CONVERSION METHODS FROM PHASE-VARIABLE TO CANONICAL FORMS

By

F. CSÁKI
Department of Automation, Technical University, Budapest

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With the advent and extensive use of state-variable formulations for dynamical systems, a great deal of interest has appeared for similitary transformations. Many papers have treated the problem of obtaining the phase-variable canonical form [1 . . . 10]. Nowadays, other transformation problems are in the focus of attention. One of the basic problems is perhaps the transformation from phase-variable form to canonical forms with explicit eigenvalues. TOU [11, 12] has shown that, when starting with the phase-variable form, transformation by means of the VANDERMONDE matrix results in the desired diagonal form in the system matrix for the case of distinct eigenvalues. Moreover, TOU has also given formulas for determining the inverse of the VANDERMONDE matrix. Many other papers were dealing withe the same problem [13 . . . 17] giving some improvements to the method. For the cases involving repeated eigenvalues, the modified (or confluent) VANDERMONDE matrix transforms the system matrix to a JORDAN canonical form. Now, the question arises how to determine the inverse of the confluent VANDERMONDE matrix.

In many previous papers, *e.g.* in [18, 19, 20], it was shown that a complete analysis of the dynamical system involved required the determination of the inverse modal matrices. Many suggestions were made for the computation of the confluent inverse VANDERMONDE matrix [18, 26], the VANDERMONDE matrix being a modal matrix for the companion matrix, that is, for the system matrix in phase-variable form. Although inversion can be accomplished by usual methods, a direct evaluation of the inverse matrices in terms of system parameters is desirable. The inverse matrices could then be evaluated from certain general forms without resorting to inversion methods. At the same time an insight into the composition of matrix elements would be possible. In some cases one or another method may also have considerable practical adventage.

The purpose of this paper is to present a comparison between the suggested methods. At the same time we will endeavour to point out the connection links between the various methods.

1. Problem formulation

Let us start from the well-known phase-variable form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdot & \cdot & \cdot & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u \quad (1)$$

$$y = [\, b_0, \quad b_1, \quad b_2, \quad \cdot \quad \cdot \quad \cdot, \quad b_{n-2}, \quad b_{n-1} \,] \; \mathbf{x} .$$

The corresponding signal-flow graph is depicted in Fig. 1. Eq. (1) put in short-hand notation reads

$$\dot{\mathbf{x}} = \mathbf{A}_0 \mathbf{x} + \mathbf{b}_0 u$$
$$y = \mathbf{c}^{\mathrm{T}} \mathbf{x} . \qquad (2)$$

We omitted from the last row of Eq. (1) or (2) the term $d_0 u$ for the sake of simplicity. It should be mentioned that the latter term means a parallel branch from node u to node v in Fig. 1 not influencing otherwise the main part of the signal-flow graphs or the corresponding equations.

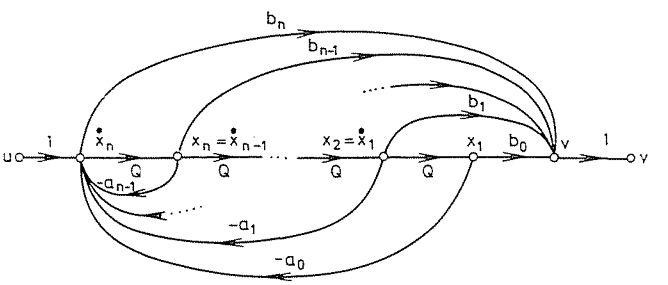

Fig. 1. State diagram of Eq. (1.) Q means here an integrator

It is to be emphasized that many other phase-variable forms exist. The form given in Eq. (1) will be called the *principle variant*, whereas any other *particular variant* may be obtained by an appropriate transformation $\mathbf{x} = \mathbf{K}\,\mathbf{x}'$ or by renumbering the phase variables.

By introducing an appropriate linear transformation

$$\mathbf{x} = \mathbf{L}\,\mathbf{z}, \quad \mathbf{z} = \mathbf{L}^{-1}\mathbf{x} \qquad (3)$$

from Eq. (2) we may obtain the JORDAN canonical form:

$$\dot{\mathbf{z}} = \mathbf{J}\,\mathbf{z} + \mathbf{b}\,u \qquad (4)$$
$$y = \mathbf{c}^{\mathrm{T}}\mathbf{z}\,.$$

Here, the JORDAN matrix is given in pseudodiagonal form: as a hypermatrix

$$\mathbf{J} = \mathbf{L}^{-1}\mathbf{A}_0\mathbf{L} = \mathbf{diag}\,[\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_m] \qquad (5)$$

where $k_1 + k_2 + \ldots k_m = n$. Furthermore,

$$\mathbf{b} = \mathbf{L}^{-1}\mathbf{b}_0, \quad \mathbf{c}^{\mathrm{T}} = c_0^{\mathrm{T}}\mathbf{L}\,. \qquad (6)$$

In Eq. (5) the $k_i \times k_i$ matrix

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & & & \\ & \lambda_i & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \lambda_i & 1 \\ & & & & & \lambda_i \end{bmatrix} \quad (i = 1,2, \ldots, m) \qquad (7)$$

is a so-called JORDAN block containing the eigenvalue λ_i with multiplicity k_i.

If some of the eigenvalues is distinct ($k_i = 1\ \exists i$) then the correponding JORDAN block reduces to a scalar quantity which is the eigenvalue itself, $\mathbf{J}_i = \lambda_i, \exists i$.

For short, transformation (3) with result (5) will be called *modal transformation*, and $\mathbf{L}$ is then called a *modal matrix* $\mathbf{M}$, $\mathbf{L} = \mathbf{M}$.

Attention is focused on the problem how to obtain $\mathbf{L}$ and furthermore $\mathbf{L}^{-1}$ in order to get Eq. (5).

As it is well known [27, 37] the original VANDERMONDE matrix is a modal matrix yielding a diagonal form, furthermore an appropriate modal transformation matrix for satisfying Eq. (5) is the confluent VANDERMONDE matrix:

$$\mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_m] \qquad (8)$$

where

$$\mathbf{V}_i = \begin{bmatrix} 1 & 0 & 0 & \ldots \\ \lambda_i & 1 & 0 & \\ & & & \ldots \\ \lambda_i^2 & 2\lambda_i & 1 & \ldots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \lambda_i^{n-1} & (n-1)\lambda_i^{n-2} & \frac{1}{2\,!}(n-1)(n-2)\lambda^{n-3} & \ldots \end{bmatrix} \qquad (9a)$$

$$\mathbf{V}_i = [\mathbf{v}_i, \mathbf{v}_i', \frac{1}{2\,!}\,\mathbf{v}_i'', \ldots] \qquad (9b)$$

is an $n \times k_i$ VANDERMONDE block with corresponding $n \times 1$ vectors $\mathbf{v}_i$ and its derivatives.

Relatively few is spoken about $\mathbf{b}$, that is, how to determine $\mathbf{L}$ and $\mathbf{L}^{-1}$ in order to obtain $\mathbf{b}$ in a certain special form at the same time to satisfy Eq. (5). The latter problem was treated in [18] in some extent. It can be shown that $\mathbf{b}$ can get the very simple form:

$$\mathbf{b} = \begin{bmatrix} \mathbf{j}_1 \\ \mathbf{j}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{j}_{m-1} \\ \mathbf{j}_m \end{bmatrix} = \mathbf{j}; \quad \mathbf{j}_i = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \qquad (i = 1, 2, \ldots, m) \tag{10}$$

where $\mathbf{j}_i$ is a $k_i \times 1$ matrix, alternatively $\mathbf{b}$ may be expressed in the somewhat more complicated form:

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ 1 \end{bmatrix} = \mathbf{1} \tag{11}$$

where $\mathbf{b}$ is a $n \times 1$ matrix.

For Eqs (10), (5) and (7), we obtain the *principal variant* of JORDAN form. Any other case will be called a *particular case.*

It should, however, be emphasized that VANDERMONDE matrices $\mathbf{V}$, or other modal matrices $\mathbf{M}$, do not fulfil automatically neither Eq. (10) nor Eq. (11):

$$\mathbf{V}^{-1}\mathbf{b}_0 \neq \mathbf{b}, \ \mathbf{M}^{-1}\mathbf{b}_0 \neq \mathbf{b}\,.$$

For this purpose special $\mathbf{T}$ commutativity matrices must be introduced such that in general

$$\mathbf{L} = \mathbf{MT}, \ \ \mathbf{L}^{-1} = \mathbf{T}^{-1}\mathbf{M}^{-1} \tag{12}$$

or specifically

$$\mathbf{L} = \mathbf{VT}, \ \ \mathbf{L}^{-1} = \mathbf{T}^{-1}\mathbf{V}^{-1}$$

with VANDERMONDE matrix $\mathbf{V}$ as given in Eq. (8). In such cases

$$\mathbf{T}^{-1}\mathbf{M}^{-1}\mathbf{b}_0 = \mathbf{b} \ \text{ or } \ \mathbf{T}^{-1}\mathbf{V}^{-1}\mathbf{b}_0 = \mathbf{b} \tag{13}$$

where $\mathbf{b}$ is given in Eq. (10) or Eq. (11), see [18].

After this introductory remark we shall concentrate our attention on the determination of the JORDAN canonical form and especially on the computation of $\mathbf{V}^{-1}$.

2. First Method

Let us apply first LAPLACE transformation technique on Eq. (2) neglecting initial conditions.

The suggested method is one of the most common ones. In textbooks, see for example [29, 30, 32, 36], this method was applied for a single repeated eigenvalue with many distinct eigenvalues. Here we shall analyse the case of many repeated eigenvalues, that is, in the most general form when a distinct eigenvalue constitutes the special case of a repeated one. With the transformation technique in Eq. (2):

$$\begin{aligned} s\mathbf{X}(s) &= \mathbf{A}_0\,\mathbf{X}(s) + \mathbf{b}_0\,U(s) \\ Y(s) &= \mathbf{c}_0^{\mathrm{T}}\,\mathbf{X}(s)\,. \end{aligned} \tag{14}$$

After some rearrangement the transfer function of the dynamical system may be obtained in the form:

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{c}_0^{\mathrm{T}}\,[s\mathbf{I} - \mathbf{A}_0]^{-1}\,\mathbf{b}_0. \tag{15}$$

The latter may be expressed as the ratio of two polynomials,

$$G(s) = \frac{N(s)}{D(s)} \tag{16}$$

where the order of $D(s)$ is n, and the order of $N(s)$ is at most $n-1$. Without loss of generality the so-called characteristic polynomial, that is, the denominator polynomial $D(s)$ can be expressed as

$$D(s) = (s-\lambda_1)^{k_1}(s-\lambda_2)^{k_2}\ldots(s-\lambda_m)^{k_m} \tag{17}$$

where the zeros λ_i of the latter are the repeated eigenvalues of multiplicity k_i in the system matrix $\mathbf{A}_0$. The partial fraction form of $G(s)$ in Eq. (16) can be written as

$$\begin{aligned} G(s) = {} & \frac{C_1^{(1)}}{(s-\lambda_1)^{k_1}} + \frac{C_2^{(1)}}{(s-\lambda_1)^{k_1-1}} + \ldots + \frac{C_{k_1}^{(1)}}{(s-\lambda_1)} + \ldots + \\ & + \frac{C_1^{(m)}}{(s-\lambda_m)^{k_m}} + \frac{C_2^{(m)}}{(s-\lambda_m)^{k_m-1}} + \ldots + \frac{C_{k_m}^{(m)}}{(s-\lambda_m)} \end{aligned} \tag{18}$$

or

$$G(s) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \frac{C_j^{(i)}}{(s - \lambda_i)^{k_i - j + 1}} \tag{19}$$

where the coefficients can be obtained from

$$C_j^{(i)} = \lim_{s \to \lambda_i} \frac{1}{(j-1)!} \frac{d^{(j-1)}}{ds^{(j-1)}} \left[(s - \lambda_i)^{k_i} \frac{N(s)}{D(s)} \right] \tag{20}$$

which, for a distinct eigenvalue ($k_i = 1$) reduces to

$$C_1^{(i)} = \lim_{s \to \lambda_i} (s - \lambda_i) \frac{N(s)}{D(s)} \, . \tag{21}$$

According to Eq. (15) the output is

$$Y(s) = G(s) U(s) \tag{22}$$

which, with (19) in mind, may be given as

$$Y(s) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \frac{C_j^{(i)} U(s)}{(s - \lambda_i)^{k_i - j + 1}} = \sum_{i=1}^{m} \sum_{j=1}^{k} C_j^{(i)} Z_j^{(i)}(s) \tag{23}$$

where

$$Z_j^{(i)}(s) = \frac{U(s)}{(s - \lambda_i)^{k_i - j + 1}} \qquad \begin{array}{l} (i = 1, 2, \ldots, m) \\ (j = 1, 2, \ldots, k_i - 1) . \end{array} \tag{24}$$

From the latter expressions the following relationships are easily obtained:

$$Z_j^{(i)}(s) = \frac{1}{(s - \lambda_i)} Z_{j+1}^{(i)}(s) \qquad \begin{array}{l} (j = 1, 2 \ldots, k_i - 1) \\ (i = 1, 2 \ldots, m) . \end{array} \tag{25}$$

Applying inverse Laplace transforms to Eqs (24) and (25) we have the state equations

$$\begin{aligned} \dot{z}_1^{(i)} &= \lambda_1 z_1^{(i)} + z_2^{(i)} \\ \dot{z}_2^{(i)} &= \lambda_1 z_2^{(i)} + z_3^{(i)} \qquad\qquad (i = 1, 2, \ldots, m) \\ &\;\;\vdots \\ \dot{z}_{k_i}^{(i)} &= \lambda_i z_{k_i} + u \end{aligned} \tag{26}$$

whereas Eq. (23) yields

$$y = \sum_{i=1}^{m} \sum_{j=1}^{k_i} C_j^{(i)} z_j^{(i)} . \tag{27}$$

Eqs (26) and (27) are exactly in the form as prescribed by Eqs (5), (7) and (10) whereas vector $\mathbf{c}^{\mathrm{T}}$ can be expressed by Eq. (27) as

$$\mathbf{c}^{\mathrm{T}} = [\mathbf{c}_1^{\mathrm{T}}, \mathbf{c}_2^{\mathrm{T}}, \ldots, \mathbf{c}_m^{\mathrm{T}}] \tag{28}$$

with

$$\mathbf{c}^{\mathrm{T}} = [C_1^{(i)}, C_2^{(i)}, \ldots, C_{ki}^{(i)}] \tag{29}$$

As a consequence of Eqs (26) and (27) the state diagram in form of a signal-flow diagram can be plotted as shown in Fig. 2.

