic canonic form is obtained from the canonic form (10) by the following transformations of independent and dependent variables.

New independent variables l, y are introduced in place of t, x, such as:

$$\begin{aligned} l &= wt + x \\ y &= wt - x \end{aligned} \tag{15}$$

The above co-ordinate transformation images a new variable from $\varphi(t, x)$, i.e.:

$$\chi(l, y)\,.$$

Hereafter new dependent variable $z(l, y)$ is introduced in place of $\chi(l, y)$, such as:

$$z(l, y) = \chi(l, y) e^{\frac{k}{4w}(l+y)} \tag{16}$$

4.2.1. *Transformation of the differential equation*

By transformations (15) and (16), — omitting details, — differential equation (10) becomes:

$$\frac{\partial^2 z}{\partial l \partial y} - Bz = 0 \tag{17}$$

where

$$B = \left(\frac{k}{4w}\right)^2$$

4.2.2. *Transformation of the boundary conditions*

For the solution the value of $z(l, y)$ should be known on the original time axis (now the line $y = l$), as well as the difference of its partial derivatives, of course expressed in terms of φ_1, ψ_1.

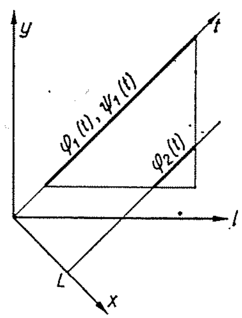

Fig. 3

Making use of transformation equations (15) and (16), and omitting details, these will be:

$$z(l, y)\Big|_{y=l} = z(l, l) = \varphi_1\left(\frac{l}{w}\right) e^{\frac{k \cdot l}{2w}} \tag{18}$$

$$\frac{\partial z}{\partial l}(l, l) - \frac{\partial z}{\partial y}(l, l) = \psi_1\left(\frac{l}{w}\right) e^{\frac{k \cdot l}{2w}} \tag{19}$$

Differential equation (17) has to be solved on the basis of boundary conditions (18) and (19).

On account of assumptions the set problem can be stated to have an unambiguous solution in the range limited by lines drawn from the terminal points of the examined time interval parallel to axes y and l, and the t axis [7] (Fig. 3). We shall use only the values of the solution, assumed on the line $x = L$ i.e. $y = l - 2L$.

4.3. *Solution by the method of Riemann*

The Riemann function of Eq. (17) is known.

$$R(\xi, \eta, l, y) = I_0(2\sqrt{(\xi - l)(\eta - y)}) \tag{20}$$

where I_0 is the zero order Bessel function of the first kind with pure imaginary variables.

The expression of the Riemann function suited for calculations is

$$R(\xi, \eta, l, y) = \lim_{V \to \infty} \sum_{m=0}^{V} \frac{[B(\xi - l)(\eta - y)]^m}{m!^2}$$

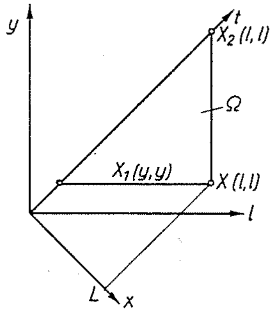

Fig. 4

In Eq. (20) ξ and η are the auxiliary integration variables for l and y, respectively.

$$0 \leq \xi \leq l$$

$$0 \leq \eta \leq y$$

The solution of the homogeneous differential equation can be written directly by means of Riemann's formula [7]. At an arbitrary point $X(l, y)$ (Fig. 4):

$$z(X) = \frac{1}{2}\left[z(X_1)R(X_1, X) + z(X_2)R(X_2, X)\right] - \int_{\overrightarrow{X_1 X_2}} (P d\eta - Q d\xi), \tag{22}$$

where

$$P = \frac{1}{2}\left[\frac{\partial z}{\partial \eta}(\xi, \eta) \cdot R - z \cdot \frac{\partial R}{\partial \eta}\right].$$

$$Q = \frac{1}{2}\left[\frac{\partial z}{\partial \xi}(\xi, \eta) \cdot R - z \cdot \frac{\partial R}{\partial \xi}\right].$$

Substituting $y = l - 2L$ into (22), on account of the fixed flow pipe length, from (21:)

$$R(X_1, X) = R(y, y, l, y) = 1$$

$$R(X_2, X) = R(l, l, l, y) = 1,$$

we find:

$$z(l, l-2L) = \frac{1}{2}\left[z(l-2L, l-2L) + z(l, l)\right] - \int\limits_{\overrightarrow{X_1X_2}} (Pd\eta - Qd\xi)$$

