OPTIMUM CONTROL OF DISTRIBUTED PARAMETER SYSTEMS BY DIGITAL COMPUTER

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Introduction

The recent extremely accelerated development of control theory allows for the investigation of systems the dynamic behaviour of which is adequately described by ordinary differential equations. However, the increasing complexity of processes to be controlled requires the development of highly accurate mathematic models to simulate the process. In case the mathematic, or other, model considers also the space and time dependence of parameters determining the dynamic behaviour, the model used to describe the process as a distributed parameter model, that is, in case of mathematic models the dynamic behaviour is described by partial differential equations rather than by ordinary ones [3, 6].

On the other hand, the systems developed must be optimum from a certain defined aspect, an important requirement in control techniques. Exact determination of the aspect from which a system must be optimum is very important, as a system highly satisfying a particular requirement might display less favourable behaviour in other respects. The aim of optimization is usually expressed by a criterion that can be fulfilled through determining the extremes, usually the minimum, less frequently the maximum, of a functional. One of the simplest, and most frequent, optimizing problems is to minimize a reasonably selected square integral criterion. Generally speaking, the task is to determine one particular control vector that satisfies a defined condition with respect to the system, with the system, that is the functional relationship between the known state and control vector $\mathbf{x}(\mathbf{y}, t)$ and $\mathbf{u}(\mathbf{y}, t)$ resp. (Fig. 1), [3, 6, 19]. As the goal is usually to minimize a functional, the methods of variation calculus concerned with the same fundamental problem have found extended use in solving optimum control problems. A deviation from the classic form of variation calculus is only represented here by the fact that the control vector $\mathbf{u}(\mathbf{y}, t)$ is introduced into the functional to be minimized, that is, into the cost function. However, this deviation set no limits to the deduction of the optimiz-

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ing problem from one of the three fundamental problems of the classic variation calculus, the choice of the problem from which deduction is made depending on the nature of the problem. It should be noted that there are other methods known to solve optimizing problems, namely, the optimizing principles suggested by PONTRJAGIN et al. and BELLMAN et al. [6] which are, however, not discussed here. PONTRJAGIN's method is based on a generalization of the variation calculus while that of BELLMAN uses the so-called "dynamic programming".

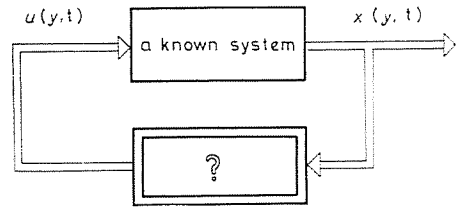

Fig. 1

A disadvantage of the available methods to solve optimizing problems is that only the simplest systems of lumped parameters can be handled manually, more complex systems would require the use of analogue or digital computers. However, the distributed parameter approach requires the use of computers even in case of simplest systems [2, 3, 6, 18, 24].

It must be noted that optimizing problems and techniques have widely been used in several fields of system theory such as system identification, adaptive control, stochastic problems etc., in addition to their use in optimum control problems involving optimum regulation discussed in this study [1, 2, 3, 5, 9, 11, 15, 17, 20, 25].

Symbols used

Ω — closed domain of the Euclidean m-dimensional space
ω — boundary of the closed region
$\mathbf{y}$ — space vector in the m-dimension space
t — time
$\mathbf{x}(\mathbf{y}, t)$ — state vector of n element, $\mathbf{x} \in \Omega$
$\mathbf{u}(\mathbf{y}, t)$ — control vector of r element, $\mathbf{u} \in \Omega$
λ — Lagrangian multiplicator

1. Problem statement

Prior to be engaged in discussing of the distributed parameter systems, it seems reasonable to formulate the problem for continuous lumped parameter systems, in particular for processes where the initial state is known and no constraints are set on state and control vectors in the course of the process or in the final state.

Equation system to describe the process is [6, 24]:

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, t) \tag{1}$$

where $\dot{\mathbf{x}}$ — denotes the derivate of $\mathbf{x}$ with respect to t,
$\mathbf{x}$ — the n-vector state, and
$\mathbf{u}$ — the control vector with r elements.

