A SHORT INTRODUCTION TO STATE-SPACE METHODS

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Introduction

Two basic trends can be observed in science history, specialization and integration. With the advent of *cybernetics* and *digital computer applications* some similitudes of various systems have come to light. We often speak of *systems theory*. One aim of the latter is to exploit such similitudes in the field of economics, biology, psychology, sociology, tactics, engineering and so on. The application possibility of similar investigation methods in quite different fields contribute to the integration trend.

In the last two decades the analysis and synthesis methods of dynamic systems made a radical alteration. This change has some connection with the general use of optimization methods and nonlinear techniques.

The common differential and difference equation methods, as well as the Laplace, Fourier and z-transform methods in linear systems were overshadowed by the recent state-space methods. The latter, like the differential and difference equation methods, also refer to the time domain. State-space methods can be applied to linear time-invariant and time-variable systems as well as to nonlinear systems. With the complexity of systems, the state-space method, as any other method, however, loses its simplicity. In any case, the state-space method furnishes a concise description of the scrutinized system, and, at the same time gives a deep insight into the main characteristics of systems and/or their parts. For example, applying the state-space method, single-variable (single-input single-output) and multivariable (multi-input multi-output) systems can be described by the same form of equations, not speaking of minor deviations or differences.

The widespread use of digital computers and the state-space method have a close mutual effect. This interaction can be clarified by the fact that, on the one hand, digital computations often demanded the application of state-space methods for the study of dynamic systems, and on the other hand, state-space equations require the employment of digital computers in complicated cases. Therefore, modern cybernetics and system theory may be characterized by the widespread use of digital computers and the application of state-space methods.

This paper has the intention to give a brief introduction to state-space methods through simple examples.

Fundamental Problems in Connection with Dynamic Systems

As concerns dynamic systems, and especially the feedback control systems, the following problems can be mentioned.

The first step is always the construction of an appropriate model for the dynamic system under study. This step is called *model-building*. A system is considered as a regularly interacting or interdependent group of items (such as bodies, organs, devices, organizations, etc.) forming a unified whole. The essential performance of a system can be described by an abstract model often called a mathematical model. This is, of course, an approach. The model is unable in every respect to replace the real system. A well chosen mathematical model may, however, to some extent express the most important performance characteristics of an actual system.

The model building is closely related to *parameter identification*. The task of the latter is to determine the value of parameters (constants, factors, gains, time constants, transportation lags). If the structure of the system itself is altered for some reason, then, the parameter values will change, too. Often the requirement arises how to approach a complicated system by a model of simpler structure. Nonlinear systems are, for example, often replaced by linear models at least in the vicinity of a certain point of operation.

In the knowledge of model structure and parameter values the next step is to choose an appropriate controller and adjust their parameters, for example, gains, rate and reset time constants or state-variable feedback coefficients. This task is called *synthesis* in contrast to the previous steps which are, in essence, the parts of *analysis*.

The fourth problem in connection with dynamic systems, and in particular, with feedback control systems is *optimization*. The task of dynamic optimization is to choose the free parameters, e.g. the parameters of the controller, in such a manner as to obtain the most favourable transient process. (Insertion of appropriate nonlinear devices, such as relays, is also admissible.) What is meant by the most favourable process depends on the criterion of optimality. A system optimal in accordance with some chosen criterion is far from being optimal from the point of view of another criterion. One of the most difficult questions is just the choice of the proper criterion. Often the capacity of digital computers is an obstacle or hindrance to the application of more complex, but at the same time more realistic, criteria.

In model building and in optimization some a priori knowledge is necessary. Unfortunately, a full knowledge of facts is rarely at our disposal. In such cases *simulation* becomes more and more important. The essence of the latter is that we construct a certain model, adjust their parameters, and perform some experiments on the model. A comparison with the reality gives us a measure as to how well we have chosen the structure of the model and the values of their parameters. In case of significant deviations we make due changes and perform the experiments over again.