Substituting the P and Q values into the above equation, taking Eqs (18), (19), (20) into consideration, transforming the line integral along the path X_1X_2 to a common integral, further retransforming z on the basis of Eqs (15) and (16), arranging and substituting $\xi = w\tau$, $L = wT$, leads as final result to:

$$\varphi_2(t-T) = \frac{1}{2}\varphi_1(t)e^{\frac{k\cdot T}{2}} + \frac{1}{2}\varphi_1(t-2T)e^{-\frac{k\cdot T}{2}} +$$
$$+ \frac{1}{2}e^{-\frac{k}{2}t + \frac{k\cdot T}{2}} \cdot w \int\limits_{t-2T}^{t} \Bigg\{\psi_1(\tau)I_0(2\sqrt{B(\xi - l)(\eta - y)})\Bigg|_{A_1} -$$
$$- \varphi_1(\tau)\left(\frac{\partial I_0}{\partial \xi} - \frac{\partial I_0}{\partial \eta}\right)\Bigg|_{A_1}\Bigg\} e^{\frac{k}{2}\tau}\, d\tau \qquad (23)$$

where the substitution A_1 has the following meaning:

$$A_1 = \{l = wt; \xi = \eta = w \cdot \tau; y = wt - 2wT\}$$

Eq. (23) is seen from the argument $-\psi_1(t)$ or from the lower limit of the integral to be interpreted for the time variable $-t \geq 2T$, that is

$$t \geq \frac{2L}{w}.$$

This result is in good agreement with the physical course of the wave phenomenon.

For $t < 2T$, the change generated at 1 by $\psi_1(t)$ and $\psi_1(t)$ do not fill out the range Ω_2 shown in Fig. 5. Thus, no $\varphi_2(t)$ interpreted at the place $x = L$ can be obtained from (23).

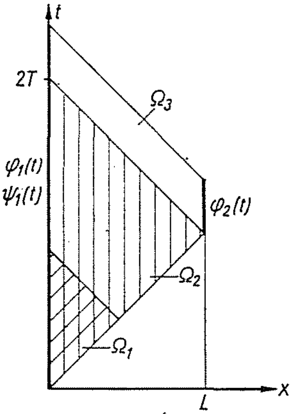

Fig. 5

Eq. (23) can be made valid also for the condition $t < \frac{2L}{w}$, – for a range of type – Ω_1, – if the L value is chosen to correspond to the pipe length where

$$\varphi_1(t) \text{ and } \psi_1(t)$$

determine unambiguous conditions in the given range of interpretation of t. This condition is expressed by the equality $T = \frac{t}{2}$.

Substituting this into (23), and remembering of the existence of an undisturbed initial value, namely 0 at $x < L$ in the left-hand side of Eq. (23), further of $\varphi_1(0) = 0$, we obtain:

$$0 = \varphi_1(t) \cdot e^{\frac{k \cdot t}{4}} + we^{-\frac{k \cdot t}{4}} \int_0^t \Bigg\{ \psi_1(\tau) I_0(2\sqrt{B(\xi - l)(\eta - y)}) \Big|_{A_2} - \psi_1(\tau) \left(\frac{\partial I_0}{\partial \xi} - \frac{\partial I_0}{\partial \eta} \right) \Big|_{A_2} \Bigg\} e^{\frac{k \cdot \tau}{2}} \, d\tau \tag{24}$$

$$\text{for} \quad t < \frac{2L}{w} = 2T,$$

$$\text{where} \quad A_2 = \{ l = wt; \xi = \eta = w\tau, y = 0 \}.$$

Integral equations (23) and (24) altogether give the required implicit functional relationship (12) for any time range. The expression of any of parameters $\varphi_1(t)$, $\psi_1(t)$ from Eqs (23) and (24), respectively, means the solution of the integral equation. The solution requires no further restrictions beyond the quoted assumptions.

5. Approximative solution of the integral equations

Completing the previous assumptions for functions $\varphi(t)$ and $\psi(t)$ by supposing $\psi(t)$ to be differentiable, it is evident that functions $\varphi(t)$ and $\psi(t)$ can be given with any required accuracy as the linear combination of their values assumed at discrete points of time.