Let the investigation be performed in interval $t \in (t_0, t_f)$, with the known state of the system at instant t_0 being

$$\mathbf{x}(t_0) = \mathbf{x}_0 . \tag{2}$$

As a criterion, assume that the expression known from the most general, that is, the BOLZA-problem of variation calculus, has to be minimized [5]:

$$I = \Theta [\mathbf{x}(t), t]\Big|_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} \varphi [\mathbf{x}(t), \mathbf{u}(t), t] \, dt \tag{3}$$

where Θ, φ — scalar functions of the given arguments, defined in the closed m-dimensional domain Ω of the space and at boundary ω of this domain.

The functional, called cost function, can be described by the help of the multiplicator λ, similarly known from variation calculus, as follows [5, 6, 20]:

$$I = \Theta [\mathbf{x}(t), t]\Big|_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} \{\varphi [\mathbf{x}(t), \mathbf{u}(t), t] + \boldsymbol{\lambda}^T(t) [f[\mathbf{x}(t), \mathbf{u}(t), t] - \dot{\mathbf{x}}]\} \, dt \tag{4}$$

and the scalar Hamilton function as:

$$H [\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t),] = \varphi [\mathbf{x}(t), \mathbf{u}(t), t] + \boldsymbol{\lambda}(t) [f[\mathbf{x}(t), \mathbf{u}(t), t]]. \tag{5}$$

By the help of this function, the cost function is:

$$I = \Theta [\mathbf{x}(t), t]\Big|_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} \{H [\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] - \boldsymbol{\lambda}^T(t)\dot{\mathbf{x}}\} \, dt \tag{6}$$

and

$$I = \{\Theta [\mathbf{x}(t), t] - \boldsymbol{\lambda}^T(t) \mathbf{x}(t)\}\Big|_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} \{H [\mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), t] + \dot{\boldsymbol{\lambda}}^T \mathbf{x}(t)\} \, dt. \tag{7}$$

The optimum is determined using the first variation of I:

$$\delta I = \left\{\delta \mathbf{x}^T \left[\frac{\partial \Theta}{\partial \mathbf{x}} - \boldsymbol{\lambda}\right]\right\}\Big|_{t=t_0}^{t=t_f} + \int_{t_0}^{t_f} \left\{\delta \mathbf{x}^T \left[\frac{\partial H}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}\right] + \delta \mathbf{u}^T \left[\frac{\partial H}{\partial \mathbf{u}}\right]\right\} dt. \tag{8}$$

From this, the relationships giving the necessary conditions for minimum are:

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

$$\dot{x} = f(\mathbf{x}, \mathbf{u}, t) = \frac{\partial H}{\partial \lambda} \tag{9}$$

$$\frac{\partial H}{\partial \mathbf{u}} = 0$$

and in the initial state:

$$\mathbf{x}(t_0) = \mathbf{x}_0 \tag{10}$$

while in the final state:

$$\lambda(t_f) = \frac{\partial \Theta[\mathbf{x}(t_f), t_f]}{\partial \mathbf{x}(t_f)} . \tag{11}$$

From the equation system the optimum $\mathbf{u}(t)$ can be determined.

2. Distributed parameter systems

(Formulation of distributed system problems)

Let the same problem be investigated for the case of distributed parameter systems to be described by partial differential equations which must always be completed by auxiliary (initial and boundary) conditions.

Derivate the m-dimensional spatial co-ordinate vector $\mathbf{y}$, defined in the closed domain Ω of the space [2, 11]:

$$\mathbf{y}^T = [y_1, y_2, \ldots y_m] \tag{12}$$

where the superscript T denotes matrix transpose.