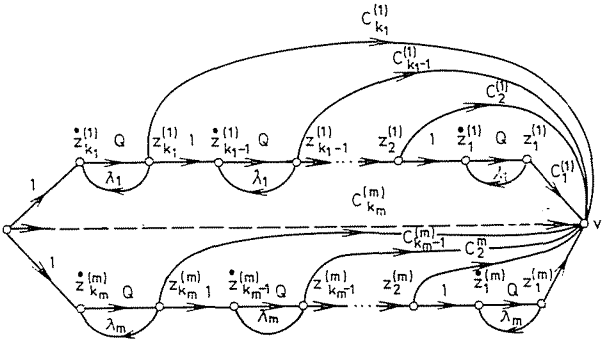

Fig. 2. State diagram of Eqs (26) and (27). Q means here an integrator

By the way, let us remark that if Eq. (11), instead of Eq. (10) should be satisfied, then the state variables are to be defined in a somewhat different form:

$$Z_j^{(i)}(s) = \sum_{l=1}^{k_i-j+1} \frac{U(s)}{(s-\lambda_i)^l} \tag{24*}$$

whereas Eq. (29) is replaced by

$$C_i^{\mathrm{T}} = [C_1^{(i)}, C_2^{(i)} - C_1^{(i)}, \ldots, C_{ki}^{(i)} - C_{ki-1}^{(i)}]\,. \tag{29*}$$

It should be emphasized that according to Eq. (15) the determination of transfer function $G(s)$ seems to involve a matrix inversion. The latter can, however, easily be avoided in practice. Taking the LAPLACE transform of each row separately in Eq. (1), and neglecting initial conditions, we obtain from the first $(n-1)$ rows

$$X_j(s) = s^{j-1}X_1(s) \qquad (j = 2, 3, \ldots, n) \tag{30}$$

and form the n-th row

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \tag{31}$$

whereas the $(n+1)$th row yields

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_{n-1}s^{n-1}+\ldots+b_1 s+b_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}. \tag{32}$$

3. First Conclusion

The suggested method for obtaining the JORDAN canonical form seems to be the easiest and most straightforward one. Neither matrix inversions nor complicated matrix manipulation are necessary. The number of eigenvalues or their multiplicity is not constrained. It supposes, however, the knowledge of LAPLACE transformation techniques.

The suggested method is not quite new; for a single repeated eigenvalue and many distinct eigenvalues it was given for example in [29, 30, 32, 36], for many multiple eigenvalues the method is, however, formulated perhaps here for the first time.

The JORDAN canonical form was seen to be directly obtained and the determination of the inverse VANDERMONDE matrix or any other modal matrices to be completely avoided. The latter is an advantage from the point of view of the practice but spoils our pleasure in determining JORDAN forms through the VANDERMONDE matrix and its inverse.

4. Second Method

Recently a simple method was suggested for the determination of the inverse VANDERMONDE matrix which applied also LAPLACE transforms. BECK and LANCE [19] based their method on forced systems assuming a special input matrix $\mathbf{B}$ and input vectors u, the former had to be an identity matrix while the latter special vectors with all zero components but one. This is, however, not a necessity.

In the opinion of the author of this paper the method is clearer and more natural by starting with unforced systems subject to initial conditions, as modal and inverse modal matrices are originally defined for system matrices $\mathbf{A}$ and have nothing to do with input matrices $\mathbf{B}$.

Taking the LAPLACE transform of the unforced system $(u=0)$ in Eq. (1) subject to non-zero initial conditions we obtain

$$s\mathbf{X}(s)=\mathbf{A}_0\,\mathbf{X}(s)+\mathbf{x}(0) \tag{33}$$

and, applying the modal transformation given in Eq. (3) with $\mathbf{L}=\mathbf{V}$ we may write

$$s\mathbf{Z}(s)=\mathbf{J}\;\mathbf{Z}(s)+\mathbf{V}^{-1}\,\mathbf{x}(0) \tag{34}$$

where $\mathbf{J} = \mathbf{V}^{-1}\,\mathbf{A}\,\mathbf{V}$. After rearrangement

$$\mathbf{Z}(s) = [s\mathbf{I} - \mathbf{J}]^{-1}\,\mathbf{V}^{-1}\mathbf{x}(0) \tag{35}$$

and

$$\mathbf{X}(s) = \mathbf{V}\,\mathbf{Z}(s) = \mathbf{V}[s\mathbf{I} - \mathbf{J}]^{-1}\mathbf{V}^{-1}\,\mathbf{x}(0). \tag{36}$$

Let us see one of the JORDAN blocks in Eq. (34)

$$s\,\mathbf{Z}^{(i)}\,(s) = \mathbf{J}_i\,\mathbf{Z}^{(i)}\,(s) + \mathbf{W}^{(i)}\,\mathbf{x}(0) \tag{37}$$

where $\mathbf{W}^{(i)}$ is a corresponding $k_i \times n$ submatrix of $\mathbf{V}^{-1}$. Eq. (37) written out in detail reads

$$\begin{aligned} sZ_1^{(i)}\,(s) &= \lambda_i\,Z_1^{(i)}\,(s) + Z_2^{(i)}\,(s) + \mathbf{w}_1^{\mathbf{T}(i)}\,\mathbf{x}(0) \\ sZ_2^{(i)}\,(s) &= \lambda_i\,Z_2^{(i)}\,(s) + Z_3^{(i)}\,(s) + \mathbf{w}_2^{\mathbf{T}(i)}\,\mathbf{x}(0) \\ &\;\;\vdots \\ sZ_{k_i}^{(i)}\,(s) &= \lambda_1 Z_{k_i}^{(i)}\,(s) + \mathbf{w}_{k_i}^{\mathbf{T}(i)}\,\mathbf{x}(0) \end{aligned} \tag{38}$$

where $\mathbf{w}_j^{\mathbf{T}(i)}$ is the j-th row vector of $\mathbf{W}^{(i)}$.

Successive substitutions and some algebraic manipulations yield

$$Z_1^{(i)}(s) = \sum_{j=1}^{k_i} \frac{\mathbf{w}_j^{\mathbf{T}(i)}\,\mathbf{x}(0)}{(s - \lambda_i)^j}\,. \tag{39}$$

For a distinct eigenvalue this would reduce to

$$Z_1^{(i)}(s) = \frac{\mathbf{w}_1^{\mathbf{T}(i)}\,\mathbf{x}(0)}{(s - \lambda_i)}\,. \tag{40}$$

Now, Eqs (8), (9) and (36) show that

$$X_1(s) = \sum_{i=1}^{m} Z_1^{(i)}(s) \tag{41}$$

or by Eq. (39)

$$X_1(s) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \frac{\mathbf{w}_j^{\mathbf{T}(i)}\,\mathbf{x}(0)}{(s - \lambda_i)^j}\,. \tag{42}$$

By the way, the latter equation has a close resemblance to Eq. (23) when taking also Eq. (24) into consideration.

Eq. (42) written in detailed form yields

$$X_1(s) = \sum_{i=1}^{m} \sum_{i=1}^{k_i} \frac{w_{j1}^{(i)}\,x_1(0) + \ldots + w_{jn}^{(i)}\,x_n(0)}{(s - \lambda_i)^j} \tag{43}$$

where $w_{j_1}^{(i)}, \ldots, w_{jn}^{(i)}$ are the components of row vector $\mathbf{w}_j^{\mathrm{T}(i)}$ or in other words the corresponding elements in the $(k_1 + \ldots k_{i-1} + j)$-th row of the inverse modal matrix $\mathbf{V}^{-1} = \mathbf{W}$. Our task is just to obtain elements $w_{jl}^{(i)}$.

In Eq. (42), $X_1(s)$ is the first component of $\mathbf{X}(s)$, thus, it could be obtained from Eq. (36) or more directly from

$$\mathbf{X}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\, \mathbf{x}(0). \tag{44}$$

This expression looks somewhat complicated involving matrix inversion. The latter, however, can completely be avoided introducing successive substitutions similar to Eqs (30), (31), (32). Instead of Eq. (30) we have now

$$X_j(s) = s^{j-1} X_1(s) = \sum_{k=1}^{j-1} s^{j-k-1}\, x_k(0) \qquad (j = 2, 3, \ldots, n) \tag{45}$$

and from the n-th row of the Laplace transform of Eq. (1), assuming $u = 0$, we obtain

$$X_1(s) = \frac{\sum\limits_{k=1}^{n} \sum\limits_{h=k}^{n} a_h\, s^{h-k}\, x_k(0)}{\sum\limits_{h=0}^{n} a_h\, s^h} = \frac{\sum\limits_{k=1}^{n} \sum\limits_{h=k}^{n} a_h\, s^{h-k}\, x_k(0)}{s^n + a_{n-1}\, s^{n-1} + \ldots + a_1\, s + a_0} \tag{46}$$

where $a_n = 1$. The relationship in Eq. (46) can also be expressed as

$$\frac{\sum\limits_{k=1}^{n} \sum\limits_{h=k}^{n} a_h\, s^{h-k}\, x_k(0)}{(s - \lambda_1)^{k_1}\ (s - \lambda_2)^{k_2} \ldots (s - \lambda)^{k_i}}, \quad (a_n = 1). \tag{47}$$

It should be emphasized that the latter expression can directly be described by inspection. Comparing Eqs (47) and (43) it is easily seen that the elements of the inverse Vandermonde matrix $\mathbf{V}^{-1} = \mathbf{W}$ can be calculated by direct conventional methods based on partial fraction expansion technique. Setting for example

$$x_k(0) \begin{cases} \neq 0, & \text{for} \quad k = l \\ = 0, & \text{for} \quad k \neq l \end{cases} \tag{48}$$

for Eqs (47) and (43)

$$\frac{X_1(s)}{x_l(0)} = \frac{s^{n-l} + a_{n-1}\, s^{n-l-1} + \ldots + a_{l+1}\, s + a_l}{s^n + a_{n-1}\, s^{n-1} + \ldots + a_1\, s + a_0} = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \frac{w_{jl}^{(i)}}{(s - \lambda_i)^j} \tag{49}$$

where

$$w_{jl}^{(i)} = \lim_{s \to \lambda_i} \frac{1}{(k_i - j)!} \frac{d^{(k_i-j)}}{ds^{(k_i-j)}} (s - \lambda_i)^{k_i} \frac{N_l(s)}{D(s)}$$
$$(j = 1, 2, \ldots, k_i) \tag{50}$$

with special polynomials

$$N_l(s) = s^{n-l} + a_{n-1} s^{n-l-1} + \ldots + a_{l+1} s + a_l \quad (l = 1, 2, \ldots, n). \tag{51}$$

According to Horner's scheme $N_l(s)$ can also be expressed as

$$N_l(s) = [((s + a_{n-1})s + a_{n-2}) s + \ldots + a_{l-1}]s + a_l .$$

It is easily seen that $N_l(s)$ is at the same time a truncated polynomial of the denumerator polynomial $D(s)$ divided by s^l. Furthermore, according to Eq. (49) the elements $w_{jl}^{(i)}$ can be obtained by partial fraction expansion, $w_{jl}^{(i)}$ being the coefficient belonging to $(s - \lambda_i)^{-j}$.
The l-th column of the inverse Vandermonde matrix $\mathbf{W}$ is, then,

$$\nu_l = \begin{bmatrix} \nu_l^{(1)} \\ \nu_l^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ \nu_l^{(m)} \end{bmatrix} \qquad \text{with } \nu_l^{(i)} = \begin{bmatrix} w_{1l}^{(i)} \\ w_{2l}^{(i)} \\ \cdot \\ \cdot \\ \cdot \\ w_{k_i l}^{(i)} \end{bmatrix} \tag{52}$$

that is, an arbitrary element in the l-th column and r-th row can be expressed as

$$w_{rl} = w_{(k_1+\ldots+k_{i-1}+j),l} = w_{jl}^{(i)} \qquad \begin{matrix} (i = 1, 2, \ldots, m) \\ (j = 1, 2, \ldots, k_i) \\ (l = 1, 2, \ldots, n) \end{matrix} \tag{53}$$

Finally, we mention that the inverse Laplace transform of Eq. (49) yields

$$x_1(t) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} w_{jl}^{(i)} \frac{t^{j-1}}{(j-1)!} e^{\lambda_i t} x_l(0). \tag{54}$$

It is readily seen that from a physical point of view $w_{jl}^{(i)}$ is the coefficient of the i-th modal transient process subject to the initial condition $x_l(0)$ and belonging to the j-th multiplicity.