Simulation becomes inevitable in such cases when experimenting in reality, are not at all, or hardly performable. This is the case, for example, in space research, in economics or in physiology. Simulation enable us to replace observation by experimentation. Although such experiments are somewhat restricted and restrained, they considerably help us in the recognition of reality.

The value of simulation rises if instead of time-invariant linear system we have to study time-variable and/or nonlinear systems. For such purposes analog, digital or hybrid computers can be applied.

Here we have no possibility of going into the details of analog simulation. All the less so as we hope that the principles of analog computers are well-known enough. For digital simulation of continuous-time processes special programming languages has been developed. We may mention MIMIC which is applied on computers IBM 7090, CDC 3300, UNIVAC 1107, 1108; DSL (Digital Simulation Language) on computers IBM 7090, IBM 1800, IBM system 360; CSMP (Continuous System Modelling Language) and CSSL (Continuous System Simulation Language) on IBM System 360 and system 370 computers. The knowledge of FORTRAN is also advisable.

Although digital simulation gives more exact results, analog simulation is often satisfactory being, at the same time, cheaper and quicker. Sometimes hybrid simulation would be the best, but for this purpose special computers are necessary which are relatively expensive.

System analysis, model building, parameter identification, system synthesis, optimization and simulation constitute the foundation of modern system theory.

In all these topics, state-space methods can give an invaluable help in the solution of more or less complicated problems.

First Introductory Example. Unforced System

The simplest linear difference equation only involves first differences $\nabla x_k = x_k - x_{k-1}$ or $\nabla x_k = x_{k+1} - x_k$ of a variable x_k. Nevertheless, it is of considerable importance, since it represents the way in which many system variables grow or decay. Consider, for example, the growth of capital fund x_0 which is bearing compound interest. If the interest rate is denoted by p (i.e. $100\,p\,\%$ is the rate of interest in percentage) then at the end of the first year the fund increases to

$$x_1 = x_0 + px_0 \tag{1}$$

and at the end of the second year to

$$x_2 = x_1 + px_1 \tag{2}$$

and so on. The general rule can be expressed as

$$x_{k+1} = x_k + px_k, \qquad (k = 0, 1, 2, \ldots). \tag{3}$$

Introducing $1 + p = F$,

$$x_{k+1} = Fx_k, \qquad (k = 0, 1, 2, \ldots) \tag{4}$$

and the solution of this simple problem can be given as

$$x_1 = Fx_0, \quad x_2 = F^2x_0, \ldots, x_k = F^kx_0 . \tag{5}$$

This is the well-known geometric sequence describing the increase of a deposit ($p > 0$, $F > 1$). If $p = 1$, $F = 2$, then we obtain the common chessboard rule of corn grains. If, on the other hand, $-1 < p < 0$, $0 < F < 1$, then the process decays describing, for example, the dying out of a population or the radiation effect of a nuclear material.

The continuous counterpart of this problem can be formulated by the simple first-order differential equation (with $\dot{x} = dx/dt$):

$$\dot{x} = Ax \tag{6}$$

whose solution, subject to initial condition $x(0) = x_0$ at $t = 0$, is

$$x(t) = x_0e^{At} . \tag{7}$$

The latter relationship can easily be verified by differentiation.

At first sight, there is a close relation between the difference and differential equations and their solutions, but this is actually not too obvious. By expressing the differential quotient for Eq. (6) we may write

$$\lim_{T\to 0} \frac{x(t+T) - x(t)}{T} = Ax(t). \tag{8}$$

Replacing the differential quotient by a difference quotient

$$\frac{x(t+T)-x(t)}{T}=Ax(t) \tag{9}$$

which, after some algebraic manipulation yields

$$x(t+T)=(1+AT)x(t)\,. \tag{10}$$

Taking $t=t_k$ and $t+T=t_{k+1}$ into account comparison of Eqs (4) and (10) suggests