State of the system at any instant of time t is described by the state vector $\mathbf{x}$:

$$\mathbf{x}(\mathbf{y}, t) = \begin{bmatrix} x_1(y_1, y_2 \ldots y_m, t) \\ x_2(y_1, y_2 \ldots y_m, t) \\ \cdot \\ \cdot \\ \cdot \\ x_n(y_1, y_2 \ldots y_m, t) \end{bmatrix} . \tag{13}$$

The space and time dependent control vector is:

$$\mathbf{u}(\mathbf{y}, t) = \begin{bmatrix} u_1(y_1, y_2 \ldots y_m, t) \\ u_2(y_1, y_2 \ldots y_m, t) \\ \cdot \\ \cdot \\ \cdot \\ u_r(y_1, y_2 \ldots y_m, t) \end{bmatrix} \tag{14}$$

[$\mathbf{u}(\mathbf{y}, t)$ may run over the whole domain Ω.] Let us consider the systems which are described by the following vector differential equation:

$$\frac{\partial \mathbf{x}(\mathbf{y}, t)}{\partial t} = f\left[\mathbf{y}, \mathbf{x}(\mathbf{y}, t), \frac{\partial \mathbf{x}(\mathbf{y}, t)}{\partial y_1} \ldots \frac{\partial \mathbf{x}(\mathbf{y}, t)}{\partial y_m} \ldots \frac{\partial^K \mathbf{x}(\mathbf{y}, t)}{\partial y_1^{k_1} \partial y_2^{k_2} \ldots \partial y_m^{k_m}}, \ldots \right.$$
$$\left. \ldots \mathbf{u}(\mathbf{y}, t), t\right] \tag{15}$$

where:

$$K = \sum_{L=1}^{m} k_L \tag{16}$$

and:

$$\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{bmatrix}. \tag{17}$$

($\mathbf{f}$ denoting the function vector).

Both the initial and boundary conditions are known at boundary ω and at instant t_0, that is

$$\mathbf{x}(\mathbf{y}, t_0) \tag{18}$$

and the values

$$\left.\frac{\partial \mathbf{x}(\mathbf{y}, t)}{\partial y_1}\right|_{\omega}, \ldots \left.\frac{\partial \mathbf{x}(\mathbf{y}, t)}{\partial y_m}\right|_{\omega}, \ldots \left.\frac{\partial^K \mathbf{x}(\mathbf{y}, t)}{\partial y_1^{k_1}, \partial y_2^{k_2} \ldots y_m^{k_m}}\right|_{\omega} \tag{19}$$

are known, too.

Introducing simplifications, the equation will take following shape:

$$\frac{\partial \mathbf{x}(\mathbf{y}, t)}{\partial t} = f\left[\mathbf{y}, \mathbf{x}(\mathbf{y}, t), \frac{\partial^{\Sigma k} \mathbf{x}(\mathbf{y}, t)}{\partial y^{\Sigma k}}, \mathbf{u}(\mathbf{y}, t), t\right]. \tag{20}$$

Now, the problem is to determine the control vector, $\mathbf{u}(\mathbf{y}, t)$ where the cost function given below is minimum:

$$I = \int_{\Omega} \Theta [\mathbf{x}(\mathbf{y}, t_f), t_f] \, d\Omega + \int_{t_0}^{t_f} \int_{\Omega} \varphi \left[\mathbf{y}, \mathbf{x}(\mathbf{y}, t), \frac{\partial^{\Sigma k} \mathbf{x}(\mathbf{y}, t)}{\partial y^{\Sigma k}}, \mathbf{u}(\mathbf{y}, t), t \right] d\Omega \, dt \quad (21)$$

(where Θ and φ are, as usual, scalar functions of the given arguments).

Deriving the Hamilton function in the way used in case of lumped parameter systems, that is, by the help of the Lagrangian multiplicator λ,

$$H[\mathbf{x}, \mathbf{u}, \lambda, t] = \varphi[\mathbf{x}, \mathbf{u}, t] + \lambda^T(y, t) \cdot f(\mathbf{x}, \mathbf{u}, t), \quad (22)$$

the cost function used as a criterion can be described as follows:

$$I = \int_{\Omega} \Theta d\Omega + \int_{t_0}^{t_f} \int_{\Omega} \left[H - \lambda^T(\mathbf{y}, t) \frac{\partial \mathbf{x}(\mathbf{y}, t)}{\partial t} \right] d\Omega \, dt. \quad (23)$$

With constant t_f and Ω, the problem of determining extremes can be solved by formulating the first variation of I.