5. Second Conclusion

The second method allows evaluation of the elements of inverse VANDERMONDE matrix $\mathbf{W}$ from explicit analytic forms. This has a practical value, but the insight to be gained from the identification of matrix elements $w_{jl}^{(i)}$ with system behaviour is of even greater interest. The elements in the l-th column are exclusively due to the initial condition of the l-th state variable, $x_l(0)$. Furthermore $w_{jl}^{(i)}$ is the coefficient of the i-th mode connected with eigenvalue λ_i, yielding the transient response of the first state variable due to multiplicity j, see Eqs (49) and (54). Of course, this interpretation holds true to the elements w_{rl} of the inverse VANDERMORE matrix regardless of how they were calculated.

The second method has serious disadvantages compared to the first method. Here, partial fraction expansion is separately required for every column of the inverse VANDERMONDE matrix, whereas in the first method one partial fraction expansion is enough. Thus, the second method is at least n-times more laborious than the first one. Even the computation work is increased because after having the inverse VANDERMONDE matrix $\mathbf{V}^{-1}$, further matrix manipulations are to be made for obtaining the canonical form, Eqs (5) and (6) including the determination of an appropriate matrix $\mathbf{T}$ and its inverse $\mathbf{T}^{-1}$ to obtain $\mathbf{b}$ in form of Eq. (10), whereas the first method eschews all these problems by giving directly the final solution.

6. A Supplementary Remark

The elements in the last column of the inverse VANDERMONDE matrix play a very specific and important role. These elements can be computed for distinct eigenvalues by

$$w_{in} = \operatorname*{Res}_{s=\lambda_i} \frac{1}{D(s)} = \frac{s-\lambda_i}{D(s)}\bigg|_{s=\lambda_i} = \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\lambda_i - \lambda_j} \tag{55}$$

whereas for multiple eigenvalues by

$$w_{jn}^{(i)} = \lim_{s\to\lambda_i} \frac{1}{(k_i-j)!} \frac{d^{(k_i-j)}}{ds^{(k_i-j)}} \left[(s-\lambda_i)^{k_i} \frac{1}{D(s)} \right] \tag{56}$$

which means for $j = k_i$:

$$w_{k_i n}^{(i)} = \operatorname*{Res}_{s=\lambda_i} \frac{1}{D(s)} = \frac{(s-\lambda_i)^{k_i}}{D(s)}\bigg|_{s=\lambda_i} = \prod_{\substack{j=1\\ j\neq i}}^{m} \frac{1}{(\lambda_i-\lambda_j)^{k_j}} \,. \tag{57}$$

By the way, introducing $D_i(s)$ for $D(s)/(s-\lambda_i)^{k_i}$ Eqs (57) and (58) can also be expressed as

$$w_{jn}^{(i)} = \lim_{s\to\lambda_i} \frac{1}{(k_i-j)!} \frac{d^{(k_i-j)}}{ds^{(k_i-j)}} \frac{1}{D_i(s)} \tag{57*}$$

and

$$w_{k_i n} = \frac{1}{D_i(\lambda_i)} . \tag{58*}$$

Choosing $x_n(0)\,\delta(t) = u(t)$, that is, $X_n(0) = U(s)$, Eq. (49) yields

$$X_1(s) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \frac{w_{jn}^{(i)}}{(s-\lambda_i)^j} U(s); \tag{58}$$

furthermore from Eq. (1), with zero initial conditions we have as in Eq. (30):

$$X_k(s) = s^{k-1} X_1(s), \qquad (k = 2, 3, \ldots, n) . \tag{59}$$

Thus, the LAPLACE transform of the output is given by

$$Y(s) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \frac{w_{jn}^{(i)}}{(s-\lambda_i)^j} [b_0 + b_1 s + \ldots + b_{n-1} s^{n-1}] U(s) . \tag{60}$$

From the latter the transfer function is

$$G(s) = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \frac{w_{jn}^{(i)}}{(s-\lambda_i)^j} [b_{n-1} s^{n-1} + \ldots + b_1 s + b_0] . \tag{61}$$

A comparison with Eq. (19) or applying Eq. (20) yields the connection between the corresponding coefficients in the form of

$$C_j^{(i)} = \lim_{s\to\lambda_i} \frac{1}{(j-1)!} \frac{d^{(j-1)}}{ds^{(j-1)}} \left[(s-\lambda_i)^{k_i} \sum_{j'=1}^{k_i} \frac{w_{j'n}^{(i)}}{(s-\lambda_i)^{j'}} \sum_{k=0}^{n-1} b_k s^k \right] . \tag{62}$$

This can also be expressed as

$$C_j^{(i)} = \lim_{s\to\lambda_i} \frac{1}{(j-1)!} \frac{d^{(j-1)}}{ds^{(j-1)}} \sum_{j'=1}^{k_i} w_{j'n}^{(i)} (s-\lambda_i)^{k_i-j'} N(s) . \tag{62*}$$

If there is no numerator dynamics then the last sum reduces to b_0. For distinct eigenvalues Eq. (62) reduces to

$$C_1^{(i)} = w_{1n}^{(i)} \sum_{k=0}^{n-1} b_k \lambda_i^k = w_{1n} \sum_{k=0}^{n-1} b_k \lambda_i^k \tag{63}$$

and for no numerator dynamics to

$$C_1^{(i)} = w_{in}^{(i)}\, b_0 = w_{in}\, b_0. \tag{64}$$

Eqs (62), (63) and (64) yield in closed form the coefficients $C_j^{(i)}$ or $C_1^{(i)}$ in terms of elements $w_{jn}^{(i)}$ and coefficients b_k.

7. Determination of Commutativity Matrices

We have seen in Eq (12) that a special commutativity matrix $\mathbf{T}$ and its inverse $\mathbf{T}^{-1}$ must be determined in order to satisfy Eq. (13) together with e.g. Eq. (10). Now, let us concentrate to the determination of matrices mentioned.

The commutativity character is expressed by

$$\mathbf{T}^{-1}\mathbf{J}\,\mathbf{T} = \mathbf{J} \tag{65}$$

or

$$\mathbf{J}\,\mathbf{T} = \mathbf{T}\,\mathbf{J}, \qquad \mathbf{T}^{-1}\mathbf{J} = \mathbf{J}\,\mathbf{T}^{-1} \tag{66}$$

which shows that $\mathbf{T}$ and $\mathbf{J}$, as well as $\mathbf{T}^{-1}$ and $\mathbf{J}$ must commute. As the JORDAN matrix $\mathbf{J}$ is given in pseudodiagonal form, see Eq. (5), according to Eq. (66) so are $\mathbf{T}$ and $\mathbf{T}^{-1}$:

$$\mathbf{T} = \mathbf{diag}\ [\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}]; \quad \mathbf{T}^{-1} = \mathbf{diag}\ [\mathbf{T}_1^{-1}, \mathbf{T}_2^{-1}, \ldots, \mathbf{T}_m^{-1}]$$

moreover the inverse of $\mathbf{T}_i$ is even $\mathbf{T}_i^{-1}$, where $\mathbf{T}_i$ and $\mathbf{T}_i^{-1}$ are $k_i \times k_i$ matrices. Thus, Eqs (56) and (66) are also valid for each JORDAN block

$$\mathbf{T}_i^{-1}\mathbf{J}_i\,\mathbf{T}_i = \mathbf{J}_i \tag{67}$$

and

$$\mathbf{J}_i\mathbf{T} = \mathbf{T}_i\mathbf{J}_i, \qquad \mathbf{T}_i^{-1}\mathbf{J}_i = \mathbf{J}_i\mathbf{T}_i^{-1}. \tag{68}$$

Evaluating the latter relationships, for the elements $t_{kl}^{(i)}$ of $\mathbf{T}_i$ and elements $\tau_{kl}^{(i)}$ of $\mathbf{T}_i^{-1}$ it becomes obvious that both $\mathbf{T}_i$ and $\mathbf{T}_i^{-1}$ are upper triangular matrices. Moreover

$$\begin{aligned}
&t_{11}^{(i)} = t_{22}^{(i)} = \ldots = t_{k_i-1,k_i-1}^{(i)} = t_{k_i,k_i}^{(i)} \triangleq t_1^{(i)} \\
&t_{12}^{(i)} = t_{23}^{(i)} = \qquad = t_{k_i-1,k_i}^{(i)} \triangleq t_2^{(i)} \\
&\ldots\ldots \\
&t_{1,k_i-1}^{(i)} = t_{2,k_i}^{(i)} \triangleq t_{k_i-1}^{(i)} \\
&t_{1,k_i}^{(i)} \triangleq t_{k_i}^{(i)}
\end{aligned} \tag{69}$$

and similarly

$$
\begin{aligned}
&\tau_{11}^{(i)} = \tau_{22}^{(i)} = \ldots = \tau_{k_i-1,k_i-1}^{(i)} = \tau_{k_i,k_i}^{(i)} \triangleq \tau_1^{(i)} \\
&\tau_{12}^{(i)} = \tau_{23}^{(i)} = \ldots = \tau_{k_i-1,k_i}^{(i)} \triangleq \tau_2^{(i)} \\
&\ldots \\
&\tau_{1,k_i-1}^{(i)} = \tau_{k_i}^{(i)} \triangleq \tau_{k_i-1}^{(i)} \\
&\tau_{1,k_i}^{(i)} \triangleq \tau_{k_i}^{(i)}
\end{aligned}
\tag{70}
$$

where $\triangleq$ means "by definition" and serves to introduce new quantities. Hence, the elements in each diagonal are equal. Furthermore, according to Eqs (2), (13) and (52)

$$
\mathbf{T}^{-1}(\mathbf{M}^{-1}\mathbf{b}_0) = \mathbf{T}^{-1}\,\nu_n = \mathbf{b} \tag{71}
$$

or for the i-th JORDAN block by Eqs (10) and (52)

$$
\mathbf{T}_i^{-1}\,\nu_n^{(i)} = \mathbf{j}_i. \tag{72}
$$

Eq. (72) written out in detailed form yields the simultaneous set of equations:

$$
\begin{aligned}
\tau_1^{(i)}\,w_{k_i,n}^{(i)} &= 1 \\
\tau_1^{(i)}\,w_{k_i-1,n}^{(i)} + \tau_2^{(i)}\,w_{k_i,n}^{(i)} &= 0 \\
\tau_1^{(i)}\,w_{k_i-2,n}^{(i)} + \tau_2^{(i)}\,w_{k_i-1,n}^{(i)} + \tau_3^{(i)}\,w_{k_i,n}^{(i)} &= 0
\end{aligned}
\tag{73}
$$

and so on. By successive substitutions Eqs (73) can easily be solved, yielding the elements $\tau_j^{(i)}$ in terms of elements $w_{j,n}^{(i)}$. Instead of this we put first Eq. (72) into the form

$$
\nu_n^{(i)} = \mathbf{T}_i\,\mathbf{j}_i \tag{74}
$$

which means in detailed form

$$
\begin{aligned}
w_{1n}^{(i)} &= t_{k_i} \\
w_{2n}^{(i)} &= t_{k_i-1} \\
&\;\vdots \\
w_{k_i-1,n} &= t_2 \\
w_{k_i,n} &= t_1\,.
\end{aligned}
$$

Hence, the elements of the commutativity matrix $\mathbf{T}$ are uniquely expressed by the elements of the last column of $\mathbf{W}$. Now, matrix inversion would yield elements $\tau_j^{(i)}$ of the inverse commutativity matrix $\mathbf{T}^{-1}$. Instead, we remark that on the basis of $\mathbf{TT}^{-1} = \mathbf{I}$ we can write

$$\begin{aligned} &t_1\tau_1 = 1 \\ &t_1\tau_2 + t_2\tau_1 = 0 \\ &t_1\tau_3 + t_2\tau_2 + t_3\tau_1 = 0 \\ &\cdots\cdots \end{aligned} \tag{76}$$

and so on. Solving explicitly this simultaneous set of equations by successive substitutions the first six results are summarized in Table 1.