Accuracy depends on the density of the discrete points of time, depending also on the linear combination.

In the fixed time intervals, $\varphi(t)$ and $\psi(t)$ are given by some, arbitrarily chosen, interpolation which can be expressed by linear combination.

Linearizing $\varphi(t)$ and $\psi(t)$, integral equations (23) and (24) can be written in the linear operator form. By fixing the operator basis, i.e. the system of discrete times, the linear operator matrix can be determined. In this way

Eqs (23) and (24) assume the form of matrix equations. A single equation is obtained since Eq. (23) is the "continuation" in time of Eq. (24). Our equation will be approximative since $\varphi(t)$ and $\psi(t)$ are approximated by previously chosen function forms.

5.1. *Choosing the approximating function form $\varphi(t)$ and $\psi(t)$*

With respect to the wave character of the examined phenomenon it is advisable to allow a break in $\varphi(t)$ and $\psi(t)$. Therefore the approximating function form will be a broken line function (Fig. 6).

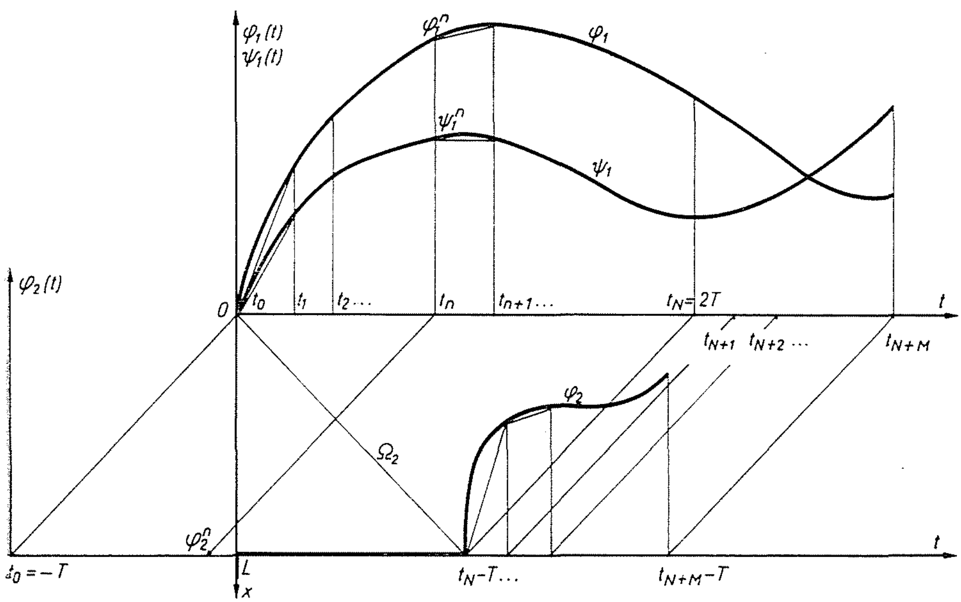

Fig. 6

The functions start from a homogeneous initial condition, from 0.

Let the examined time interval at $x = 0$ be $[0, t_{N+M}]$. Denote the end of the time interval belonging to the range Ω_2, by t_N:

$$t_N = 2\,T.$$

Intervals $[0, t_N]$ and $[t_N, t_{N+M}]$ are divided to N and M partial intervals, respectively.

The range of interpretation for $\varphi_2(t)$ is the time interval $[t_N - T, t_{N+M} - T]$, to be formally extended to the interval $[-T, t_{N+M} - T]$, such as:

$$\text{for } t \in (-T,\ t_N - T),$$

$$\varphi_2(t) = 0.$$

Thereby $\varphi_2(t_n)$ can be achieved to figure with as many values as $\varphi_1(t_n)$ and $\psi_1(t_n)$

The values at point t_n of broken line functions approximating the functions $\varphi_1(t)$, $\psi_1(t)$, $\varphi_2(t)$ are marked with a subscript.

$$\varphi_1(t_n) = \varphi_1^n,\ \varphi_2(t_n - T) = \varphi_2^n,\ \psi_1(t_n) = \psi_1^n$$

In this way the boundary values can be given by the following vectors.