3. Computation method

So far only continuous functions have been used in expressions describing the system. However, in investigating distributed parameter systems, the differential equation(s) described the system must usually be replaced by difference equations, that is, discretized using one of the difference methods. To do so, two possibilities are available. First, the differential equations describing the system are discretized to provide thus difference equations, and, on the other hand, the process itself can be resolved, in space and/or in time, into final terms [4, 7, 8, 10]. Models so obtained are approximations which have to satisfy some convergence and stability criteria. However, this problem is not dealt with here in details.

It follows that the formulae obtained in the foregoing should reasonably be described by discrete variables in a discrete form in order to make them available for direct use in computers, and, actually, this is possible according to the rules of variation calculus.

Let the n-element state vector and the r-element input signal vector of a discrete or even nonlinear system be $\mathbf{x}_k$ and $\mathbf{u}_k$, respectively, at k-th instant of time.

At $(k + 1)$-th instant, the state of the system can be characterized with the relationship:

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, k). \quad (24)$$

Here the formulation of the problem is quite similar to that described previously the task being here the determination of extremes, too, that is, to minimize the cost function

$$I = \Theta [\mathbf{x}_k, k] \Big|_{k=k_0}^{k=k_f} + \sum_{k=k_0}^{k_f-1} \varphi [\mathbf{x}_k, \mathbf{u}_k, k]. \tag{25}$$

Initial and final points of the process are k_0 and k_f, respectively.

Making use of the Lagrangian multiplicator $\boldsymbol{\lambda}_k$, the cost function can be rewritten as:

$$I = \Theta [\mathbf{x}_k, k] \Big|_{k=k_0}^{k=k_f} + \sum_{k=k_0}^{k_f-1} \varphi [\mathbf{x}_k, \mathbf{u}_k, k] - \boldsymbol{\lambda}_{k+1}^T [\mathbf{x}_{k+1} - f(\mathbf{x}_k, \mathbf{u}_k, k)]. \tag{26}$$

The Hamilton function is then formed as:

$$H [\mathbf{x}_k, \mathbf{u}_k, \boldsymbol{\lambda}_{k+1}, k] = H_k = \varphi [\mathbf{x}_k, \mathbf{u}_k, k] + \boldsymbol{\lambda}_{k+1}^T f(\mathbf{x}_k, \mathbf{u}_k, k) \tag{27}$$

and thus the cost function is:

$$I = \Theta [\mathbf{x}_k, k] \Big|_{k=k_0}^{k=k_f} + \sum_{k=k_0}^{k_f-1} [H_k - \boldsymbol{\lambda}_{k+1}^T \mathbf{x}_{k+1}]. \tag{28}$$

Similarly to the method used in case of continuous systems, the solution proceeds as follows: The difference equation system, describing the distributed parameter linear system, is formulated in matrix form, using a discretizing in both space and time:

$$\mathbf{x}_{k+1} = \mathbf{A} \cdot \mathbf{x}_k + \mathbf{B} \cdot \mathbf{u}_k \tag{29}$$

and

$$\mathbf{x}(0) = \mathbf{x}_0$$
$$k = 0, 1, 2 \ldots k_f \tag{30}$$

where matrices $\mathbf{A}$ and $\mathbf{B}$ may be both space and time dependent.

With introducing reasonably selected weighting matrices, $\mathbf{Q}$ and $\mathbf{R}$, which may be similarly the functions of space and of time level k, the Hamilton function will be:

$$H = \frac{1}{2} [\mathbf{x}_k^T \mathbf{Q} \mathbf{x}_k] + \frac{1}{2} [\mathbf{u}_k^T \mathbf{R} \mathbf{u}_k] + \boldsymbol{\lambda}_{k+1}^T [\mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k] \tag{31}$$

and

$$\lambda_k = \mathbf{Q}\mathbf{x}_k + \mathbf{A}^T \boldsymbol{\lambda}_{k+1}. \tag{32}$$

Thus, with $\boldsymbol{\lambda}_k$ known, the multiplicator $\boldsymbol{\lambda}_{k+1}$ of the new time level can be determined if $\mathbf{A}^{-1}$ exists. This condition is fulfilled as the transfer matrix is $\mathbf{A}$.