$$F=1+AT \quad \text{or} \quad A=\frac{1}{T}(F-1) \tag{11}$$

which is, however, a rough approach and can only be considered as a first approximation. A comparison of the two solutions, Eqs (5) and (7), that is to say, an exact coincidence for $t=t_k=kT$ yields

$$x_k=F^k x_0=x_0 e^{AkT}=x(kT)$$

from which we obtain

$$F=e^{AT} \quad \text{or} \quad A=\frac{1}{T}\ln F. \tag{12}$$

It can readily be seen that the first approximation in Eq. (11) contains only the first two terms in the Taylor series expansion of the exponential or logarithm in Eq. (12). It must be emphasized that we obtained the exact relationship between F and A by the solutions. (If the latter are not known, a conversion from a differential equation to a difference equation, or vice versa, is far from being trivial.)

Some relationships between differential and difference operators can be found in Table 1.

Second Introductory Example. Forced System

Let us look at a second simple economic example, that of redemption, which is a common problem in private life or in connection with investments. Let us assume, that a loan L is raised which is to be amortized in n equal instalments, $u_k=U=\text{const.}$ $(k=0,1,2,\ldots,n-1)$. Each instalment u_k has to be paid within the first, second, ... n-th year, but not later than the end of that year, that is, in the $(k+1)$-th interval. Thus, u_0 is the first and u_{n-1} is the last payment. Let us denote the debt or liability by x_k referring

Table 1

Relationships between Operators

Denoting the differential operator $D = d/dt$ and using the shifting operator E, and the forward and backward difference operator $\nabla = 1 - E^{-1}$ and $\triangle = E - 1$, respectively, Taylor series expansion yields

$$Ef(t) = \left[1 + TD + \frac{T^2}{2!} D^2 + \frac{T^3}{3!} D^3 + \dots\right] f(t)$$

that is

$$E = e^{TD}; \ \nabla = 1 - e^{-TD}; \ \triangle = e^{TD} - 1;$$

$$D = \frac{1}{T} \ln E; \quad D = -\frac{1}{T} \ln (1 - \nabla); \quad D = \frac{1}{T} \ln (1 + \triangle);$$

Taylor series expansions and raising to power will result in

$$\left.\begin{matrix}\triangle \\ \nabla\end{matrix}\right\} = TD \pm \frac{T^2}{2!} D^2 + \frac{T^3}{3!} D^3 \pm \dots$$

$$\left.\begin{matrix}\triangle^2 \\ \nabla^2\end{matrix}\right\} = T^2D^2 \pm T^3D^3 + \frac{7}{12} T^4D^4 \pm \dots$$

$$\left.\begin{matrix}\triangle^3 \\ \nabla^3\end{matrix}\right\} = T^3D^3 \pm \frac{3}{2} T^4D^4 + \frac{5}{4} T^5D^5 \pm \dots$$

and

$$D = \begin{cases} \dfrac{1}{T}\left(\triangle - \dfrac{\triangle^2}{2} + \dfrac{\triangle^3}{3} - \dots\right) \\ \dfrac{1}{T}\left(\nabla + \dfrac{\nabla^2}{2} + \dfrac{\nabla^3}{3} + \dots\right) \end{cases}$$

$$D^2 = \begin{cases} \dfrac{1}{T^2}\left(\triangle^2 - \triangle^3 + \dfrac{11}{2} \triangle^4 - \dots\right) \\ \dfrac{1}{T^2}\left(\nabla^2 + \nabla^3 + \dfrac{11}{2} \nabla^4 - \dots\right) \end{cases}$$

$$D^3 = \begin{cases} \dfrac{1}{T^3}\left(\triangle - \dfrac{3}{2} \triangle^4 + \dfrac{7}{4} \triangle^5 - \dots\right) \\ \dfrac{1}{T^3}\left(\nabla^3 + \dfrac{3}{2} \nabla^4 + \dfrac{7}{4} \nabla^5 + \dots\right) \end{cases}$$