Table I

Element $\tau_j^{(i)}$ of $\mathbf{T}_i^{-1}$ in terms of elements $t_j^{(i)}$ of $\mathbf{T}_i$

$$\tau_1^{(i)} = \frac{1}{t_1^{(i)}},$$

$$\tau_2^{(i)} = -\frac{1}{t_1^{(i)}}\frac{t_2^{(i)}}{t_1^{(i)}},$$

$$\tau_3^{(i)} = \frac{1}{t_1^{(i)}}\left[\left(\frac{t_2^{(i)}}{t_1^{(i)}}\right)^2 - \frac{t_3^{(i)}}{t_1^{(i)}}\right],$$

$$\tau_4^{(i)} = -\frac{1}{t_1^{(i)}}\left[\left(\frac{t_2^{(i)}}{t_1^{(i)}}\right)^3 - 2\frac{t_2^{(i)}}{t_1^{(i)}}\frac{t_3^{(i)}}{t_1^{(i)}} + \frac{t_4^{(i)}}{t_1^{(i)}}\right],$$

$$\tau_5^{(i)} = \frac{1}{t_1^{(i)}}\left[\left(\frac{t_2^{(i)}}{t_1^{(i)}}\right)^4 - 3\left(\frac{t_2^{(i)}}{t_1^{(i)}}\right)^2\frac{t_3^{(i)}}{t_1^{(i)}} + 2\frac{t_2^{(i)}}{t_1^{(i)}}\frac{t_4^{(i)}}{t_1^{(i)}} + \left(\frac{t_3^{(i)}}{t_1^{(i)}}\right)^2 - \frac{t_5^{(i)}}{t_1^{(i)}}\right]$$

$$\begin{aligned} \tau_6^{(i)} = -\frac{1}{t_1^{(i)}}\Bigg[&\left(\frac{t_2^{(i)}}{t_1^{(i)}}\right)^5 - 4\left(\frac{t_2^{(i)}}{t_1^{(i)}}\right)^3\frac{t_3^{(i)}}{t_1^{(i)}} + 3\left(\frac{t_2^{(i)}}{t_1^{(i)}}\right)^2\frac{t_4^{(i)}}{t_1^{(i)}} + \\ &+ 3\frac{t_2^{(i)}}{t_1^{(i)}}\left(\frac{t_3^{(i)}}{t_1^{(i)}}\right)^2 - 2\frac{t_2^{(i)}}{t_1^{(i)}}\frac{t_5^{(i)}}{t_1^{(i)}} - 2\frac{t_3^{(i)}}{t_i^{(i)}}\frac{t_4^{(i)}}{t_1^{(i)}} + \frac{t_6^{(i)}}{t_1^{(i)}}\Bigg], \end{aligned}$$

$$\cdots\cdots\cdots\cdots$$

$$\tau_j^{(i)} = -\frac{1}{t_1^{(i)}}\sum_{l=1}^{j-1} t_{l+1}^{(i)}\tau_{j-l}^{(i)} \qquad (j = 2, 3, \ldots, k_i).$$

Any element $\tau_j^{(i)}$ can also be obtained by the recurrent relationship

$$\tau_j^{(i)} = \frac{1}{t_1^{(i)}}$$

$$\tau_j^{(i)} = -\frac{1}{t_1^{(i)}}\sum_{l=1}^{j-1} t_{l+1}^{(i)}\tau_{j-l}^{(i)}, \quad (j=2, 3, \ldots, k_i). \tag{77}$$

Taking also Eq. (75) into consideration:

$$\tau_1^{(i)} = \frac{1}{w_{k_i,n}^{(i)}}$$

$$\tau_j^{(i)} = -\frac{1}{w_{k_i,n}^{(i)}} \sum_{h=1}^{j-1} w_{k_i-h,n}^{(i)} \tau_{j-h}^{(i)}, \qquad (j = 2, 3, \ldots, k_i). \tag{78}$$

Although Table 1 is sufficient to determine elements $\tau_j^{(i)}$ in terms of $w_{jn}^{(i)}$ when Eq. (75) is taken into consideration, for the sake of convenience Table 2 is compiled.

Table II

Elements $\tau_j^{(i)}$ of $\mathbf{T}^{-1}$ in terms of elements $w_{jn}^{(i)}$ of $\mathbf{W}$

$$\tau_1^{(i)} = \frac{1}{w_{k_i,n}^{(i)}},$$

$$\tau_2^{(i)} = -\frac{1}{w_{k_i,n}^{(i)}} \frac{w_{k_i-1,n}^{(i)}}{w_{k_i,n}^{(i)}},$$

$$\tau_3^{(i)} = \frac{1}{w_{k_i,n}^{(i)}} \left[\left(\frac{w_{k_i-1,n}^{(i)}}{w_{k_i,n}^{(i)}} \right)^2 - \frac{w_{k_i-2,n}^{(i)}}{w_{k_i,n}^{(i)}} \right]$$

$$\tau_4^{(i)} = -\frac{1}{w_{k_i,n}^{(i)}} \left[\left(\frac{w_{k_i-1,n}^{(i)}}{w_{k_i,n}^{(i)}} \right)^3 - 2 \frac{w_{k_i-1,n}^{(i)}}{w_{k_i,n}^{(i)}} \frac{w_{k_i-2,n}^{(i)}}{w_{k_i,n}^{(i)}} + \frac{w_{k_i-3,n}^{(i)}}{w_{k_i,n}^{(i)}} \right];$$

$$\tau_5^{(i)} = \frac{1}{w_{k_i,n}^{(i)}} \left[\left(\frac{w_{k_i-1,n}^{(i)}}{w_{k_i,n}^{(i)}} \right)^4 - 3 \left(\frac{w_{k_i-1,n}^{(1)}}{w_{k_i,n}^{(1)}} \right)^2 \frac{w_{k_i-2,n}^{(i)}}{w_{k_i,n}^{(i)}} + 2 \frac{w_{k_i-1,n}^{(1)}}{w_{k_i,n}^{(i)}} \frac{w_{k_i-3}^{(i)}}{w_{k_i,n}^{(i)}} + \right.$$

$$\left. + \left(\frac{w_{k_i-2,n}^{(i)}}{w_{k_i,n}^{(i)}} \right) - \frac{w_{k_i,-4,n}^{(i)}}{w_{k_in}^{(i)}} \right],$$

$$\tau_6^{(i)} = -\frac{1}{w_{k_i,n}^{(i)}} \left[\left(\frac{w_{k_i-1,n}^{(i)}}{w_{k_i,n}^{(i)}} \right)^5 - 4 \left(\frac{w_{k_i-1,n}^{(i)}}{w_{k_i,n}^{(i)}} \right)^3 \frac{w_{k_i-2,n}^{(i)}}{w_{k_i,n}^{(i)}} + 3 \left(\frac{w_{k_i-1,n}^{(i)}}{w_{k_i,n}^{(i)}} \right)^2 \frac{w_{k_i-3,n}^{(i)}}{w_{k_i,n}^{(i)}} + \right.$$

$$\left. + 3 \frac{w_{k_i-1,n}^{(i)}}{w_{k_i,n}^{(i)}} \left(\frac{w_{k_i-2,n}^{(i)}}{w_{k_i,n}^{(i)}} \right)^2 - 2 \frac{w_{k_i-1,n}^{(i)}}{w_{k_i,(n)}^{(i)}} \frac{w_{k_i-4,n}^{(i)}}{w_{k_i,n}} - 2 \frac{w_{k_i-2}^{(i)}}{w_{k_i,n}^{(i)}} \frac{w_{k_i-3,n}^{(i)}}{w_{k_i,n}^{(i)}} + \frac{w_{k_i-5,n}^{(i)}}{w_{k_i,n}^{(i)}} \right]$$

. . . .

$$\tau_j^{(i)} = -\frac{1}{w_{k_i,n}^{(i)}} \sum_{h=1}^{j-1} w_{k_i-h}^{(i)} \tau_{j-h}^{(i)} \qquad (j = 2, 3, \ldots, k_i).$$

Finally, we remark that Eqs (6) and (12) yield

$$\mathbf{c}^{\mathrm{T}} = \mathbf{c}_0^{\mathrm{T}} \mathbf{V} \mathbf{T} \tag{79}$$

which, according to Eqs (9) and (75) results in the same relationships as Eq. (62) expressing coefficients $C_j^{(i)}$ in terms of elements $w_{jn}^{(i)}$ and coefficients b.

8. Illustrative Examples

First Example. Let us start with the phase-variable form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\alpha^3 & -3\alpha^2 & -3\alpha\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} u$$

$$y = [K, \quad 0, \quad 0]\ \mathbf{x}.$$

The corresponding transfer function is

$$G(s) = \frac{K}{(s+\alpha)^3} = \frac{K}{s^3 + 3\alpha s^2 + 3\alpha^2 s + \alpha^3}\,.$$

According to Eq. (20) in the first method we have

$$C_1^{(1)} = K; \qquad C_2^{(1)} = 0; \qquad C_3^{(1)} = 0.$$

Thus, the JORDAN canonical form can be expressed as

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix} = \begin{bmatrix}-\alpha & 1 & 0\\ 0 & -\alpha & 1\\ 0 & 0 & -\alpha\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} u$$

$$y = [\ K, \quad 0, \quad 0]\ \mathbf{z}.$$

Following the second method, from Eq. (51) we have

$$N_1(s) = s^2 + 3\alpha s + 3a^2; \quad N_2(s) = s + 3\alpha; \quad N_3(s) = 1\,,$$

and applying Eq. (50) after some calculation

$$\mathbf{V}^{-1} = \begin{bmatrix}1 & 0 & 0\\ \alpha & 1 & 0\\ \alpha^2 & 2\alpha & 1\end{bmatrix}$$

whereas, according to Eq. (9)

$$\mathbf{V} = \begin{bmatrix}1 & 0 & 0\\ -\alpha & 1 & 0\\ \alpha^2 & -2\alpha & 1\end{bmatrix}.$$

Keeping a check on, the result is

$$\mathbf{V}^{-1}\,\mathbf{V} = \mathbf{M}^{-1}\,\mathbf{M} = \mathbf{M}\,\mathbf{M}^{-1} = \mathbf{V}\,\mathbf{V}^{-1} = \mathbf{I}\,,$$

indeed. As $\mathbf{V}^{-1}\mathbf{b}_0 = \mathbf{b}$ is in the form of Eq. (10) we obtained directly the principal variant and the commutativity matrix as well, as its inverse is in this case

$$\mathbf{T} = \mathbf{I} = \mathbf{T}^{-1}\,.$$

The last equation holds for every case when the JORDAN matrix is composed by a single JORDAN block. Finally,

$$\mathbf{c}^{\mathrm{T}} = \mathbf{c}_0^{\mathrm{T}}\,\mathbf{L} = \mathbf{c}_0^{\mathrm{T}}\,\mathbf{V} = \mathbf{c}_0^{\mathrm{T}} = [K,\ 0,\ 0]\,.$$

Furthermore, Eq. (62) is also satisfied.

Second Example. Let the phase-variable form be

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha^2\beta & -2\alpha\beta - \alpha^2 & -2\alpha - \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [\ K, \quad 0, \quad 0]\ \mathbf{x}.$$

The corresponding transfer function is

$$G(s) = \frac{K}{(s+\alpha)^2\,(s+\beta)} = \frac{K}{s^3 + (2\alpha+\beta)\,s^2 + (2\alpha\beta+\alpha^2)\,s + \alpha^2\beta}\,.$$

Following the first method, according to Eq. (20) we can compute

$$C_1^{(1)} = \frac{K}{\beta-\alpha}\,,\quad C_2^{(1)} = -\frac{K}{(\beta-\alpha)^2}\,,\quad C_1^{(2)} = \frac{K}{(\beta-\alpha)^2}$$

leading to the JORDAN canonical form

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -\alpha & 1 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \left[\frac{K}{\beta-\alpha}\,,\ -\frac{K}{(\beta-\alpha)^2}\,,\ \frac{K}{(\beta-\alpha)^2}\right]\ \mathbf{z}.$$

The modal matrix is now

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 1 \\ -\alpha & 1 & -\beta \\ \alpha^2 & -2\alpha & \beta^3 \end{bmatrix}.$$

Following the second method, from Eq. (51) we have

$$N_1(s) = s^2 + (2\alpha + \beta)s + (2\alpha\beta + \alpha^2); \; N_2(s) = s + (2\alpha + \beta); \; N_3(s) = 1;$$

and, applying Eq. (50) after some computation

$$\mathbf{V}^{-1} = \frac{1}{(\beta-\alpha)^2}\begin{bmatrix} \beta^2 - 2\alpha\beta & -2\alpha & -1 \\ \alpha\beta^2 - \alpha^2\beta & \beta^2 - \alpha^2 & \beta - \alpha \\ \alpha^2 & 2\alpha & 1 \end{bmatrix} = \begin{bmatrix} w_{11}^{(1)} & w_{12}^{(1)} & w_{13}^{(1)} \\ w_{21}^{(1)} & w_{22}^{(1)} & w_{23}^{(1)} \\ w_{11}^{(2)} & w_{12}^{(2)} & w_{13}^{(2)} \end{bmatrix}.$$

Now

$$\mathbf{V}^{-1}\mathbf{b}_0 = \left[\frac{\beta-\alpha}{(\beta-\alpha)^2}, \frac{-1}{(\beta-\alpha)^2}, \frac{1}{(\beta-\alpha)^2}\right]^{\mathrm{T}} \neq [0, \; 1, \; 1]^{\mathrm{T}},$$

Applying Table 2 with $k_1 = 2$

$$\tau_1^{(1)} = \frac{1}{w_{23}^{(1)}} = (\beta - \alpha)$$

$$\tau_2^{(1)} = -\frac{1}{w_{23}^{(1)}}\left[\frac{w_{13}^{(1)}}{w_{23}^{(1)}}\right] = 1$$

and with $k_2 = 1$

$$\tau_1^{(2)} = \frac{1}{w_{13}^{(2)}} = (\beta-\alpha)^2.$$

The appropriate inverse commutativity matrix is then

$$\mathbf{T}^{-1} = \begin{bmatrix} \beta-\alpha & 1 & 0 \\ 0 & \beta-\alpha & 0 \\ 0 & 0 & (\beta-\alpha)^2 \end{bmatrix}$$

yielding indeed the desired vector $\mathbf{b}$:

$$\mathbf{T}^{-1}\mathbf{V}^{1}\mathbf{b}_0 = [0, \quad 1, \quad 1]^{\mathrm{T}}.$$