This can be easily understood if the homogeneous term of the discrete state equation is considered:

$$\mathbf{x}(t_{k+1}) = \mathbf{A}(t_{k+1}, t_k)\,\mathbf{x}(t_k).$$

In the final state:

$$\boldsymbol{\lambda}(k_f) = \mathbf{S} \cdot \mathbf{x}(k_f)$$

$$\mathbf{S} = \frac{\partial \Theta(k_f)}{\partial \mathbf{x}(k_f)} [\mathbf{x}(k_f)]\,. \tag{33}$$

With the use of discrete maximum principle:

$$\frac{\partial H}{\partial \mathbf{u}_k} = \mathbf{0} = \mathbf{R}\mathbf{u}_k + \mathbf{B}^T \boldsymbol{\lambda}_{k+1}. \tag{34}$$

To accomplish optimatization, the following linear difference equation system must be solved (Fig. 2):

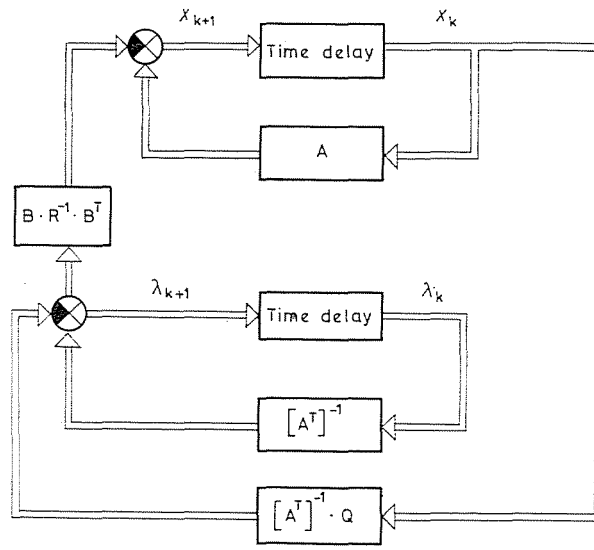

Fig. 2

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \boldsymbol{\lambda}_{k+1} \tag{35}$$

$$\boldsymbol{\lambda}_k = \mathbf{Q}\mathbf{x}_k + \mathbf{A}^T \boldsymbol{\lambda}_{k+1}$$

at the initial time:

$$\mathbf{x}(k_0) = \mathbf{x}_0 \tag{36}$$

at the final time:

$$\boldsymbol{\lambda}(k_f) = \mathbf{S}\mathbf{x}_k\,. \tag{37}$$

Now, assume that

$$\boldsymbol{\lambda}_k = \mathbf{P}_k \mathbf{x}_k \,. \tag{38}$$

Rearranging and substituting:

$$\begin{aligned} \mathbf{P}_k \mathbf{x}_k &= \mathbf{Q}\mathbf{x}_k + \mathbf{A}^T \mathbf{P}_{k+1} [\mathbf{I} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \mathbf{P}_{k+1}]^{-1} \cdot \mathbf{A}\mathbf{x}_k \\ \mathbf{P}_k &= \mathbf{Q} + \mathbf{A}^T [\mathbf{P}_{k+1}^{-1} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T]^{-1} \cdot \mathbf{A} \,. \end{aligned} \tag{39}$$

In the final state:

$$\mathbf{P}_{kf} = \mathbf{S} \,. \tag{40}$$

With the matrices describing systems **A** and **B** and the weighting matrices **Q** and **R** known, $\mathbf{P}(k)$ can be determined if the above equation is solved proceeding backwards, from $k = k_f$ to $k = 0$. With the knowledge of this, the relation between $\mathbf{x}_k$ and the optimum $\mathbf{u}_k$ at the k-th time level will be:

$$\mathbf{u}_k = -\mathbf{R}^{-1}\mathbf{B}^T [\mathbf{P}_{k+1}^{-1} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T]^{-1}\mathbf{A}\mathbf{x}_k \tag{41}$$

(where a necessary condition of this result is that **Q** and **S** be negative while **R** positive definites). Thus, input signals of the system are obtained in a way similar to the function of the conventional control device, that is, the optimum control vector has been produced.