$$\varphi_1 = \begin{bmatrix} \varphi_1{}^1 \\ \\ \varphi_1{}^2 \\ \vdots \\ \varphi_1^N \\ \vdots \\ \varphi_1^{N+M} \end{bmatrix} \quad \varphi_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \varphi_2^{N+1} \\ \vdots \\ \varphi_2^{N+M} \end{bmatrix} \quad \psi_1 = \begin{bmatrix} \psi_1{}^1 \\ \\ \psi_2{}^2 \\ \vdots \\ \psi_1^N \\ \vdots \\ \psi_1^{N+M} \end{bmatrix}$$

In the interval $[t_n, t_{n+1}]$ the functions are substituted by a straight line. For $t \in [t_n, t_{n+1}]$

$$\varphi_1(t) = \varphi_1^n + \frac{\varphi_1^{n+1} - \varphi_1^n}{t_{n+1} - t_n}(t - t_n).$$

$\psi_1(t)$ and $\varphi_2(t)$ can be written similarly to the above relationship.

Calculating the integrals in the right-hand side of integral equations (24) and (23) at times $t = t_0, \ldots, t_{N+M}$ substituting the above values, and making use of the theorem of the additivity of integrals, replacing (21) for the Riemann function, applying the binomial theorem relative to the raising to power, further the theorem relative to partial integration, the integral equation can be written in the following form.

$$\underline{\varphi_2} = \Psi \cdot \underline{\psi_1} + \Phi \cdot \underline{\varphi_1} \tag{25}$$

where ψ and Φ are $(N + M) \times (N + M)$ matrices of elements to be calculated as:

$$\Psi = [a_{ij}];\ i, j \in [1, N + M]$$

for $j > i$; $a_{ij} = 0$

for $1 \leq i \leq N$

$$j = i;\quad a_{ij} = e^{-\frac{k \cdot t_i}{4}} D_1(t_i, t_i, j - 1)$$

$$j < i;\quad a_{ij} = e^{-\frac{k \cdot t_i}{4}} \left[D_1(t_i, t_i, j - l) + C_1(t_i, t_i, j)\right]$$

for $N < i \leq N + M$

$$j = i \quad ; a_{ij} = e^{-\frac{k \cdot t_i}{2} + \frac{k \cdot T}{2}} D_1(t_i, 2 \cdot T, j-1)$$

$$i - N < j < i; \; a_{ij} = e^{-\frac{k \cdot t_i}{2} + \frac{k \cdot T}{2}} [D_1(t_i, 2 \cdot T, j-1) + C_1(t_i, 2 \cdot T, j)]$$

$$j = i - N; \quad a_{ij} = e^{-\frac{k \cdot t_i}{2} + \frac{k \cdot T}{2}} C_1(t_i, 2T, j)$$

$$j < i - N; \; a_{ij} = 0$$

$$\Phi = [b_{ij}]; \; i, j \in [1, N + M]$$

for $j > i; \; b_{ij} = 0$

for $1 \leq i \leq N$

$$j = i; \quad b_{ij} = \frac{1}{2} e^{\frac{k \cdot t_i}{4}} + e^{-\frac{k \cdot t_1}{4}} D_2(t_i, t_i, j-1)$$

$$j < i; \quad b_{ij} = e^{-\frac{k \cdot t_i}{4}} [D_2(t_i, t_i, j-1) + C_2(t_i, t_i, j)]$$

for $N < i \leq N + M$

$$j = i; \quad b_{ij} = e^{-\frac{k \cdot t_i}{2} + \frac{k \cdot T}{2}} D_2(t_i, 2 \cdot T, j-1) + \frac{1}{2} e^{\frac{k \cdot T}{2}}$$

$$i - N < j < i; \quad b_{ij} = e^{-\frac{k \cdot t_i}{2} + \frac{k \cdot T}{2}} [D_2(t_i, 2 \cdot T, j-1) + C_2(t_i, 2 \cdot T, j)]$$

$$j = i - N; \quad b_{ij} = \frac{1}{2} e^{-\frac{k \cdot T}{2}} + e^{-\frac{k \cdot t_i}{2} + \frac{k \cdot T}{2}} C_2(t_i, 2 \cdot T, j)$$

$$j < i - N; \; b_{ij} = 0$$

The three variable functions C_1, C_2, D_1, D_2 in the above relationship can be calculated as follows.