Furthermore from Eq. (75) for $i = 1$ we have

$$t_2^{(1)} = w_{1n}^{(1)} = -\frac{1}{(\beta-\alpha)^2}$$

$$t_1^{(1)} = w_{2n}^{(1)} = \frac{\beta-\alpha}{(\beta-\alpha)^2} = \frac{1}{\beta-\alpha}$$

and for $i = 2$

$$t_1^{(2)} = w_{1n}^{(2)} = \frac{1}{(\beta+\alpha)^2}.$$

The commutativity matrix itself is then

$$\mathbf{T} = \begin{bmatrix} \dfrac{1}{\beta-\alpha} & -\dfrac{1}{(\beta-\alpha)^2} & 0 \\ 0 & \dfrac{1}{\beta-\alpha} & 0 \\ 0 & 0 & \dfrac{1}{(\beta-\alpha)^2} \end{bmatrix}.$$

To keep a check on the commutativity matrix

$$\mathbf{TJ} = \begin{bmatrix} -\dfrac{\alpha}{\beta-\alpha} & \dfrac{\beta}{(\beta-\alpha)^2} & 0 \\ 0 & -\dfrac{\alpha}{\beta-\alpha} & 0 \\ 0 & 0 & -\dfrac{\beta}{(\beta-\alpha)^2} \end{bmatrix} = \mathbf{J\,T}$$

so $\mathbf{T}^{-1}\mathbf{J}\,\mathbf{T} = \mathbf{J}$ is satisfied, indeed. Finally

$$\mathbf{c}^{\mathrm{T}} = \mathbf{c}_0^{\mathrm{T}}\,\mathbf{VT} = [K, 0, K]\,\mathbf{T} = \left[\frac{K}{\beta-\alpha}, -\frac{K}{(\beta-\alpha)^2}, \frac{K}{(\beta-\alpha)^2}\right].$$

The latter result, which can also be obtained from Eq. (62), corresponds that obtained by the first method.

Third Example. In [21] it has been shown that for the VANDERMONDE matrix

$$\mathbf{V} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & -2 & 1 & -3 \\ 1 & -2 & 1 & 4 & -4 & 9 \\ -1 & 3 & -3 & -8 & 12 & -27 \\ 1 & -4 & 6 & 16 & -32 & 81 \\ -1 & 5 & -10 & -32 & 80 & -234 \end{bmatrix}$$

by an appropriate algorithm, see Chapter 12, the inverse matrix can be obtained as:

$$\mathbf{V}^{-1} = \begin{bmatrix} 16.5 & 58.0 & 83.125 & 56.375 & 17.875 & 2.125 \\ -9.0 & -36.0 & -52.25 & -34.75 & -10.75 & -1.25 \\ 6.0 & 20.0 & 25.5 & 15.5 & 4.5 & 0.5 \\ -15.0 & -56.0 & -80.0 & -54.0 & -17.0 & -2.0 \\ -6.0 & -23.0 & -34.0 & -24.0 & -8.0 & -1.0 \\ -0.5 & -2.0 & -3.125 & -2.375 & -0.875 & -0.125 \end{bmatrix}$$

Now, the characteristic equation is

$$D(s) = (s+1)^3(s+2)^2(s+3) = \\ = s^6 + 10s^5 + 40s^4 + 82s^3 + 91s^2 + 52s + 12 = 0 .$$

Thus, the numerator polynomials are

$$\begin{aligned} N_1(s) &= s^5 + 10s^4 + 40s^3 + 82s^2 + 91s + 52 \\ N_2(s) &= s^4 + 10s^3 + 40s^2 + 82s + 91 \\ N_3(s) &= s^3 + 10s^2 + 40s + 82 \\ N_4(s) &= s^2 + 10s + 40 \\ N_5(s) &= s + 10 \\ N_6(s) &= 1. \end{aligned}$$

For example, the last column of $\mathbf{V}^{-1}$ can be computed by means of Eq. (50) as follows:

$$w_{36}^{(1)} = (s+1)^3 \left.\frac{N_6(s)}{D(s)}\right|_{s=-1} = \left.\frac{N_6(s)}{D_1(s)}\right|_{s=-1} = \left.\frac{1}{(s+2)^2(s+3)}\right|_{s=-1} = \frac{1}{2} = 0.5$$

$$w_{26}^{(1)} = \frac{d}{ds}\left.\frac{N_6(s)}{D_1(s)}\right|_{s=-1} = -\left.\frac{(s+2)(3s+8)}{(s+2)^4(s+3)^2}\right|_{s=-1} = -\frac{5}{4} = -1{,}25$$

$$w_{16}^{(1)} = -\frac{1}{2}\,\frac{(3s+8)+3(s+2)(s+2)^4(s+3)^2 - [4(s+2)^3(s+3)^2+2(s+2)^4}{(s+2)^8(s+3)^4}\ldots$$

$$\ldots\left.\frac{(s+3)](s+2)(3s+8)}{}\right|_{s=-1} = \frac{68}{32} = 2.125$$

$$w_{26}^{(2)} = (s+2)^2\left.\frac{N_6(s)}{D(s)}\right|_{s=-2} = \left.\frac{N_6(s)}{D_2(s)}\right|_{s=-2} = \left.\frac{1}{(s+1)^3(s+3)}\right|_{s=-2} = -1$$

$$w_{16}^{(2)} = \frac{d}{ds}\left.\frac{N_6(s)}{D_2(s)}\right|_{s=-2} = -\left.\frac{3(s+1)^2(s+3)+(s+1)^3}{(s+1)^6(s+3)^2}\right|_{s=-2} = -\frac{2}{1} = -2$$

$$w_{16}^{(3)} = (s+3)\left.\frac{N_6(s)}{D_3(s)}\right|_{s=-3}\left.\frac{N_6(s)}{D_3(s)}\right|_{s=-3} = \left.\frac{1}{(s+1)^3(s+2)^2}\right|_{s=-3} = \frac{1}{8} = -0.125.$$

Then, Eq. (75) yields

$$t_1^{(1)} = 0.5;\ t_2^{(1)} = -1.25;\quad t_3^{(1)} = 2.125$$

$$t_1^{(2)} = -1;\quad t_2^{(2)} = -2;$$

$$t_1^{(3)} = -0.125$$

and from Table 1 we obtain

$$\tau_1^{(1)} = 2; \quad \tau_2^{(1)} = 5; \quad \tau_3^{(1)} = 4$$
$$\tau_1^{(2)} = -1; \quad \tau_2^{(2)} = 2;$$
$$\tau_1^{(3)} = -8.$$

9. Third Method

It is well known *e.g* [11, 12] that for distinct eigenvalues the elements w_{rl} of the inverse VANDERMONDE matrix $\mathbf{W} = \mathbf{V}^{-1}$ can be obtained from the r-th LAGRANGE polynomial

$$P_r(\lambda) = \sum_{l=1}^{n} w_{rl}\, \lambda^{l-1} \quad (r = 1, 2 \ldots, n) \tag{80}$$

as the coefficients corresponding to λ^{l-1}, where

$$P_r(\lambda) = \prod_{\substack{l=1 \\ l \neq r}}^{n} \frac{\lambda - \lambda_l}{\lambda_r - \lambda_l} \qquad (r = 1, 2, \ldots, n). \tag{81}$$

This statement is clarified by the fact that according to Eq. (80), $P_r(\lambda_j)$ expresses the scalar or inner product of the r-th row of $\mathbf{W}$ and the j-th column of $\mathbf{V}$ which, according to Eq. (81), is equal to δ_{rj}, where δ_{rj} is the KRONECKER symbol ($\delta_{rj} = 0$, for $r \neq j$ and $\delta_{rj} = 1$, for $r = j$) yielding $\mathbf{WV} = \mathbf{I}$.

Now, the question arises whether a similar method is also applicable to multiple eigenvalues. The answ er is yes, but LAGRANGE interpolation polynomials have to be replaced by HERMITIAN ones.

Let the row vectors of $\mathbf{W}$ be $\mathbf{w}_j^{\mathrm{T}(i)}$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k_i$ and the column vectors of $\mathbf{V}$ as given in Eq. (9). Then, the property $\mathbf{W}\,\mathbf{V} = \mathbf{I}$ prescribes

$$\mathbf{w}_j^{\mathrm{T}(i)} \frac{1}{(h-1)!} \mathbf{v}_{\mathrm{k}}^{(\mathrm{h}-1)} = \delta_{ik}\, \delta_{lh}, \quad \begin{matrix}(i = 1, 2, \ldots, m) \\ (h, j, k = 1, 2, \ldots, k_i).\end{matrix} \tag{82}$$

Suppose $P_{j(i)}(\lambda)$ are appropriate polynomials. Then, Eq. (80) must be further valid:

$$P_{j(i)}(\lambda) = \sum_{l=1}^{n} w_{jl}^{(i)}\, \lambda^{l-1}. \tag{83}$$

For $\lambda = \lambda_k$, Eqs (83) and (82) lead to

$$P_{j(i)}(\lambda_k) = \delta_{ik}\, \delta_{j1}. \tag{84}$$

Differentiating Eq. (83) with respect to λ yields

$$P'_{j(i)}(\lambda) = \sum_{l=2}^{n} w_{jl}^{(i)} (l-1) \lambda^{l-2}. \tag{85}$$

For $\lambda = \lambda_k$, Eqs (85) and (82) have to lead to

$$P'_{j(i)}(\lambda_k) = \delta_{ik} \delta_{j2} . \tag{86}$$

Differentiating once more

$$P''_{j(i)}(\lambda) = \sum_{l=3}^{n} w_{jl}^{(i)} (l-1)(l-2) \lambda^{l-3}. \tag{87}$$

For $\lambda = \lambda_k$, Eqs (87) and (82) have to yield

$$P''_{j(i)}(\lambda) = 2\,!\, \delta_{ik} \delta_{j3} . \tag{88}$$

In general

$$P_{j(i)}^{(q)}(\lambda) = \sum_{l=q+1}^{n} w_{jl}^{(i)} \frac{(l-1)!}{(l-q-1)!} \lambda^{l-q-1} \tag{89}$$

to yield

$$\mathrm{P}_{j(i)}^{(q)}(\lambda_k) = q!\, \delta_{ik} \delta_{j,q+1} \qquad \begin{matrix} (i, k = 1, 2, \ldots, m) \\ (j = 1, 2, \ldots, k_i) \\ (q = 0, 1, \ldots, k_i - 1) \end{matrix} \tag{90}$$

A polynomial assuming, together with its derivatives, prescribed values, is called HERMITE interpolation polynomial [38, 39]. Owing to the properties expressed in Eq. (90) we will term $P_{j(i)}(\lambda)$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k_i$ the HERMITE—KRONECKER interpolation polynomials.

We emphasize that in contrast to the classical HERMITE problem, where only one polynomial is to be determined, here a set of polynomials are to be obtained.

In accordance with Eq. (17) let us define

$$D(\lambda) = (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \ldots (\lambda - \lambda_m)^{k_m} . \tag{91}$$

Suppose:

$$\frac{P_{j(i)}}{D(\lambda)} = \sum_{i'=1}^{m} \sum_{j'=1}^{k_{i'}} \frac{A_{j,j'}^{(i,i')}}{(\lambda - \lambda_{i'})^{k_{i'} - j' + 1}} \tag{92}$$

or

$$P_{j(i)}(\lambda) = \sum_{i'=1}^{m} \sum_{j'=1}^{k_{i'}} A_{j,j'}^{(i,i')} (\lambda - \lambda_{i'})^{j'-1} D_{i'}(\lambda) \tag{93}$$

where

$$D_{i'}(\lambda) = \frac{D(\lambda)}{(\lambda - \lambda_{i'})^{k_{i'}}} \,. \tag{94}$$

Eq. (92) is a direct generalization of JOHANSEN's formula [40], valid here for a complete set of polynomials.