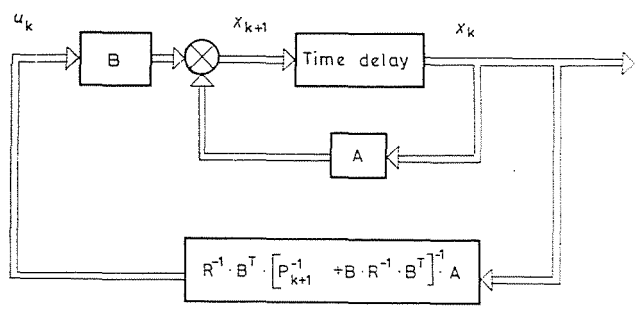

Fig. 3

4. Applications

4.1 *Thermal applications — temperature control*

A common problem in thermal engineering is the control of spatially distributed temperature of media flowing in ducts.

The typical optimization problem can be formulated as follows [15, 16, 18]:

With given medium flowing in a duct of negligible heat capacity, heated from outside, assume that the heating, i.e. the heat flux density affecting

the duct or the outer temperature having a known relation to heat flux density can be modified as a function of space and time. Temperature of the medium is assumed to be a controlled characteristic while the difference between internal and external temperatures is the control signal. For the sake of simplicity, the example chosen is a one-dimensional investigation (Fig. 4).

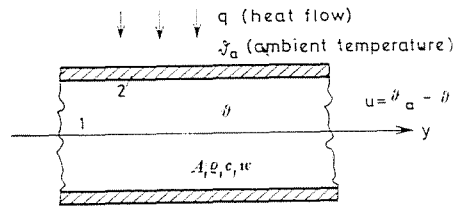

Fig. 4

Differential equation for the medium flowing in a duct is:

$$-a\frac{\partial \vartheta(y,t)}{\partial y} + b\,u(y,t) = \frac{\partial \vartheta(y,t)}{\partial t}\,. \tag{42}$$

Let the initial condition be

$$\vartheta(y,0) = 1 + \delta \cdot y; \quad \delta > 0, \quad 0 \leq y \leq y_f. \tag{43}$$

Boundary conditions:

$$\begin{aligned} &\frac{\partial \vartheta(y,t)}{\partial y} = 0 \text{ for } y = 0 \text{ and } 0 < t \leq t_f \\ &\frac{\partial \vartheta(y,t)}{\partial y} = 0 \text{ for } y = y_f \text{ and } 0 < t \leq t_f\,. \end{aligned} \tag{44}$$

Coefficients of the equation:

$$a = w$$

$$b = \frac{\alpha \cdot u^*}{A \cdot \varrho \cdot c}$$

where w — the flow rate of the medium, [m/sec]
u^* — internal perimeter of the tube (assuming a thin-walled pipe compared with its diameter) [m]
A — flow cross-section [m^2]
ϱ — density of flowing medium [kg/m^3]
c — specific heat of the medium [kcal/kg °C]
α — heat transfer coefficient [kcal/m^2 sec °C].

The task is to modify the difference ($u[y, t]$) between inner and outer temperature so as to minimize functional

$$I = \frac{1}{2}\int_0^t \int_0^{y_f} [Q_1 \vartheta^2(y,t) + R_1 u^2(y,t)]\, dy\, dt \tag{45}$$

Q_1 and R_1 are reasonably selected weighting factors.

There are several methods available to solve the problem numerically, such as to discretize in space and time the process, that is, to divide investigation time K and the tube length (i.e. the section investigated) into L parts and rewrite the differential equation into difference equation with forming a grid of time and space increments of Δt and Δy, respectively (e.g. on the basis of centered difference method).