to the beginning of the $(k + 1)$-th interval, that is, at the k-th instant, $(k = 0, 1, \dots, n - 1)$. Assuming an interest rate p, the amortization process can be visualized by the difference equation

$$x_{k+1} = x_k + px_k - u_k \tag{13}$$

or

$$x_{k+1} = Fx_k - u_k \tag{14}$$

subject to the initial condition $x_0 = L$. The liability decreases and at the very end vanishes, $x_n = 0$. The final solution can be expressed as

$$x_k = F^k L - \frac{F^k - 1}{F - 1} U \ . \tag{15}$$

(The latter can easily be obtained by using Table 3.) Condition $x_n = 0$ results in

$$U = F^n \frac{F - 1}{F^n - 1} L \tag{16}$$

which yields the value of the instalments. Choosing $n = 5$ and $p = 0.05$ a redemption schedule is shown in Table 2.

Table 2

Amortization Schedule

Time k	Year $k+1$	Liability x_k	Redemption u_k	Interest redemption px_k	Loan redemption $u_k - px_k$
0	1	100.00	23.10	5.00	18.10
1	2	81.90	23.10	4.10	19.00
2	3	62.90	23.10	3.14	19.96
3	4	42.94	23.10	2.15	20.95
4	5	21.99	23.10	1.10	22.00
5	6	0.00			
Sum total			115.50	15.49	100.01

Simple State Difference Equations

After some generalization of Eq. (14) we may write

$$x_{k+1} = Fx_k + Gu_k \tag{17}$$

where F and G are appropriate constants. This is the common form of the *simplest state difference equation*. We often have an algebraic auxiliary equation as well

$$y_k = Hx_k + Ku_k \tag{18}$$

with H and K constants.

Here u_k is the *input*, x_k is the *state variable*, y_k is the *output*. Subscript k refers to the start of the $(k+1)$-th interval, that is, to the k-th instant. Because the solution of algebraic equations is simple we concentrate our attention to the solution of the state difference equation in Eq. (17). Applying the initial conditions x_0, u_0 we may write down the following recurrence relations:

$$\begin{aligned} x_1 &= Fx_0 + Gu_0 ; \\ x_2 &= Fx_1 + Gu_1 = F^2x_0 + Gu_1 + FGu_0 ; \\ x_3 &= Fx_2 + Gu_2 = F^3x_0 + Gu_2 + FGu_1 + F^2Gu_0 . \end{aligned} \tag{19}$$

These lead us to the general form

$$x_k = F^k x_0 + \sum_{j=0}^{k-1} F^{k-j-1} Gu_j . \tag{20}$$

It can readily be seen that the state difference equation in its original form, Eq. (17), assigns a *solution algorithm* facilitating the program preparation for a digital computer. Owing to these circumstances we often endeavour to convert state differential equations into state difference equations when studying dynamic systems.

For some special cases which often occur the solutions of state difference equations are summarized in Table 3. Samples of discrete-time processes are outlined in Fig. 1 for the case $u_k = 1$.

Table 3

Solutions of Special Cases

If $u_k = C^k$ (a geometric sequence, $C \neq 1$), then

$$x_k = \begin{cases} F^k x_0 + \dfrac{C^k - F^k}{C - F} G, & \text{for } F \neq C \\ F^k x_0 + kF^{k-1}G, & \text{for } F = C . \end{cases}$$

If $C = 1, u_k = 1$, then

$$x_k = \begin{cases} F^k x_0 + \dfrac{1 - F^k}{1 - F} G, & \text{for } F \neq 1 \\ x_0 + kG, & \text{for } F = 1 . \end{cases}$$

Introducing $X = G/(C-F)$, for $C \neq 1$ and $X = G/(1-F)$, for $C = 1$, respectively, we obtain

$$x_k = \begin{cases} F^k(x_0 - X) + C^k X, & \text{for } F \neq C \neq 1 \\ F^k x_0 + kF^{k-1}G, & \text{for } F = C \neq 1 \end{cases}$$

and

$$x_k = \begin{cases} F^k(x_0 - X) + X, & \text{for } F \neq C = 1 \\ x_0 + kG, & \text{for } F = C = 1 . \end{cases}$$

If $|F| < 1$, then the processes are asymptotically stable, that is, decaying; if $|F| = 1$, then the processes are stable but not asymptotically stable, that is, they are constant or oscillatory, in short, bounded; if $|F| > 1$, then the processes are unstable, that is, unbounded.