If the polynomials $P_{j(i)}(\lambda)$ were known then coefficients $A_{j,j'}^{(i,i')}$ could easily be obtained as

$$A_{j,j'}^{(i,i')} = \lim_{s \to \lambda_i} \frac{1}{(j'-1)} \frac{d^{(j'-1)}}{d\lambda^{(j'-1)}} \left[(\lambda - \lambda_{i'})^{k_{i'}} \frac{P_{j(i)}(\lambda)}{D(\lambda)} \right] \tag{95}$$

$$(i, i' = 1, 2, \ldots, m; \; j = 1, 2, \ldots, k_{i}; \; j' = 1, 2, \ldots, k_{i'}) \,.$$

Here we have, however, an inverse problem, *i.e.* how to determine coefficients $A_{j,j'}^{(i,i')}$ in order to satisfy the requirements in Eq. (9).
First we examine $P_{1(i)}(\lambda)$. From Eqs (93) and (84) we obtain

$$P_{1(i)}(\lambda_i) = A_{11}^{(i,i)} D_i(\lambda_i) = 1$$

that is,

$$A_{11}^{(i,i)} = \frac{1}{D_i(\lambda_i)} \tag{96}$$

Eq. (86) and the first derivative of Eq. (93) yield

$$P'_{1(i)}(\lambda_i) = A_{11}^{(i,i)} D'_i(\lambda_i) + A_{12}^{(i,i)} D_i(\lambda_i) = 0$$

that is,

$$A_{12}^{(i,i)} = - \frac{A_{11}^{(i,i)} D'_i(\lambda_i)}{D_i(\lambda_i)} = - \frac{D'_i(\lambda_i)}{D_i^2(\lambda_i)} \,. \tag{97}$$

Eq. (88) and the second derivative of Eq. (93) yield

$$P''_{1(i)}(\lambda_i) = A_{11}^{(i,i)} D''_i(\lambda_i) + 2A_{12}^{(i,i)} D'_i(\lambda_i) + 2A_{13} D_i(\lambda_i) = 0$$

that is,

$$A_{13}^{(i,i)} = \frac{A_{11}^{(i,i)} D''_i(\lambda_i) + 2A_{12}^{(i,i)} D'(\lambda_i)}{2D_i(\lambda_i)} = - \frac{D''_i(\lambda_i)}{2D_i^2(\lambda_i)} + \frac{D_i'^2(\lambda_i)}{D_i^3(\lambda_i)} \,. \tag{98}$$

The procedure can be continued if necessary. The coefficients $A_{1,j}^{(i,i')}$ being zero for $i \neq i'$, the final form of Eq. (93) for $j = 1$ becomes

$$P_{1(i)}(\lambda) = A_{11}^{(i,i)}\, D_i(\lambda) + A_{12}^{(i,i)}\,(\lambda-\lambda_i)\, D_i(\lambda) + A_{13}^{(i,i)}\,(\lambda-\lambda_i)^2\, D_i(\lambda) + \ldots =$$
$$= \frac{D_i(\lambda)}{D_i(\lambda_i)} - \frac{D'(\lambda_i)}{D_i(\lambda_i)}\, D_i(\lambda)\,(\lambda-\lambda_i) +$$
$$+ \left[-\frac{D_i''(\lambda_i)}{2D_i^2(\lambda_i)} + \frac{D_i'^2(\lambda_i)}{D_i^3(\lambda_i)}\right] D_i(\lambda)\,(\lambda-\lambda_i)^2 + \ldots \tag{99}$$

It is to be emphasized that in order to get the elements in the r-th row inverse matrix $\mathbf{W} = \mathbf{V}^{-1}$, where $r = k_1 + \ldots + k_{i-1} + 1$, $P_{1(i)}$ should be arranged in terms of powers of λ.

Next, we analyse $P_{2(i)}(\lambda)$. From Eqs (93) and (84) we obtain

$$P_{2(i)}(\lambda_i) = A_{21}^{(i,i)}\, D_i(\lambda_i) = 0$$

that is

$$A_{21}^{(i,i)} = 0\,. \tag{100}$$

Eq. (86) and the first derivative of Eq. (93) yield

$$P'_{2(i)}(\lambda_i) = A_{22}^{(i,i)}\, D_i(\lambda_i) = 1$$

that is,

$$A_{22}^{(i,i)} = \frac{1}{D_i(\lambda_i)} = A_{11}^{(i,i)}. \tag{101}$$

Eq. (88) and the second derivative of Eq. (93) yield

$$P''_{2(i)}(\lambda_i) = 2A_{22}^{(i,i)}\, D_i'(\lambda_i) + 2A_{23}^{(i,i)}\, D_i(\lambda_i) = 0$$

that is

$$A_{23}^{(i,i)} = -\frac{A_{22}^{(i,i)}\, D_i'(\lambda_i)}{D_i(\lambda_i)} = -\frac{D_i'(\lambda_i)}{D_i^2(\lambda_i)} = A_{12}^{(i,i)}. \tag{102}$$

Now

$$P_{2(i)}(\lambda) = A_{22}^{(i,i)}\,(\lambda-\lambda_i)\, D_i(\lambda) + A_{23}^{(i,i)}\,(\lambda-\lambda_i)^2\, D_i(\lambda) + \ldots =$$
$$= \frac{D_i(\lambda)}{D_i(\lambda_i)}\,(\lambda-\lambda_i) - \frac{D_i'(\lambda_i)}{D_i^2(\lambda_i)}\, D_i(\lambda)\,(\lambda-\lambda_i)^2 + \ldots \tag{103}$$

Finally, we analyze $P_{3(i)}(\lambda)$. From Eqs (93) and (84) as well as from Eq. (86) and the first derivative of Eq. (93) we obtain

$$A_{31}^{(i,i)} = 0, \qquad A_{32}^{(i,i)} = 0 \tag{104}$$

whereas Eq. (88) and the second derivative of Eq. (93) yield

$$2A_{33}^{(i,i)}\, D_i(\lambda_i) = 2\,!$$

that is,

$$A_{33}^{(i,i)} = \frac{1}{D_i(\lambda_i)} = A_{11}^{(i,i)} = A_{22}^{(i,i)}. \tag{105}$$

Thus,

$$P_{3(i)}(\lambda) = \frac{D_i(\lambda)}{D_i(\lambda_i)}(\lambda - \lambda_i)^2 + \dots \tag{106}$$

The coefficients of these polynomials are the elements of the corresponding row. In case of a single repeated eigenvalue the HERMITE—KRONECKER polynomials reduce to binomial polynomials:

$$\begin{aligned}
P_{1(1)}(\lambda) &= 1 \\
P_{2(1)}(\lambda) &= \lambda - \lambda_1 \\
&\;\vdots \\
P_{j(1)}(\lambda) &= (\lambda - \lambda_1)^{j-1} \\
&\;\vdots \\
P_{n(1)}(\lambda) &= (\lambda - \lambda_1)^{n-1}.
\end{aligned} \tag{107}$$

The structure of the latter are given in Eqs (99), (103) and (106). They remain the same.

10. Third Conclusion

The third method is a direct one avoiding LAPLACE transforms. It gives a deep insight into the character of the inverse VANDERMONDE matrix. It supplies the rows of the latter by constructing appropriate HERMITE—KRONECKER interpolation polynomials. For distinct eigenvalues or a single repeated eigenvalue the third method seems to be the simplest, see Eqs. (81) and (107).

The third method has, however, serious inconveniences. The computation of the coefficients $A_{j,j'}^{(i,i)}$ requires generally more work than does the calculation of coefficients $w_{jl}^{(i)}$ in the second method. Although here we have only to differentiate polynomials in factored form, whereas in the second method, ratios of polynomials are to be differentiated, but at the very end the desired polynomials $P_{j(i)}$ must be obtained in terms of powers of λ requiring complicated multiplications. (The multiplications become, however, simpler if the eigenvalues are given in numerical form.)

We remark that SCHAPPELLE [23, 24] has given closed-form formulas but they are very complicated and unlikely to be applied, except if a digital computer is available.

We do also remark that the HERMITE—KRONECKER polynomials can be compiled for certain structures. An example is shown in Table 3. After substituting numerical values the final form is relatively easy to be obtained. In case of distinct eigenvalues, HERMITE—KRONECKER interpolation polynomials reduce to LAGRANGE polynomials. In case of a single repeated eigenvalue the HERMITE—KRONECKER polynomials become binomial polynomials.

Table III

HERMITE-KRONECKER interpolation polynomials

For $k_1 = 2$, $k_2 = 1$:

$$P_{1(1)}(\lambda) = \frac{\lambda - \lambda_2}{1 - 2} - \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{(\lambda_1 - \lambda_2)^2}; \qquad P_{1(2)} = \frac{(\lambda - \lambda_1)^2}{(\lambda_1 - \lambda_2)^2}$$

$$P_{2(1)}(\lambda) = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{(\lambda_1 - \lambda_2)^2};$$

For $k_1 = 3$, $k_2 = 1$:

$$P_{1(1)}(\lambda) = \frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} - \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{(\lambda_1 - \lambda_2)^2} + \frac{(\lambda - \lambda_1)^2(\lambda - \lambda_2)}{(\lambda_1 - \lambda_2)^3}$$

$$P_{2(1)}(\lambda) = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{(\lambda_1 - \lambda_2)} - \frac{(\lambda - \lambda_1)^2(\lambda - \lambda_2)}{(\lambda_1 - \lambda_2)^2}$$

$$P_{3(1)}(\lambda) = \frac{(\lambda - \lambda_1)^2(\lambda - \lambda_2)}{\lambda - \lambda_2}$$

$$P_{1(2)}(\lambda) = -\frac{(\lambda - \lambda_1)^3}{(\lambda_1 - \lambda_2)^2}$$

For $k_1 = 2$, $k_2 = 2$

$$P_{1(1)}(\lambda) = \frac{(\lambda - \lambda_2)^2}{(\lambda_1 - \lambda_2)^2} - 2\frac{(\lambda - \lambda_1)(\lambda - \lambda_2)^2}{(\lambda_1 - \lambda_2)^2}$$

$$P_{2(1)}(\lambda) = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{(\lambda_1 - \lambda_2)^2}$$

$$P_{1(2)}(\lambda) = \frac{(\lambda - \lambda_1)^2}{(\lambda_2 - \lambda_1)^2} - 2\frac{(\lambda - \lambda_2)(\lambda - \lambda_1)^2}{(\lambda_2 - \lambda_1)^2}$$

$$P_{2(2)}(\lambda) = \frac{(\lambda - \lambda_2)(\lambda - \lambda_1)}{(\lambda_2 - \lambda_1)^2}$$

Such precalculated polynomials may considerably facilitate the inversion of the VANDERMONDE matrix.

11. Illustrative Examples

We will solve the same problems as above by the third method.

First Example. If $\lambda_1 = -\alpha$ is a single repeated eigenvalue of multiplicity 3, then, according to Eq. (107)

$$P_{1(1)}(\lambda) = 1; \quad P_{2(1)}(\lambda) = \alpha + \lambda; \quad P_{3(1)}(\lambda) = \alpha^2 + 2\alpha\lambda + \lambda^2$$

yielding

$$\mathbf{W} = \mathbf{V}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \alpha^2 & 2\alpha & 1 \end{bmatrix}$$

Second Example. Let us choose $\lambda_1 = -\alpha$ as a twofold eigenvalue and $\lambda_2 = -\beta$ as a distinct eigenvalue. Then Table 3 yields:

$$P_{1(1)}(\lambda) = \frac{(\lambda + \alpha)}{\beta - \alpha} - \frac{(\lambda + \alpha)(\lambda + \beta)}{(\beta - \alpha)^2} = \frac{1}{(\beta - \alpha)^2}[\beta^2 - 2\alpha\beta - 2\alpha\lambda - \lambda^2]$$

$$P_{2(1)}(\lambda) = \frac{(\lambda + \alpha)(\lambda + \beta)}{(\beta - \alpha)} = \frac{1}{(\beta - \alpha)^2}[\alpha\beta^2 - \alpha^2\beta + (\beta^2 - \alpha^2)\lambda + (\beta - \alpha)\lambda^2]$$

$$P_{1(2)}(\lambda) = \frac{(\lambda + \alpha)^2}{(\beta - \alpha)^2} = \frac{1}{(\beta - \alpha)^2}[\alpha^2 + 2\alpha\beta\lambda + \lambda^2]$$

leading to the same inverse matrix as in Chapter 8.

Third Example. Let us determine, for example, the first and third rows of the inverse VANDERMONDE matrix for $\lambda_1 = -1$, $k_1 = 3$; $\lambda_2 = -2$, $k_2 = 2$, $\lambda_3 = -3$, $k_3 = 1$. Now,

$$D(\lambda) = (\lambda + 1)^3(\lambda + 2)^2(\lambda + 3)$$

and

$$D_1(\lambda) = (\lambda + 2)^2(\lambda + 3);$$

$$D_2(\lambda) = (\lambda + 1)^3(\lambda + 3);$$

$$D_3(\lambda) = (\lambda + 1)^3(\lambda + 2)^2.$$

For the first row, from Eqs (96), (97) and (98):

$$A_{11}^{(11)} = \frac{1}{2}; \; A_{12}^{(11)} = -\frac{5}{4} \; ; A_{13}^{(11)} = -\frac{8}{8} + \frac{25}{8} = \frac{17}{8}$$

and

$$A_{11}^{(12)} = 0, \; A_{12}^{(12)} = 0; \; A_{11}^{(13)} = 0.$$

From Eq. (99)

$$P_{1(1)}(\lambda) = \frac{1}{2}(\lambda + 2)^2(\lambda + 3) - \frac{5}{4}(\lambda + 1)(\lambda + 2)^2(\lambda + 3) -$$

$$- \frac{17}{8}(\lambda + 1)^2(\lambda + 2)^2(\lambda + 3) = 16.5 + 58\lambda + 83.125\lambda^2 + 56.375\lambda^4 +$$
$$+ 17.875\lambda^4 + 2.125\lambda^6 .$$

Similarly, for the third row, from Eq. (106)

$$P_{3(1)}(\lambda) = \frac{1}{2}(\lambda + 1)^2(\lambda + 2)^2(\lambda + 3) = 6 + 20\lambda + 25.5\lambda^2 + 15.5\lambda^4 +$$
$$+ 4.5\lambda^4 + 0.5\lambda^6 .$$

12. Fourth Method

For the sake of completeness we remark that in [21], KAUFMAN suggested a procedure of obtaining the inverse of the VANDERMONDE matrix.