Thus

$$\begin{aligned} 0 &\leq k \leq K \\ 0 &\leq 1 \leq L \end{aligned} \tag{46}$$

so that the following matrix equation is obtained:

$$\vartheta(k+1) = \mathbf{A}\vartheta(k) + \mathbf{B}\mathbf{u}(k) \tag{47}$$

where

$$\begin{aligned} \vartheta(k)^T &= [\vartheta_{0,k}, \quad \vartheta_{1,k} \ldots \vartheta_{L,k}] \\ \mathbf{u}(k)^T &= [u_{0,k}, \quad u_{1,k} \ldots u_{L,k}] \end{aligned} \tag{48}$$

A and **B** are bidiagonal and diagonal matrices containing coefficients a and b, respectively. The functional to be minimized is:

$$I = \frac{1}{2}\Delta y\, \Delta t \sum_{k=0}^{K-1} [\vartheta^T(k)\, \mathbf{Q}\, \vartheta(k) + \mathbf{u}^T(k)\, \mathbf{R}\, \mathbf{u}(k)] \tag{49}$$

and the Hamilton function:

$$H(k) = \frac{1}{2}\Delta y\, \Delta t\, [\vartheta^T(k)\, \mathbf{Q}\vartheta(k) + \mathbf{u}^T(k)\, \mathbf{R}\, \mathbf{u}(k)] + \lambda^T(k+1)\, [\mathbf{A}\vartheta(k) + \mathbf{B}\, \mathbf{u}(k)]. \tag{50}$$

Equation system to be solved:

$$\vartheta(k) = \left[\mathbf{I} + \frac{1}{\Delta y\, \Delta t}\, \mathbf{B}\, \mathbf{R}^{-1}\, \mathbf{B}^T\, \mathbf{P}(k)\right]^{-1} \mathbf{A}\vartheta(k-1) \tag{51}$$

$$\mathbf{u}(k) = -\frac{1}{\Delta y\, \Delta t}\, \mathbf{R}^{-1}\, \mathbf{B}^T\, \mathbf{A}^{-T}\, [\mathbf{P}(k) - \mathbf{Q}]\, \vartheta(k)\,. \tag{52}$$

Relationship to be written for $\mathbf{P}(k)$:

$$\mathbf{P}(k) = \Delta y \, \Delta t \, \mathbf{Q} + \mathbf{A}^T \mathbf{P}(k+1) \cdot \left[\mathbf{A}^{-1} + \frac{1}{\Delta y \, \Delta t} \mathbf{A}^{-1} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^T \mathbf{P}(k+1) \right]^{-1} \tag{53}$$

and

$$\mathbf{P}(K) = \mathbf{0}.$$

Initial condition as a function of time:

$$\vartheta(0) = 1 + \delta \cdot l \cdot y_f / L. \tag{54}$$

Let the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ be reasonably:

$$\mathbf{Q} = \mathbf{R} = \text{diag} < 1/2 \;\; 1 \ldots 1 \;\; 1/2 >.$$

To obtain a numerical solution, the grid of the domain investigated must be assumed, that is, the values of Δy and Δt must be determined. Here, considerations in relation with the method of finite differences apply (stability and convergence conditions).

Thus, on the basis of relations given above, the digital computer program to solve the distributed optimum problem can be formulated with ease.

4.2 *Thermal load of solid dielectrics*

Another example of applications is the problem of insulations frequently used in high voltage technique. Fig. 5 shows the homogeneous one-dimensional plane variation of the problem. Using this setup, transformer insulations can be simulated.

In this setup, the two electrodes having an a.c. voltage of an effective $U_0(t)$ value between them are insulated with homogeneous material. As

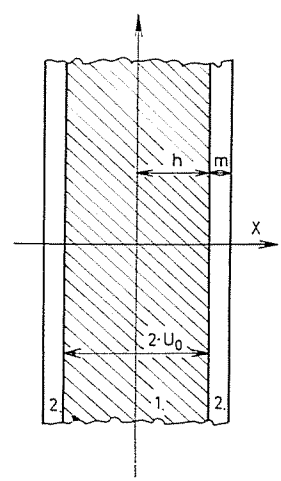

Fig. 5

known, a loss may occur in the dielectric, partly emerging into the ambient and, on the other hand, increasing the temperature of the dielectric. As the maximum temperature rise of dielectrics is determined by the so-called thermal groups, no higher temperature than that specified as a maximum for the dielectrics is permitted to develop in the dielectric. At the same time, economic reasons require that the operating temperature of the insulation be possibly close to the maximum value specified. Thus, in the present case, the maximum temperature of the dielectric can be considered as the controlled characteristic, the voltage applied to the dielectric being the control signal. Let us investigate now the relationships between these characteristics.