$$C_1(t_s, t_q, n) = \left(1 + \frac{t_n}{t_{n+1} - t_n}\right) G_1(t_s, t_q, n) - \frac{1}{t_{n+1} - t_n} H_1(t_s, t_q, n)$$

$$C_2(t_s, t_q, n) = \left(1 + \frac{t_n}{t_{n+1} - t_n}\right) G_2(t_s, t_q, n) - \frac{1}{t_{n+1} - t_n} H_2(t_s, t_q, n)$$

$$D_1(t_s, t_q, n) + \frac{1}{t_{n+1} - t_n} H_1(t_s, t_q, n) - \frac{t_n}{t_{n+1} - t_n} G_1(t_s, t_q, n)$$

$$D_2(t_s, t_q, n) = \frac{1}{t_{n+1} - t_n} H_2(t_s, t_q, n) - \frac{t_n}{t_{n+1} - t_n} G_2(t_s, t_q, n)$$

where

$$G_1(t_s, t_q, n) =$$

$$= \sum_{m=0}^{V} \frac{B^m w^{2m+1}}{2 \cdot m\,!^2} \sum_{i=0}^{m} \binom{m}{i} t_q^i \sum_{j=0}^{2m-i} \binom{2m-i}{j} t_s^j\,(-1)^j \sum_{r=0}^{2m-i-j} \frac{1}{\left(\frac{k}{2}\right)^{r+1}} \cdot$$

$$\cdot \binom{2m-i-j}{r} r\,!\,(-1)^r \left[e^{\frac{k\cdot\tau}{2}} \left(\tau^{2m-i-j-r}\right) \right]_{tn}^{t_{n+1}}$$

$$H_1(t_s, t_q, n) =$$

$$= \sum_{m=0}^{V} \frac{B^m w^{2m+1}}{2 \cdot m\,!^2} \sum_{i=0}^{m} \binom{m}{i} t_q^i \sum_{j=0}^{2m-i} \binom{2m-i}{j} t_s^j (-1)^j \sum_{r=0}^{2m-i-j+1} \frac{1}{\left(\frac{k}{2}\right)^{r+1}} \cdot$$

$$\cdot \binom{2m-i-j+1}{r} r\,!\,(-1)^r \left[e^{\frac{k\cdot\tau}{2}} \left(\tau^{2m-i-j-r+1}\right) \right]_{tn}^{t_{n+1}}$$

$$G_2(t_s, t_q, n) =$$

$$= -\sum_{m=0}^{V} \frac{B^m m w^{2m}}{2m\,!^2} \cdot t_q \sum_{i=0}^{m-1} \binom{m-1}{i} t_q^i \sum_{j=0}^{2m-i-2} \binom{2m-i-2}{j} t_s^j (-1)^j \cdot$$

$$\cdot \sum_{r=0}^{2m-i-j-2} \frac{1}{\left(\frac{k}{2}\right)^{r+1}} r\,!(-1)^r \left[e^{\frac{k\cdot\tau}{2}} \left(\tau^{2m-i-j-r-2}\right) \right]_{tn}^{t_{n+1}}$$

$$H_2(t_s, t_q, n) =$$

$$= -\sum_{m=0}^{V} \frac{B^m m w^{2m}}{2 \cdot m\,!^2} t_q \sum_{i=0}^{m-1} \binom{m-1}{j} t_q^i \sum_{j=0}^{2m-i-2} \binom{2m-i-2}{j} t_s^j (-1)^j \cdot$$

$$\cdot \sum_{r=0}^{2m-i-j-1} \frac{1}{\left(\frac{k}{2}\right)^{r+1}} \binom{2m-i-j-1}{r} r\,!(-1)^r \left[e^{\frac{k\tau}{2}} \left(\tau^{2m-i-j-r-1}\right) \right]_{tn}^{t_{n+1}}$$

The function in square brackets is meant as the difference between the indicated limits.

Matrices Ψ and Φ can only be computer determined. The rather time-consuming calculation is worth while if a given pipeline — liquid system is examined, to find the response functions of various disturbances.* Namely

* A part of the present paper was prepared, in connection with the research work entitled "New type of pressure control system" [4], under the leadership of Dr. Imre SZABÓ, for the control problems of the oil pipeline "Friendship II".

matrices depend on y on the known constant parameters of pipe flow, but do not depend on ψ and φ, i.e. the values of pressure and velocity at both ends of the flow pipe.

The data necessary for determining matrices Φ and Ψ are

$$k, w, T \quad t_1, t_2 \dots t_{N+M}$$

There are V terms in the series of the Riemann functions taken into consideration. Concrete calculations proved the accuracy of matrices obtained for $V = 10$ to be sufficient.