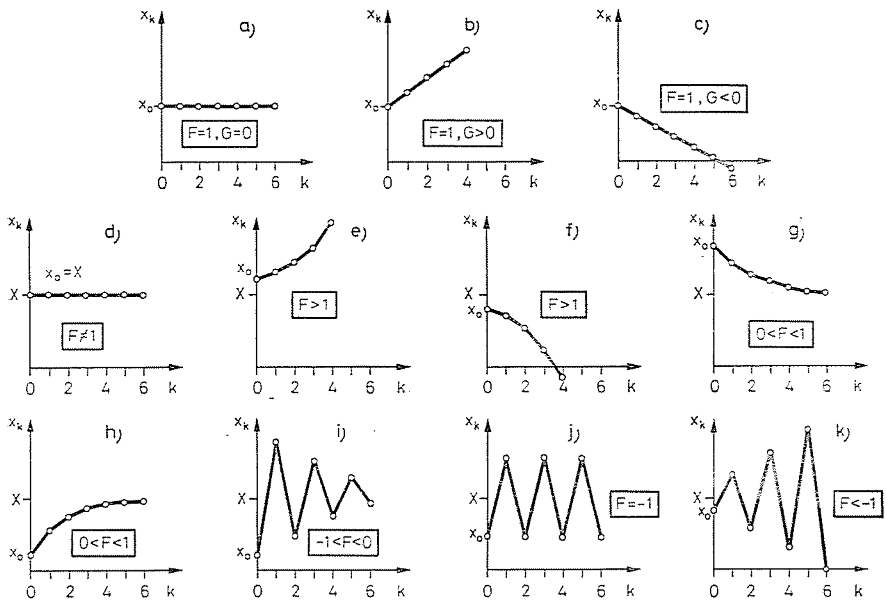

Fig. 1

An Application Example: Forecasting

The state difference equations can also be applied to forecasting. Any quantitative forecasting method serves to smooth out fluctuations in a process. In this case $F = 1 - \alpha$ and $G = \alpha$, where $0 < \alpha < 1$ is the so-called smoothing factor, and, x_{k+1} is the forecast for the next period, x_k the forecast for the present period, u_k the actual value for the present period. As

$$x_{k+1} = (1 - \alpha)x_k + \alpha u_k \tag{21}$$

can also be written in the form

$$x_{k+1} = x_k + \alpha(u_k - x_k) \tag{22}$$

the smoothed forecast is equal to the present smoothed forecast plus some fraction α of the difference between the forecasted and actual values during the present period. The selection of α seems to be of crucial importance, in general,

α is some value between 0 and 1, often between 0.1 and 0.3. A response to a rapidly changing process improves with higher smoothing constant. If $\alpha = 0$, no data except the original forecast are included. When $\alpha = 1$, the next forecast is the same as the present actual value.

Taking the general solution formula, Eq. (20), into consideration and assuming the initial historical value u_0 equivalent to the starting forecast x_0, that is, by $u_0 = x_0$, we can easily obtain

$$x_k = (1-\alpha)^k u_0 + \sum_{j=0}^{k-1} (1-\alpha)^{k-j-1} \alpha u_j \ . \tag{23}$$

Let us investigate, for example, a time sequence $x_0 = u_0 = 216$, $u_1 = 238$, $u_2 = 220$, $u_3 = 244$, $u_4 = 260$. Linear regression by least squares fitting would suggest $x_5 = 264$.

Setting the smoothing constant to $\alpha = 0.8$ the above forecasting formula in Eq. (23) yields $x_5 = 256$. Taking on the other hand $\alpha = 0.2$ we obtain $x_5 = 232$.