This method is a generalization of the REIS [16] and WERTZ [14] methods relating to inversion of VANDERMONDE matrix with distinct eigenvalues. The computation is based on

$$\mathbf{W} = \mathbf{V}^{-1} = \mathbf{D}^{-1}\mathbf{Q} \tag{108}$$

where for distinct eigenvalues

$$\mathbf{Q} = \begin{bmatrix} q_{n-1}(\lambda_1) & \cdots & q_2(\lambda_1) & q_1(\lambda_1) & 1 \\ q_{n-1}(\lambda_2) & \cdots & q_2(\lambda_2) & q_1(\lambda_2) & 1 \\ q_{n-1}(\lambda_n) & \cdots & q_2(\lambda_n) & q_1(\lambda_n) & 1 \end{bmatrix} \tag{109}$$

and

$$\mathbf{D}^{-1} = \mathbf{diag}\left[\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_n}\right]. \tag{110}$$

Defining the characteristic equation in form

$$D(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i) = \sum_{i=0}^{n} \bar{a}_i \lambda^{n-i} \tag{111}$$

where the bar denotes a distinction, the elements of matrix $\mathbf{Q}$ are given in a simple algorithm by HORNER's scheme:

$$q_{k+1}(\lambda_i) = q_k(\lambda_i)\lambda_i + \bar{a}_{k+1} \qquad \begin{matrix}(i = 1, 2, \ldots, n) \\ (k = 0, \ldots, n-2)\end{matrix} \tag{112}$$

and

$$q_1(\lambda_i) = 1 \tag{113}$$

where subscript k refers to the degree in λ_i. The elements of matrix $\mathbf{D}$ are

$$d_i = \frac{d}{d\lambda} D(\lambda)\Big|_{\lambda=\lambda_i} = \prod_{\substack{t=1\\ j\neq i}}^{n} (\lambda_i - \lambda_j). \tag{114}$$

For multiple eigenvalues Eq. (108) further holds with a pseudodiagonal matrix

$$\mathbf{D}^{-1} = \mathbf{diag}[\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_m] \tag{115}$$

and a hypermatrix

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \cdot \\ \cdot \\ \mathbf{Q}_m \end{bmatrix} \tag{116}$$

where the $k_i \times n$ matrices $\mathbf{Q}_i$ are given as

$$\mathbf{Q}_i = \begin{bmatrix} q_{n-1}(\lambda_i) & \ldots & q_2(\lambda_i) & q_1(\lambda) & 1 \\ q_{n-1}^{(1)}(\lambda_i) & \ldots & q_2^{(1)}(\lambda_i) & 1 & 0 \\ \cdot & & & & \\ \cdot & & & & \\ q_{n-1}^{(k_i-1)}(\lambda_i) \ldots & & \ldots & 0 & 0 \end{bmatrix} \tag{117}$$

where $1, \ldots k_i - 1$ are superscripts and do not refer to derivation. The $k_i \times k_i$ matrices $\mathbf{D}_i$ have the following form

$$\mathbf{D}_i = \begin{bmatrix} & & & d_1^{(i)} \\ & & d_1^{(i)} & d_2^{(i)} \\ & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot \\ d_1^{(i)} & d_2^{(i)} & \ldots & d_{k_i}^{(i)} \end{bmatrix} \tag{118}$$

where

$$d_j^{(i)} = \frac{D^{k_i+j-i}(\lambda)|_{\lambda=\lambda_i}}{(k_i + j - 1)!}, \quad (j = 1, 2, \ldots, k_i). \tag{119}$$

Notice that the inverse $\mathbf{D}^{-1}$ of $\mathbf{D}$ is given as

$$\mathbf{D}_i^{-1} = \begin{bmatrix} \delta_{k_i}^{(i)} \ldots & \ldots \delta_2^{(i)} & \delta_1^{(i)} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot\ \cdot & \cdot & \\ \delta_2^{(i)} & \delta_1^{(i)} & \\ \delta_1^{(i)} & & \end{bmatrix} \tag{120}$$

where

$$\delta_j^{(i)} = -\delta_1^{(i)} \sum_{s=1}^{j-1} d_{j+1-s}^{(i)} \delta_s^{(i)}, \quad (j = 2, 3, \ldots, k_i) \tag{121}$$

and

$$\delta_1^{(i)} = \frac{1}{d_1^{(i)}} . \tag{122}$$

The elements of the matrix **Q** can be obtained from Eqs (112), (113) and, using the following recursive algorithm

$$q_{k+1}^{(p)}(\lambda_i) = q_k^{(p)}(\lambda_i)\, \lambda_i + q_k^{(p-1)}(\lambda_i) \qquad \begin{matrix} (p = 1, \ldots, k_i - 1) \\ (k = p, \ldots, n-2) \end{matrix} \tag{123}$$

also

$$q_p^{(i)}(\lambda_i) = 1. \tag{124}$$

Thus Eqs (108) and (115) ... (124) can effectively be used in obtaining the inverse $\mathbf{W} = \mathbf{V}^{-1}$ of $\mathbf{V}$.

It is to be emphasized that on the one hand Eq. (119) can be replaced by

$$d_j^{(i)} = \frac{1}{(j-1)!} D_i^{(j-1)}(\lambda_i) \tag{125}$$

where $(j-1)$ means differentiation, and on the other hand,

$$\delta_1^{(i)} = A_{11}^{(i,i)} = A_{22}^{(i,i)} = A_{33}^{(i,i)} = \ldots = \frac{1}{D_i(\lambda_i)}$$

$$\delta_2^{(i)} = A_{12}^{(i,i)} = A_{23}^{(i,i)} = A_{34}^{(i,i)} = \ldots = \frac{1}{d\lambda_i}\left[\frac{1}{D_i(\lambda_i)}\right] = -\frac{D_i'(\lambda_i)}{D_i^2(\lambda_i)} \tag{126}$$

$$\delta_3^{(i)} = A_{13}^{(i,i)} = A_{24}^{(i,i)} = A_{35}^{(i,i)} = \ldots = \frac{1}{2!}\frac{d^2}{d\lambda_i^2}\left[\frac{1}{D_i(\lambda_i)}\right] = \frac{D_i'^2(\lambda_i)}{D_i^3(\lambda_i)} - \frac{D_i''(\lambda_i)}{2D_i^2(\lambda_i)}$$

$$\delta_p^{(i)} = A_{1p}^{(i,i)} = A_{2,p+1}^{(i,i)} = A_{3,p+2}^{(i,i)} = \frac{1}{(p-1)!}\frac{d^{p-1}}{d\lambda_i^{p-1}}\left[\frac{1}{D_i(\lambda_i)}\right].$$

Furthermore,

$$w_{jl}^{(i)} = \sum_{h=0}^{k_i-j} \delta_{k_i-j-h+1}\, q_{n-l}^{(h)}(\lambda_i) \qquad \begin{matrix} (j = 1, 2, \ldots, k_i) \\ (l = 1, 2, \ldots, n). \end{matrix} \tag{127}$$

The latter remarks show that in principle, the fourth method is equivalent to the third method yielding the corresponding matrix elements $w_{jl}^{(i)}$ by appropriate recursive formulas. Eq. (127) also connects the fourth and the second method.

We also remark that polynomials $q_k^{(s)}(\lambda)$, $(s = 0, 1, \ldots, k_i - 1,\ k = s, \ldots n-1)$ may be expressed in terms of $N_{n-k}(\lambda)$ and its derivatives, where $N_l(\lambda_i)$ $(l=0, 1, 2, \ldots n)$ are given in Eq. (51). The elements in the $(n-k)$-th column of matrix $\mathbf{Q}$ can be expressed as

$$\begin{aligned} q_k(\lambda) &= N_{n-k}(\lambda) \\ q_k^{(1)}(\lambda) &= \sum_{h=0}^{k-1} N_{n-h}(\lambda)\, \lambda^{k-h-1} \\ q_k^{(2)}(\lambda) &= \sum_{h=1}^{k-1} N'_{n-h}(\lambda)\, \lambda^{k-h-1} \qquad (k = 1, \ldots, n-1) \quad (128) \\ q_k^{(3)}(\lambda) &= \frac{1}{2!} \sum_{h=2}^{k-1} N''_{n-h}(\lambda)\, \lambda^{k-h-1} \end{aligned}$$

and in general

$$q_k^{(p)}(\lambda) = \frac{1}{(p-1)!} \sum_{h=p-1}^{k-1} N_{n-h}^{(p-1)}(\lambda)\, \lambda^{k-h-1}\,. \tag{129}$$

Here

$$N_n(\lambda) = N'_{n-1}(\lambda) = \frac{1}{2!} N''_{n-2}(\lambda) = \ldots = 1 \tag{130}$$

and

$$\begin{aligned} N_{n-k}(\lambda) &= \lambda^k + a_{n-1}\lambda^{k-1} + \ldots + a_{n-k+1}\lambda + a_{n-k} = \\ &= \lambda^k + \bar{a}_1\lambda^{k-1} + \ldots + \bar{a}_{k-1}\lambda + \bar{a}_k\,. \end{aligned} \tag{131}$$

Previously we observed that polynomials $N_l(\lambda_i)$ could also be obtained by HORNER's scheme. The same is valid also for $N'_l(\lambda_i)$, $(1/2)\,N''_l(\lambda_i)$ if coefficients $\bar{a}_{l'}$ are replaced by

$$\bar{a}_{l'}^{(1)} = N_{l'}(\lambda_i) \tag{132}$$

and

$$\bar{a}_{l'}^{(2)} = N'_{l'}(\lambda_i), \tag{133}$$

respectively. The procedure can be continued for higher derivatives.

Eqs (128) ... (131) put the connection between the second and fourth method in a new light.

13. Fourth Conclusion

This method avoids partial fraction expansion technique as well as LAPLACE transforms. Only simple recursive algorithms are used. For obtaining $\mathbf{D}$ the method requires repeated differentiation of the characteristic equation, which, however, is not very complicated and can further be facilited by Eq.

(125). The inverse matrix $\mathbf{D}^{-1}$ of $\mathbf{D}$ is also relatively easy to obtain. The polynomials $q_k^s(\lambda)$ involved are of the same structure, only various substitutions are necessary.

The drawback of the method is shown in Eq. (108) referring to matrix multiplication. Also some other numerical computations are necessary.

The fourth method can be regarded as a variant of the third method. Principally it gives perhaps less insight into the essence of the inverse VANDERMONDE matrix but at the same time it is more suitable for numerical computations and easily programmable for digital computers.

The insight can, however, be somewhat enhanced by the relationships in Eqs (125), (126) and (127) as well as (128) . . . (130).

14. An Illustrative Example

For the VANDERMONDE matrix given in the third example (in Chapter 8) by Eqs (112), (113); (123) and (124) the matrix $\mathbf{Q}$ can be obtained as

$$\mathbf{Q} = \begin{bmatrix} 12 & 40 & 51 & 31 & 9 & 1 \\ 12 & 28 & 23 & 8 & 1 & 0 \\ 12 & 16 & 7 & 1 & 0 & 0 \\ 6 & 23 & 34 & 23 & 8 & 1 \\ 3 & 10 & 12 & 6 & 1 & 0 \\ 4 & 16 & 25 & 19 & 7 & 1 \end{bmatrix}.$$

Applying Eq. (119)

$$d_1^{(1)} = 2,\ d_2^{(1)} = 5;\ d_3^{(1)} = 4;\ d_1^{(2)} = -1,\ d_2^{(2)} = 2;\ d_1^{(3)} = -8$$

yielding

$$\mathbf{D}_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 5 \\ 2 & 5 & 4 \end{bmatrix};\quad \mathbf{D}_2 = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix};\quad \mathbf{D}_3 = [-8].$$

From Eqs (121), (122)

$$\delta_1^{(1)} = 0.5,\ \delta_2^{(1)} = -1.25;\ \delta_3^{(1)} = 2.125;\ \delta_1^{(2)} = -1,\ \delta_2^{(2)} = -2,\ \delta_1^{(3)} = -0.125$$

yielding

$$\mathbf{D}^{-1} = \begin{bmatrix} 2.125 & -1.25 & 0.5 & 0 & 0 & 0 \\ -1.25 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.125 \end{bmatrix}.$$

Through Eq. (108) $\mathbf{W} = \mathbf{V}^{-1}$ is obtained. It is worth mentioning that

$$\mathbf{T}^{-1}\mathbf{D}^{-1} = \left[\begin{array}{ccc|cc|c} 0 & 0 & 1 & & & \\ 0 & 1 & 0 & & & \\ 1 & 0 & 0 & & & \\ \hline & & & 0 & 1 & \\ & & & 1 & 0 & \\ \hline & & & & & 1 \end{array}\right]$$

and premultiplying matrix $\mathbf{Q}$ with the latter means the change of rows in $\mathbf{Q}$, so finally

$$\mathbf{L}^{-1} = \mathbf{T}^{-1}\mathbf{D}^{-1}\mathbf{Q} = \begin{bmatrix} 12 & 16 & 7 & 1 & 0 & 0 \\ 12 & 28 & 23 & 8 & 1 & 0 \\ 12 & 40 & 51 & 31 & 9 & 1 \\ 3 & 10 & 12 & 6 & 1 & 0 \\ 6 & 23 & 34 & 24 & 8 & 1 \\ 4 & 16 & 25 & 19 & 7 & 1 \end{bmatrix}$$

is the inverse of the modal matrix

$$\mathbf{L} = \mathbf{T}\,\mathbf{V}$$

yielding the principal variant and satisfying Eq. (10). (For the elements of $\mathbf{T}$ and $\mathbf{T}^{-1}$ see the third example in Chapter 8.)