6. Dimension analysis

Divide Eq. (28) throughout by the dimension of φ and write the dimensions obtained in this way. Consider that

$$\psi = \frac{\partial \varphi}{\partial x}$$

$$\underline{\varphi}_2[1] - \Phi[1]\underline{\varphi}_1[1] + \Psi[\text{length}]\underline{\psi}_1\left[\frac{1}{\text{length}}\right]$$

Examine the data determining matrices Φ and Ψ.

$$\Phi, \Psi = f\left(k, w, \{t_n\}_1^{N+M}\right)$$

$$\text{Since } k = \left[\frac{1}{\text{time}}\right], \quad w = \left[\frac{\text{Length}}{\text{time}}\right], \quad t_n = \left[\text{time}\right],$$

$$\text{thus}\left[\text{length}\right] = \frac{w}{k}.$$

Factoring out $\frac{w}{k}$ from matrix Ψ, multiplying all the elements of the matrix by $\frac{k}{w}$, and designating the new matrix by Ψ', we obtain:

$$\underline{\varphi}_2 = \Phi \cdot \underline{\varphi}_1 + \frac{w}{k}\Psi' \cdot \underline{\psi}_1 \tag{26}$$

where

$$\Psi' = \left[\frac{k}{w}a_{ij}\right] \tag{27}$$

Matrices Φ and Ψ' in Eq. (26) are relatively dimensionless.

To eliminate the dimension of time, let us express $t_1, \dots, t_n \dots, t_{N+M}$ as the relation of a real set of numbers, $\{\omega_n\}$, and of k.

$$t_n = \frac{\omega_n}{k} \quad n = 1, \dots N + M \tag{28}$$

In place of time the dimensionless ω_n can be given. It is important to note that for $t_n = 2T = \frac{\omega_N}{k}$ actually Eq. (24) is converted to (23). This appears in matrices Φ and Ψ. N is the number of non-zero elements in the first column of the matrices.

As a conclusion it can be stated, that it is sufficient to replace $\Phi, \Psi, k, w, \{t_n\}_1^{N+M}$ by

$$\Phi, \Psi, \{\omega_n\}_1^{N+M}$$

For an arbitrary group of data k, w, $\{t_n\}_1^{N+M}$, Eq. (26) is valid, the dividing points of the examined time interval can be calculated from Eq. (28).

N can be read off some of the matrices, and evident from the range of interpretation of Eq. (24) that the relationship

$$L = WT = \frac{\omega_N}{2k}$$

is valid for the length of the examined pipe section.

7. Example

This method will be illustrated on computer determined matrices Φ and Ψ, belonging to a time interval and $\{\omega_n\}$. Matrix elements are given in Table II. The pertaining $\{\omega_n\}$ is the following.

$$\{\omega_n\} = \{0.8\ 1.6\ 2.4\ 3.2\ 4\ 4.8\ 5.6\ 6.4\ 7.2\ 8\}$$

Evidently, $N = 10$, $M = 0$.

The above will be applied for determining the shutting off of a liquid flow.
Data: $d = 0.6$ [m]
$w = 1000$ [m/s]
$L = 10^5$ [m]
$\varrho = 880$ [kg/m^3]
$\nu = 0.41 \cdot 10^{-4}$ [m^2/s]

Initial velocity before the final control element, at the end of the pipeline is $v = 2$ [m/s].

Velocity has to drop to 0 during 40 [s]. This problem is calculated by superposition, as a velocity reduction from 0 to 2 [m/s] (Fig. 7).

(6/a) yields $k = 0.04$ [1/s]. The time interval calculated with the matrices with the above data ($N = 10$, $M = 0$) is

$$t_N = 2T = \frac{\omega_N}{k} = 200\ [s].$$

Accordingly the length of pipe that can be examined is $L = wT = 10^5$ [m] at the maximum. Time division is seen from $\{\omega_n\}$ to be uniform, one division corresponds to 20 [s].

Denoting the end of the pipeline by subscript 1, and taking into consideration that $M = 0$ causes the examined time interval to be [0.2 T] hence v_2, p_2 are known, and identical with the unchanged homogeneous initial condition, the substitution $v_1 = \psi_1$ can be selected from Table I.