As any extrapolation is loaded with errors so is the forecast (and also the regression) technique.

Thus far we tacitly assumed that the processes can be described by a state difference equation containing only first-order differences. This is, however, a serious limitation. What to do if we have a higher-order process then is shown in the next paragraph.

Third Introductory Example

Let us start out from the simultaneous set of homogeneous difference equations

$$\begin{aligned} p_{k+1} &= (1-\alpha)p_k + \beta q_k ; \\ q_{k+1} &= \alpha p_k + (1-\beta) q_k . \end{aligned} \tag{24}$$

Many applications of such equations can be mentioned. Let us first see a neuron network in the nerve system. We shall confine ourselves to two neurons. If the first neuron gets an impulse, then after a certain synaptic lag, the second neuron also gets the stimulus. Let us assume that the first neuron has a feedback, so after a period, the first neuron becomes reexcited. The second impulse also arrives to the other neuron. Thus, an impulse sequence is produced.

This memory network may, however, have some faults. Let us assume that the transition from the excited state to the unexcited one can be characterized by the probability α, whereas to the reverse transition belongs a probability β. The endurance probabilities of the excited and unexcited states are, then, $1 - \alpha$ and $1 - \beta$, respectively. Obviously $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$.

The probability of the excited and unexcited states are denoted by p and q, respectively, where $q = 1 - p$. At the next interval the system will be in the excited state if at present the system is in excited state and the latter survive, or the system is in the unexcited state and the state changes. This statement is expressed by the first relation of Eq. (24). The second relation can similarly be explained.

The same equations also express a certain model of behaviour. During public opinion poll a series of questions are posed. Let the probability of the endurance of reply "yes" be $1 - \alpha$, that of the response "not" $1 - \beta$. Transition from "yes" to "not" and vice versa have probabilities α and β, respectively. The first difference equation in Eq. (24) expresses the probability of affirmative answers, whereas the second that of the negative answers.

The same equations are also valid for some teaching or learning processes.

Such processes are called MARKOV processes. The probabilities at t_{k+1} or $k + 1$, do depend only on the probabilities of time t_k or k. MARKOV processes are important in many random systems. They are easily describable by state-space methods.

Matrix Notations

Let us introduce the so-called system matrix by

$$F = \begin{bmatrix} 1 - \alpha & \beta \\ \alpha & 1 - \beta \end{bmatrix}. \tag{25}$$

If we introduce state vectors by column matrices,

$$x_{k+1} = \begin{bmatrix} p_{k+1} \\ q_{k+1} \end{bmatrix}, \qquad \text{and} \qquad x_k = \begin{bmatrix} p_k \\ q_k \end{bmatrix} \tag{26}$$

then the simultaneous set of state difference equations in Eq. (24) can be expressed by one matrix equation

$$x_{k+1} = F x_k \, . \tag{27}$$

The latter is also called a state-vector difference equation. The solution of Eq. (27) subject to initial condition

$$x_0 = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} = [p_0, q_0]^T \tag{28}$$

can be expressed as

$$x_k = F^k x_0 . \tag{29}$$

A comparison of Eq. (5) and Eq. (29) is a clear explanation for the advantage of matrix notations. The k-th power of matrix F can be written in the form

$$F^k = \frac{1}{\alpha+\beta}\begin{bmatrix}\beta & \beta\\ \alpha & \alpha\end{bmatrix} + \frac{(1-\alpha-\beta)^k}{\alpha+\beta}\begin{bmatrix}\alpha & -\beta\\ -\alpha & \beta\end{bmatrix}. \tag{30}$$

Taking $q_0 = (1 - p_0)$ and Eq. (30) into consideration we obtain:

$$\begin{aligned} p_k &= \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^k \left(p_0 - \frac{\beta}{\alpha+\beta}\right) \\ q_k &= \frac{\alpha}{\alpha+\beta} + (1-\alpha-\beta)^k \left(q_0 - \frac{\alpha}{\alpha+\beta}\right) \end{aligned} \tag{31}$$

$$(k = 0, 1, 2, \ldots)$$

for a detailed final solution.