Let the partial differential equation describing the temperature distribution be simulated [21, 22, 23, 25]:

$$c_f \varrho_f \frac{\partial \vartheta(y,t)}{\partial t} = \lambda_1 \frac{\partial^2 \vartheta(y,t)}{\partial y^2} + q_b(y,t) \tag{55}$$

where c_f — the specific heat of the dielectric, [kcal/kg °C]
ϱ_f — the density of the dielectric, [kg/m^3]
λ_1 — the thermal conductivity, [kcal/m sec °C]
q_b — the internal heat source per unit volume and per unit time [kcal/m^3 sec].

Assuming dielectric losses alone [21, 22, 23]:

$$q_b(y,t) = p_0 e^{b[\vartheta(y,t)-\vartheta_0]} \text{ [kcal/sec m}^3\text{]} \tag{56}$$

where

$$p_0 = ku^2(t).$$

Necessary initial and boundary conditions are:

$$\vartheta(y,0) = \vartheta_0 = \text{const.} \qquad 0 \leq y \leq y_f \tag{57}$$

$$\left.\frac{\partial \vartheta(y,t)}{\partial y}\right|_{y=0} = 0 \qquad \forall t \in [t, t_f] \tag{58}$$

$$\lambda_1 F \frac{\partial \vartheta(y,t)}{\partial y} = \alpha \cdot F [\vartheta(y,t) - \vartheta_a]$$

$$\text{for } y = y_f \text{ and } \forall t \in [t, t_f] \tag{59}$$

where F — the heat transmission surface [m^2]
α — the heat transfer coefficient [kcal/m^2 sec C°]
ϑ_a — the constant ambient temperature [C°].

Several methods are offered by the literature to solve equations of this type, in the present case, however, that preferred in the foregoing will be

used. Hence, the time co-ordinate is resolved by increments Δt and the space by increments Δy. Introducing the symbols

$$\vartheta^T(k) = [\vartheta_{0,k}, \vartheta_{1,k}, \ldots, \vartheta_{L,k}] \tag{60}$$

and

$$\mathbf{u}_*^T(k) = [u_{0,k'}, u_{1,k}, \ldots, u_{L,k}] \tag{61}$$

where $u_* = U^2$, that is, the square of the voltage vector is considered to be the control vector, and, introducing the symbols given in [55] the following formula will be obtained:

$$\vartheta(k+1) = \mathbf{A}\vartheta(k) + \mathbf{A}^* \vartheta(k) + \mathbf{B}\,\mathbf{u}(k) \tag{62}$$

where

$$\mathbf{A} = \mathbf{I}$$

$$\mathbf{A}^* = \begin{bmatrix} b_0 & c_0 & 0 \ldots & & 0 \\ a_1 & b_1 & c_1 \ldots & & 0 \\ 0 & a_2 & b_2 & c_2 & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & a_{L-1} & b_{L-1} & c_{L-1} \\ 0 & & \ldots & a_L & b_L \end{bmatrix} \tag{63}$$

$$a_i = c_i = \frac{\lambda_1}{c_f \gamma_f} \frac{\Delta t}{(\Delta y)^2}; \qquad b_i = -2a_i, \quad i = 2, \ldots L-1,$$

$$B = Ke^{b[\vartheta(k)-\vartheta_0]}. \tag{64}$$

Although the expression (62) can be rewritten as the well known expression (47), there are still difficulties as the matrix **B** is in exponentional relation with the temperature vector according to (64). This means that in solving the control problem a non-linear equation system has to be solved at each k-th internal of time. As a cost function, again (49) can be chosen.

Boundary conditions are contained by the corresponding elements of the individual coefficient matrices.

Using the above symbols and solutions, the expression of the algorithm to minimize the cost function is completely analogue with the example given in 4.1.

Summary

In recent years the problem of optimizing control systems has become highly significant. In the case of systems of distributed parameters the discrete variational calculus and discrete maximum principle is used to give a calculation algorithm which can be directly programmed.

The calculation method described finds a wide application in solving the problem of optimum control of different physical processes, that is, with the knowledge of parameters determining the system properties the optimum control vector as a function of space and time can be calculated.

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