ON THE SIMILARITY INVARIANTS OF THE CLASSICAL ELECTRODYNAMICS

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Stratton gives *two* similarity invariants obtained from the *Maxwell's* equations for homogeneous isotropic conductors. [1] These are

$$\Pi_1 = \mu\varepsilon\left(\frac{l_0}{t_0}\right)^2; \qquad \Pi_2 = \mu\sigma\frac{l_0^2}{t_0}.$$

(Originally he marks them C_1 and C_2.) Here l_0 is a characteristic length and t_0 a characteristic time interval, e.g. the period.

It is easy to recognize, however, that the equations

$$\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}; \qquad \nabla \times \mathbf{H} = \sigma\mathbf{E} + \varepsilon\frac{\partial \mathbf{E}}{\partial t}$$

require *three* invariants to assure the similitude of the electrodynamic phenomena. The "similitude" means that the dimensionsless governing equations have the same form for either case.

It is known [2] that the minimum number of similarity invariants is the difference between the number of the physical quantities necessary to describe the investigated phenomenon and the number of the independent basic units.

There are 7 physical variables in the above mentioned equations i.e. **E**, **H**, ε, μ, σ, l, and t. The number of basic units is 4. In the LTUI system this fact is self-evident. The required minimum number of invariants is therefore $7 - 4 = 3$.

To discover the error let us follow Stratton's line. At first he writes the above two equations in a dimensionless form, presupposing the geometrical similitude. i.e. the geometrical transformation of the governing equations should be homogeneous isotropic similarity transformation and the invariance of the direction of the vectors **E** and **H**. He obtains three similarity parameters* as a result:

$$\Pi_1^* = \frac{\mu l_0}{t_0}\frac{H}{E}; \qquad \Pi_2^* = \frac{\varepsilon l_0}{t_0}\frac{E}{H}; \qquad \Pi_3^* = \sigma l_0\frac{E}{H}.$$

* (In the original work $\Pi_1^* = \alpha K_m$; $\Pi_2^* = \beta K_e$; $\Pi_3^* = \gamma s$).

The result is correct. These quantities can be similarity invariants as they are dimensionless. They form a complete set of invariants because the equations describing two different phenomena remain unchanged if these three quantities do not change in either case. This statement is found in Stratton's work.

But the next step is incorrect. I quote it: "Upon eliminating the common ratio E/H, . . . the condition of similitude requires *two* characteristic parameters C_1 and C_2 be invariant to a change of scale." $C_1 = \Pi_1$ and $C_2 = \Pi_2$ are really similarity invariants because they are obtained as power-products of similarity parameters:

$$\Pi_1 = \Pi_1^* \, \Pi_2^*; \quad \Pi_2 = \Pi_1^* \, \Pi_3^* .$$

But in the quoted text implicitly we find that E/H is invariant too. And this statement is incorrect. One cannot obtain this ratio as a power-product of the original invariants. This ratio is not dimensionless in the LTUI system. Therefore it cannot be an invariant — even if it remains unchanged. We need the required *third* invariant to obtain a complete set of them which contains the E/H ratio *too*. The invariance of Π_1 and Π_2 is necessary but not sufficient to assure the similitude.

Let us consider Stratton's example. "Suppose that the characteristic length l_0 is halved. C_1 and C_2 remain unchanged if the permeability μ at every point of the field is quadrupled." In this case E/H does not remain unchanged if similitude is required. It is evident from anyone of Π_1^*, Π_2^* and Π_3^* that the ratio E/H must be doubled. This condition of the similitude cannot be obtained from Π_1 and Π_2. The choice $\Pi_3 = \Pi_3^*$ is suitable, because it directly shows the dimension of the wave impedance.

The general transport theory is the subject which deals in detail with the similarity invariants obtainable from the similarity transformation of the governing equations. From its aspect the two *Maxwell* equations are equivalent with the *Poynting* equation which describes the transport of the electromagnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right) + \nabla(\mathbf{E} \times \mathbf{H}) = -\sigma E^2 .$$

It is easy to realize that this equation offers *three* invariants. If one considers the transport equations of the charge and that of the electromagnetic moment the number of the invariants grows. This happens in that case, too, where the medium is anisotropic and/or the geometrical transform is affin. It is impossible to assure the electromagnetic similitude with the invariance of only *two* quantities.

Summary

The paper shows that the minimum required number of the electrodynamic similarity invariants is three, in contradiction to Stratton's statement.

References

1. STRATTON, J. A., *Electromagnetic Theory*. New York: McGraw-Hill, 1941, Sec. 9.3.
2. KLINE, S. J., *Similitude and Approximation Theory*. New York: McGraw-Hill, 1965, p. 18.

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