If $0 < \alpha < 1$, $0 < \beta < 1$, then for $k \to \infty$

$$p_k \to \frac{\beta}{\alpha+\beta}; \quad q_k \to \frac{\alpha}{\alpha+\beta}. \tag{32}$$

If $\alpha = 0$, $\beta = 0$, then the process remains constant:

$$p_{k1} = p_k = p_0;\ q_{k+1} = q_k = q_0;\ (k = 0, 1, 2, \ldots). \tag{33}$$

If $\alpha = 1$, $\beta = 1$, the process is alternating:

$$p_{k+2} = q_{k+1} = p_k;\ q_{k+2} = p_{k+1} = q_k;\ (k = 0, 1, 2, \ldots). \tag{34}$$

State Equations of Discrete Systems

With the aid of matrix notation the state equations of linear, time-invariant, lumped parameter systems can be expressed as

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{F}\mathbf{x}_k + \mathbf{G}\mathbf{u}_k \\ \mathbf{y}_{k+1} &= \mathbf{H}\mathbf{x}_k + \mathbf{K}\mathbf{u}_k \end{aligned} \tag{35}$$

where $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$ are matrices, $\mathbf{u}_k$ is the input vector, $\mathbf{x}_k$ the state vector and $\mathbf{y}_k$ the output vector. By matrix notation the form of higher order processes becomes similar to that of the first order processes, compare Eqs (17), (18) and (35).

If the system is time-variable but invariably linear then the only difference manifest itself in the fact that variable matrices $\mathbf{F}_k$, $\mathbf{G}_k$, $\mathbf{H}_k$, $\mathbf{K}_k$ must be employed instead of the constant matrices $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$.

The solution of the state difference equation (by reiterated substitutions) can be given in the general form

$$\mathbf{x}_k = \prod_{i=0}^{k-1} \mathbf{F}_i \mathbf{x}_0 + \sum_{j=0}^{k-2} \prod_{i=j+1}^{k-1} \mathbf{F}_i \mathbf{G}_j \mathbf{u}_j + \mathbf{G}_{k-1} \mathbf{u}_{k-1} . \tag{36}$$

For the time-invariant case the latter reduces to

$$\mathbf{x}_k = \mathbf{F}^k \mathbf{x}_0 + \sum_{j=0}^{k-1} \mathbf{F}^{k-j-1} \mathbf{G} \mathbf{u}_j \tag{37}$$

which has a close resemblance to the scalar case, compare Eqs (20) and (37).

If the system is nonlinear (and time-variable) then the state equation can be expressed in the form

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{f}_*(\mathbf{x}_k, \mathbf{u}_k, k) \, ; \\ \mathbf{y}_k &= \mathbf{g}_*(\mathbf{x}_k, \mathbf{u}_k, k) \, . \end{aligned} \tag{38}$$

We draw the attention to the fact that discrete state difference equations immediately furnish a recurrence relationship for the solution, thus, step-by-step we obtain

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{f}_*(\mathbf{x}_0, \mathbf{u}_0, 0) \, , \\ \mathbf{x}_2 &= \mathbf{f}_*(\mathbf{x}_1, \mathbf{u}_1, 1) \, , \\ \mathbf{x}_3 &= \mathbf{f}_*(\mathbf{x}_2, \mathbf{u}_2, 2) \end{aligned} \tag{39}$$

and so on. The latter circumstance puts into relief the great advantage of discrete systems over continuous systems. This is the reason why we attempt to converse continuous-time equations into discrete-time equivalents. As we have seen earlier, the conversion procedure is not so evident.

In many applications, however, we have to manipulate with continuous-time equation. So, for the sake of completeness we shall summarize here also the forms of state differential equations.