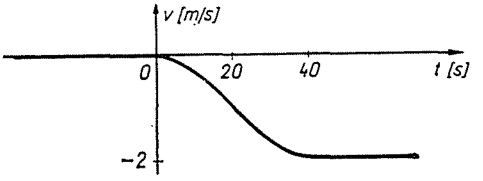

Fig. 7

From Eq. (13), substituting Eq. (5) for S, our data lead to the following vector ψ_1 approximating the velocity function in Fig. 7:

$$\underline{\psi_1} = -\varrho\left(\underline{kv} + \frac{dv}{\underline{dt}}\right) = \begin{bmatrix} 158.4 \\ 70.4 \\ 70.4 \\ \vdots \\ 70.4 \end{bmatrix}$$

φ_1 can be expressed from Eq. (26). Since $\varphi_2 = 0$, further, since x, and thus also $\frac{d\varphi_1}{dx} = \psi_1$ are negative, the pipe end being denoted by 1, we may write:

$$\underline{\varphi_1} = \frac{w}{k}\,\Phi^{-1}\Psi'\underline{\psi_1}.$$

Performing the operations, we obtain the following values for $\underline{\varphi_1} = \underline{p_1}$ in intervals of 20 [s].

$$\underline{\varphi_1} = \underline{p} = 10^5 \begin{bmatrix} 14.06 \\ 28.63 \\ 32.95 \\ 37.02 \\ 40.83 \\ 44.38 \\ 47.71 \\ 51.29 \\ 53.83 \\ 56.66 \end{bmatrix} \left[\frac{N}{m^2}\right]$$

Table II

ψ Matrix				
+.1747/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.2628/+00	+.1725/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.1686/+00	+.2529/+00	+.1703/+00	+.0000/+00	+.0000/+00
+.1081/+00	+.1553/+00	+.2433/+00	+.1682/+00	+.0000/+00
+.6922/—01	+.9496/—01	+.1425/+00	+.2338/+00	+.1661/+00
+.4429/—01	+.5780/—01	+.8263/—01	+.1303/+00	+.2246/+00
+.2831/—01	+.3499/—01	+.4730/—01	+.7107/—01	+.1186/+00
+.1807/—01	+.2105/—01	+.2663/—01	+.3767/—01	+.6023/—01
+.1153/—01	+.1258/—01	+.1468/—01	+.1916/—01	+.2887/—01
+.7341/—02	+.7448/—02	+.7866/—02	+.9131/—02	+.1250/—01

Φ Matrix				
+.4912/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
—.2672/—01	+.4826/+00	+.0000/+00	+.0000/+00	+.0000/+00
—.2630/—01	—.3934/—01	+.4740/+00	+.0000/+00	+.0000/+00
—.2300/—01	—.3371/—01	—.5147/—01	+.4656/+00	+.0000/+00
—.1886/—01	—.2706/—01	—.4049/—01	—.6313/—01	+.4573/+00
—.1484/—01	—.2085/—01	—.3053/—01	—.4666/—01	—.7433/—01
—.1135/—01	—.1560/—01	—.2235/—01	—.3344/—01	—.5225/—01
—.8509/—02	—.1143/—01	—.1600/—01	—.2341/—01	—.3583/—01
—.6276/—02	—.8239/—02	—.1126/—01	—.1608/—01	—.2407/—01
—.4571/—02	—.5860/—02	—.7810/—02	—.1088/—01	—.1589/—01

In Fig. 8 the obtained curve *a* of pressure change has been plotted, and so has been the pressure change curve *b* obtained by the method of characteristics [4], for the same data, taking into consideration a quadratic attenuation. The linear attenuation is seen to "over-attenuate" in the range of low velocities, but for relatively short time interval, at the beginning of the transient, the error is low.

Summary

For the determination of pressure and velocity transients arising during the flow of incompressible media in a pipeline a general implicit integral equation can be obtained by linearizing the basic differential equation. Approximating the transients by a broken-line function transforms the integral equation into a matrix equation. Coefficient matrices can be calculated from the previously known constant parameters of the flow and are constant for a liquid-pipeline system. The matrices are not singular, thus by considering also the determination, dynamic response functions can be established for any disturbance.