15. Final Conclusions

The purpose of this paper is to compare the various methods of conversion from phase-variable form to JORDAN canonical form, in case of multiple eigenvalues, or to diagonal form, in case of distinct eigenvalues. In such cases the confluent or original VANDERMONDE matrix may play an important role: that of a modal matrix. The determination of the inverse VANDERMONDE matrix is not a trivial one and three various methods were shown to obtain it. For sake of comparison a method completely eliminating the VANDERMONDE matrices was also shown.

The paper aims also at making a comparison and revealing the connection links between the various methods. At the same time the advantages and drawbacks of each method are also summarized. To the author's knowledge, no such a relatively complete review of the problem exists to now.

Some elaborated examples illustrate the theoretical considerations.

Besides of its reviewing and critic character, the paper pretends also to priority in some points:

1. To the author's knowledge, the paper is likely to be the first to give a complete solution of the first method for any number of repeated eigenvalues (See Eqs (5), (7), (10), (28) and (29) also Fig. 2). This method seems be the simplest and the most straightforward one.

2. The second method suggested originally by BECK and LANCE [19] is here derived supposing an unforced system subject to special initial conditions. The element $w_{jl}^{(i)}$ of the inverse VANDERMONDE matrix can be obtained by partial fraction expansion, see Eq. (49). The element $w_{jl}^{(i)}$ is the coefficient belonging to $(s-\lambda_i)^{-j}$ and simultaneously it is the coefficient of the i-th modal transient process of the first phase-variable subject to the initial condition of the l-th phase-variable $x_l(O)$ and belonging to the j-th multiplicity, see Eq. (54). The identification of the nature of the coefficients $w_{jl}^{(i)}$ gives a deep insight into the processes involved.

3. The connection between the first and the second method is manifest from Eq. (62) where elements $C_j^{(i)}$ of row vector $\mathbf{c}^T$ are given in terms of matrix elements $w_{jn}^{(i)}$ and coefficients b_k, that is, elements b_k of row vector $\mathbf{c}_0^T$. The same connection is also expressed in Eq. (79).

4. The paper shows how HERMITE interpolation, as a generalization of LAGRANGE interpolation, may facilitate the inversion of the confluent VANDERMONDE matrix. In our case, we have to do with special HERMITE—KRONECKER polynomials $P_{j(i)}(\lambda)$. The auxiliary coefficients $A_{jl}^{(i,i)}$ may be obtained from Eqs (96), (97), (98) and (101), (102) and (105). The procedure is easy to be continued if necessary.

The HERMITE—KRONECKER polynomial, here introduced perhaps for the first case, may be precalculated as shown by an example in Table 3.

Two special cases, that of distinct eigenvalues and that of a single repeated eigenvalue, are quite simplified. The first case leads to LAGRANGE interpolation, the second one to binomial interpolations.

5. For numerical computations the KAUFMAN method seems to be the best because it avoids differentiation and partial fraction expansion technique as well as LAPLACE transforms. These are replaced by relatively simple recursive algorithms.

For manual calculation, expressions with derivatives are perhaps more perspicuous, therefore we derived Eqs (125) . . . (127) as well as Eqs (128) . . . (130). These groups of equations are connection links between the second, third and fourth methods.

6. Although the second method determines the columns of the inverse VANDERMONDE matrix whereas the third and the fourth method their rows, all the three methods are essentially the same. This is pointed out by the fact that on the one hand

$$w_{jl}^{(i)} = \lim_{s\to\lambda_i} \frac{1}{(k_i-j)!} \frac{d^{(k_i-j)}}{ds^{(k_i-j)}} \frac{N_l(s)}{D_i(s)} \tag{134}$$

which yields the first method, and on the other hand, according to the LEIBNIZ rule of derivation

$$w_{jl}^{(i)} = \lim_{s\to\lambda_i} \sum_{k=0}^{k_i-j} \left[\frac{1}{(k_i-j-k)!} N_l^{k_i-j-k}(s)\right]\left[\frac{1}{k!}\frac{d^k}{ds^k}\frac{1}{D_i(s)}\right] \tag{135}$$

which yields the second or third method as

$$\delta_k^{(i)} = \lim_{s\to\lambda_i} \frac{1}{(k-1)!}\frac{d^{(k-1)}}{ds^{(k-1)}}\frac{1}{D_i(s)} = A_{1,k}^{(i,i)} = A_{2,k+1}^{(i.i)} = A_{3,k+2}^{(i.i)} \tag{136}$$

and, by the way,

$$d_k^{(i)} = \lim_{s\to\lambda_i} \frac{1}{(k-1)!}\frac{d^{(k-1)}}{ds^{(k-1)}} D_i(s). \tag{137}$$

Also

$$q_{n-l}^{(p)}(\lambda_i) = \frac{1}{p!} N_l^{(p)}. \tag{138}$$

7. In contrast to other papers dealing with the same topic such as [19, 21, 24, 25], this paper also points to the fact that by the inversion of the VANDERMONDE matrix the problem of transforming the phase-variable form into a JORDAN canonical form is but partly solved.

The paper emphasizes that VANDERMONDE matrices are not the best modal matrices. In order to obtain the simplest JORDAN form, that is, the principal variant, $\mathbf{b}$ must assume the form given in Eq. (10). This can be realized by introducing a commutativity transformation matrix $\mathbf{T}$. The commutativity matrix and its inverse are both upper triangular matrices, each having equal elements in each diagonal. Moreover the elements of $\mathbf{T}$ are given by the element $w_{jn}^{(i)}$ in the last column of $\mathbf{W} = \mathbf{V}^{-1}$, see Eq. (75). The elements of $\mathbf{T}^{-1}$ are relatively easy to compute as shown in Tables 1 and 2. Recurrent relationships are also given, see Eqs (77) and (78).
By the way

$$\tau_k^{(i)} = d_k^{(i)} = \frac{1}{(k-1)!} D_i^{(k-1)}(\lambda_i) \tag{139}$$

and

$$t_k^{(i)} = \delta_k^{(i)} = \frac{1}{(k-1)!}\left[\frac{d^{(k-1)}}{ds^{(k-1)}}\frac{1}{D_i(s)}\right]\Bigg|_{\lambda=\lambda_i}. \tag{140}$$

Summary

This paper compares four methods for the conversion from phase-variable form to canonical JORDAN form: the LAPLACE transform method, the inversion of the confluent VANDERMONDE matrix by LAPLACE transforms, by HERMITE-KRONECKER polynomials and by recursive formulas. It is emphasized that the last three methods are only partial solutions to the problem and commutativity matrices should also be introduced.

The advantages and disadvantages of each method are analyzed. Some examples are also given for the sake of illustration.

References

1. RAUE, D. S.: A Simplified Transformation to Phase Variable Form. IEEE Trans. on Automatic Control, *AC*-8 (1963), pp. 608—619.
2. JOHNSONN, C. D.—WONHAM, W. M.: A Note on the Transformation to Canonical (Phase-Variable) Forms. IEEE Trans. on Automatic Control, *AC*-9 (1964), pp. 312—313.
3. SILVERMAN, L. M.: Transformation of Time-Variable Systems to Canonical (Phase-Variable) Form. IEEE Trans. on Automatic Control, *AC*-11 (1966), pp. 300—303.
4. MUFTI, I. H.: On the Reduction of a System to Canonical (Phase-Variable) Form. IEEE Trans. on Automatic Control, *AC*-10 (1965), pp. 206—207.
5. TUEL, W. C.: Canonical Form for Linear Systems. IBM Research Rep., RJ 375, 1966.
6. TUEL, W. C.: On the Transformation to (Phase-Variable) Canonical Form. IEEE Trans. on Automatic Control, *AC*-11 (1966), p. 107.
7. CHIDAMBARA, M. R.: Comment on "On the Transformation to (Phase-Variable) Canonical Form" IEEE Trans. on Automatic Control, *AC*-11 (1966), pp. 607—608.
8. LUENBERGER, D. G.: Canonical Forms for Linear Multivariable Systems. IEEE Trans. on Automatic Control, *AC*-12 (1967), pp. 290—293.
9. RAMASWAMI, B.—RAMAR, K.: Transformation to the Phase-Variable Canonical Form. IEEE Trans. on Automatic Control, *AC*-13 (1968), pp. 746—747.
10. RAMASWAMI, B.—RAMAR, K.: On the Transformation of Time-Variable Systems to the Phase-Variable Canonical Form. IEEE Trans. on Automatic Control, *AC*-14 (1969) pp. 417—419.
11. TOU, J. T.: Determination of the Inverse Vandermonde Matrix. IEEE Trans. on Automatic Control, *AC*-9 (1964) July. (Correspondence.)
12. TOU, J. T.: Modern Control Theory. McGraw-Hill, New York 1964.
13. BRULÉ, J. D.: A Note on the Vandermonde Determinant. IEEE Trans. on Automatic Control, *AC*-9 (1964), July, pp. 314—315. (Correspondence.)
14. WERTZ, H. J.: On the Numerical Inversion of the Recurrent Problem: the Vandermonde Matrix. IEEE Trans. on Automatic Control, *AC*-10 (1965), Oct., p. 492. (Correspondence.)
15. WU, S. H.: On the Inverse of Vandermonde Matrix. IEEE Trans. on Automatic Control, *AC*-11 (1966), Oct. p. 769. (Correspondence.)
16. REIS, G. C.: A Matrix Formulation for the Inverse Vandermonde Matrix. IEEE Trans. on Automatic Control, *AC*-12 (1967), Dec., p. 793. (Correspondence.)
17. SYSLO, M.: The Inverse for the Real Similarity Transform of a Companion Matrix. IEEE Trans. on Automatic Control, *AC*-16 (1971), Oct. p. 491—492.
18. CSÁKI, F.: Some Remarks on Modal Transformations. Periodica Polytechnica, Electrical Engineering, **16** (1972), pp. 333—346.
19. BECK, R.—LANCE, G. M.: Direct and Inverse Transformations between Phase Variable and Canonical Forms. Joint Automatic Control Conference of the American Automatic Council. Stanford, 1972, pp. 133—137.
20. BECK, R.: Analysis and Pole-Zero Synthesis Using Stte Vari able Techniques. Ph. D. Dissertation, The University of Iowa, May, 1972.
21. KAUFMAN, I.: The Inversion of the Vandermonde Matrix and the Transformation to the Jordan Canonical Form. IEEE Trans. on Automatic Control, *AC*-14 (1969), Dec., pp. 774—777. (Correspondence.)
22. KAN, P. E.: An Inversion Procedure of the Generalized Vandermonde Matrix. IEEE Trans. on Automatic Control, *AC*-16 (1971), Oct. pp. 492—493. (Techn. Notes and Corresp.)
23. SCHAPPELLE, R. H.: The Inverse of the Confluent Vandermonde Matrix. IEEE Trans. on Automatic Control, *AC*-17 (1 972), Oct. 724—725.

24. SCHAPPELLE, R. H.: The Inverse of the General Confluent Vandermonde Matrix. General Dynamics Convair Aerosp. Div. Rep. GDC-BBA69-001, 1969.
25. GAUTSCHI, W.: On Inverses of Vandermonde and Confluent Vandermonde Matrices. Numer. Math. **4** (1962), pp. 117—123.
26. RAO, K. R.—AHMED, N.: Evaluation of Transition Matrices. IEEE Trans. on Automatic Control, *AC*-14 (1969), Dec., pp. 779—780.
27. ATHANS, M.—FALB, P.: Optimal Control. McGraw-Hill, New York 1966.
28. CSÁKI, F.: Modern Control Theories. Nonlinear, Optimal and Adaptive Systems. Akadémiai Kiadó, Budapest, 1972.
29. CSÁKI, F.: Die Zustandsraum-Methode in der Regelungstechnik. VDI Verlag Düsseldorf — Akadémiai Kiadó, Budapest 1973.
30. DERUSSO, P. M.—ROY, R. J.—CLOSE, C. M.: State Variables for Engineers. John Wiley, New York 1965.
31. DORF, R.: Time-Domain Analysis and Design of Control Systems. Reading. Mass. Addison-Wesley, 1965.
32. OGATA, K.: State Space Analysis of Control Systems. Englewood Cliffs, 1967.
33. SAUCEDO, R.—SCHIRING, E.: Introduction to Continuous and Digital Control Systems. McMillan, New York 1968.
34. SCHULTZ, D.—MELSA, J.: State Functions and Linear Control Systems. McGraw-Hill, New York 1967.
35. SCHWARZ, R.—FRIEDLAND, B.: Linear Systems. McGraw-Hill, New York 1965.
36. TIMOTHY, L.—BONA, B.: State Space Analysis: An Introduction. McGraw-Hill, New York 1968.
37. TAKAHASHI, Y.—RABINS, M.—AUSLANDER, D.: Control and Dynamic Systems. Reading Mass-Addison Wesley, 1970.
38. HERMITE, CH.: Sur la formule d'interpolation de Lagrange. Journal für die reine u. angewandte Mathematik, **84** (1878), p. 70.
39. SPITZBART, A.: A Generalization of Hermite's Interpolation Formula. American Mathematical Monthly, **67** (1960), pp. 42—46.
40. JOHANSEN, P.: Über osculierende Interpolation. Skandinavisk Aktuarietidskrift, 1931, Uppsala.

Prof. Dr. Frigyes CSÁKI, H-1511 Budapest