State Equations of Continuous Systems

There is some similarity between the state equations of discrete-time and continuous-time systems, but instead of difference equations differential equations play a part. We shall restrict ourselves to higher-order system because the vector equations are more general and scalar equations can easily be obtained by reduction.

The state differential equations of the linear, time-invariant, lumped-parameter, continuous-time system is expressed in the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{40}$$

whereas the auxiliary equation is

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \, . \tag{41}$$

Here $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t$, that is, the dot denotes a derivation.

If the linear system is time-variable then variable matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, $\mathbf{D}(t)$ replace the constant matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$. Nonlinear (and time-variable) systems may be represented by state equations

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \, ; \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \, . \end{aligned} \tag{42}$$

The solutions of the latter can generally be obtained only by numerical methods, that is, by the discretization of the problem.

The solution of time-variable linear systems can be expressed in the form

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\mathrm{d}\tau \, . \tag{43}$$

Here, $\mathbf{\Phi}(t, t_0)$ is called *fundamental matrix* or *state-transition matrix* and plays a similar role to that of $\mathbf{F}_0\mathbf{F}_1 \ldots \mathbf{F}_{k-1}$ in a discrete system, [compare Eq. (36) and (43)]. The determination of $\mathbf{\Phi}(t, t_0)$ is, however, principally and practically far more complicated than the computation of $\mathbf{F}_0\mathbf{F}_1 \ldots \mathbf{F}_{k-1}$.

If the scrutinized system is time-invariant as in Eq. (40), then, the state-transition matrix becomes a simple exponential matrix

$$\mathbf{\Phi}(t, 0) = e^{t\mathbf{A}} = e^{\mathbf{A}t} = \mathbf{\Phi}(t) \tag{44}$$

where for the sake of simplicity $t_0 = 0$ has been set. The determination of $\mathbf{\Phi}(t)$ is relatively easy. The solution of the state differential equation can be given in the form

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_{0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\mathrm{d}\tau \, . \tag{45}$$

Some Applications of State-Space Methods

Without aiming at completeness we shall mention here some applications of state-space methods.

The dynamic processes of electrical, mechanical, chemical, social, menagement, economic, physiological, biological or even austronautical systems are often described by state-space equations.

The stability theory of nonlinear systems is based on LJAPUNOV's indirect and direct investigation methods which both apply state-space techniques for continuous and discrete systems as well. The same can be said about perturbation methods.

The dynamic optimization methods, such as LAGRANGE—EULER methods in calculus of variations, BELLMAN's dynamic programming, PONTRYAGIN's maximum and minimum principles, functional analysis methods are also based on state-space principles.

Model building and parameter identification techniques are also making the best of state-space methods.

Analog, hybrid and digital simulation of continuous, discrete or mixed systems very often also apply state-space techniques.

One of the greatest application fields of state-space methods manifests itself in feedback control systems.

We only enumerate here these application possibilities and for details we refer to the special literature. This may be done so much the more as the author delivered at the JUREMA conference a lecture about these problems two years ago [17].

Conclusions

The wide-spread use of state-space methods undoubtedly contributed to the integration tendency of various branches of science, such as economics, biology, psychology, sociology, engineering, etc., and thus serve as a basis for a unified system theory. State-space methods enable us to describe relatively easily the dynamic processes of various systems, facilitating their study, that is, the analysis and synthesis of such systems. State-space methods promoted and will promote to the development of a unified system theory.

The state-space method is, however, no philosopher's stone and it does not give a solution for every problem. There are a lot of problems where it is not at all or only hardly applicable. Sometimes other methods, for example, frequency domain methods or transfer function methods, especially for linear systems, may prove equivalent or even more advantageous.

Summary

In recent system theory, especially in the analysis and synthesis of dynamic systems, state-space methods play ever more an important role. State-space methods often serve as a basis in model building, parameter identification, simulation, optimization. In this paper we introduce discrete-time state equations, and we show the continuous-time counterparts. Some possibilities of application are outlined as well.

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