+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.1640/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.2156/+00	+.1619/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.1073/+00	+.2068/+00	+.1599/+00	+.0000/+00	+.0000/+00
+.5009/—01	+.9657/—01	+.1982/+00	+.1579/+00	+.0000/+00
+.2084/—01	+.4063/—01	+.8628/—01	+.1898/+00	+.1558/+00

+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
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+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
+.4490/+00	+.0000/+00	+.0000/+00	+.0000/+00	+.0000/+00
—.8507/—01	+.4409/+00	+.0000/+00	+.0000/+00	+.0000/+00
—.5728/—01	—.9538/—01	+.4319/+00	+.0000/+00	+.0000/+00
—.3773/—01	—.6178/—01	—.1053/+00	+.4250/+00	+.0000/+00
—.2436/—01	—.3917/—01	—.6578/—01	—.1147/+00	+.4172/+00

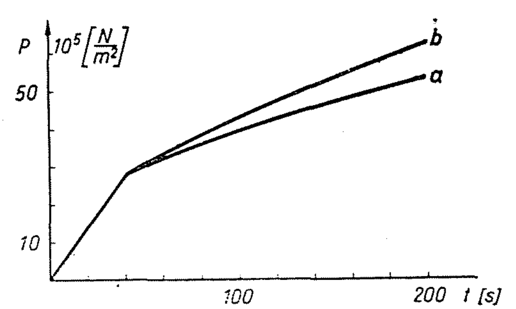

Fig. 8

Symbols

p	— pressure
v	— flow velocity affected by a sign
t	— time variable
x	— place variable
S	— pressure drop of the liquid of unit density flowing in a pipe of unit length
Re	— Reynolds' number
E	— Young's modulus
d	— pipe diameter
L	— length of the pipe section
F	— differential operator
f	— symbol of a functional relationship
k	— attenuation factor
z	— transformed pressure or velocity
B	— constant characterizing the flow
l, y	— transformed independent variable
X	— symbol of a point
r, s, q, n, m, i, j	— subscript variables
I_0	— Bessel's function
R	— Riemann's function
P, Q	— line integral arguments
A	— set symbolizing the substitution value
w	— wave propagation velocity
V	— the limit of subscript variable m — the number of terms in the series of Bessel's function
C, D, G, H	— auxiliary functions
[m]	— unit of length, meter
[s]	— unit of time, second
[kg]	— unit of mass, kilogramm
[N]	— unit of force, newton
a	— designates the elements of matrix Ψ
b	— designates the elements of matrix Φ
N, M	— number of the partial intervals of time
T	— time of propagation of an effect in the pipe section
ϱ	— density
δ	— pipe wall thickness
λ	— pipe friction coefficient
φ	— general boundary value
ψ	— partial derivative of φ with respect to x
χ	— transformed pressure or velocity
ξ, η, τ	— auxiliary integration variables
Ω	— the examined place-time range
Ψ, Φ	— matrices
$\underline{\varphi}$	— column vector formed of the general boundary value
$\underline{\psi}$	— column vector formed of function ψ
ω	— element of the set of real numbers

References

1. SZABÓ, I.: Folyadékot szállító hosszú csővezeték üzemének szabályozása. (Service control of long pipelines for liquids.) Candidate's Thesis. Budapest (1965).
2. SZABÓ, I.: Gépek és folyamatok rendszertana. (Systematization of machines and processes.) Tankönyvkiadó, Budapest (1972).
3. IMRE, L.: Műszaki hőtan és áramlástan. (Engineering thermodynamics and flow theory.) Tankönyvkiadó, Budapest (1969).

4. SZABÓ, I.—HECKER, G.—MÁTHÉ, I.—DANKÓ, GY.: Újtípusú nyomáskorlátozó rendszer. (New-type pressure control system.) (Research report, Department of Machines at the Faculty of Electrical Engineering, Budapest Technical University, Nov. 1970.)
5. STREETER, V. L.—WYLIE, E. B.: Hydraulic Transients. McGraw-Hill Book Company, New York (1967).
6. RICH, G. R.: Hydraulic Transients. Dover Publications, Inc., New York (1963).
7. CZÁCH, L.—SIMON, L.: Parciális differenciálegyenletek. (Partial differential equations.) Tankönyvkiadó, Budapest (1969).
8. MIKHLIN, S. G.: Integrálegyenletek és alkalmazásuk a mechanika, a matematika, fizika és technika egyes problémáira. (Integral equations and applications for some problems of mechanics, mathematics, physics and engineering.) Akadémiai Kiadó, Budapest (1953).
9. KISS, O.: Számítási módszerek. (Calculation methods.) Tankönyvkiadó, Budapest (1969).

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