SIMULATION OF TRANSMISSION LINES BY LADDER NETWORKS

By

A. MAGOS

Department of Theoretical Electricity,

Technical University, Budapest

Received September 20, 1972

Presented by Prof. Dr. G. FODOR

Introduction

The demand of simulating a transmission line by lumped network in a given frequency-band may arise in several cases. Partly the terminating impedances of the equivalent network, partly the geometry preferred for the network determine which of the available methods is chosen for solving the problem. Equivalent networks in form of symmetrical lattice were treated in a previous paper [3] stating that such a network is favourable if the terminating impedances are much greater or much smaller than the characteristic impedance of the transmission line. In this paper the terminating impedances are supposed to be of the order of the characteristic one. Since the lattice network has several undesirable properties, the equivalent network is sought in form of ladder rather than of lattice network.

The method of determining the equivalent network

At first the transmission line is supposed to be lossless, and so the equivalent network contains only capacitances and inductances. An equivalent network is sought for, which is valid at low frequencies, more concretely in a frequency band $[0; \omega_0]$. If the impedances terminating the transmission line are of the order of the characteristic one, the line is suitably described in terms of a scattering matrix with normalizing resistances equal to the characteristic impedance. This matrix is to be approximated by a scattering matrix realizable by a symmetrical network containing only inductances and capacitances.

If the normalizing resistances equal the characteristic impedance, the scattering matrix of a lossless uniform transmission line can be expressed as:

$$[S(s)] = \begin{bmatrix} 0 & e^{-lp(s)} \\ e^{-lp(s)} & 0 \end{bmatrix}, \tag{1}$$

where l denotes the length of the transmission line, and $p(s) = s\sqrt{L'C'}$ is the propagation coefficient; here L' and C' denote the self-inductance and capacitance per unit length, resp. The total self-inductance and the total capacitance of the transmission line are denoted by $L = lL'$ and $C = lC'$, resp. With the notation $\tau = \sqrt{LC}$ the scattering matrix can be written as:

$$[S(s)] = \begin{bmatrix} 0 & e^{-s\tau} \\ e^{-s\tau} & 0 \end{bmatrix}. \tag{1a}$$

It is known [1] that a matrix $[S(s)]$ can be a scattering matrix of a symmetrical two-port containing only inductances and capacitances if and only if it is of the following form:

$$[S(s)] = \frac{1}{d(s)} \begin{bmatrix} h(s) & f(s) \\ f(s) & h(s) \end{bmatrix}, \tag{2}$$

where $d(s)$, $f(s)$ and $h(s)$ are polynomials with real coefficients, $d(s)$ has no zero in the right half plane, the imaginary axis included, one of $f(s)$ and $h(s)$ is even, the other is odd and

$$f(s)f(-s) + h(s)h(-s) = d(s)d(-s). \tag{3}$$

If s is pure imaginary, $f(s)f(-s)$ and $h(s)h(-s)$ give the square of the absolute value of the polynomials $f(s)$ and $h(s)$, resp., so the left side of (3) can equal zero on the imaginary axis only if $f(s)$ and $h(s)$ have a common pure imaginary zero. This case must be excluded, because $d(s)$ may not have a zero on the imaginary axis. On the other hand, if $f(s)$ and $h(s)$ have no common zero there, they always define through (3) a unique polynomial $d(s)$ with real coefficients and having no zero in the right half plane, the imaginary axis included.

So the problem has the following form: polynomials $f(s)$ and $h(s)$ must be found, one of which is even, the other is odd, which have no common zero on the imaginary axis, and form with $d(s)$ uniquely defined by (3) such a triad of polynomials that the matrix (2) fairly approximates the scattering matrix of the transmission line as a function of s along the interval $[-j\omega_0; j\omega_0]$ of the imaginary axis.

All-pass equivalent network

Matrices (1a) and (2) having been compared, the polynomial $h(s)$ can be chosen to equal zero. Then the polynomial $f(s)$ may be either even or odd, but it cannot have any zero on the imaginary axis and (3) assumes the following form:

$$f(s)f(-s) = d(s)d(-s). \tag{4}$$

Accordingly the zeros of $d(s)$ are given by the left half plane zeros of $f(s)$ and its right half plane zeros multiplied by (-1). It results from the comparison of matrices (1a) and (2) that the fraction $f(s)/d(s)$ must approximate the function $\exp(-\tau s)$. The left half plane zeros of $f(s)$ are also zeros of $d(s)$, and so an all-pass transfer function is got after cancelling, the zeros of which are images of its poles with respect to the imaginary axis. The absolute value of an all-pass function equals unity along the imaginary axis, so there it presents an approximation without error in absolute value. But the approximation has to be valid as to the phase, too. If instead of the phase the time delay is considered, the problem changes into one of finding an all-pass network the time delay of which closely approximates the constant value τ in the frequency band $[0; \omega_0]$. An all-pass network can simply be realized by lattice, but not by ladder network. The problem both of the approximation and the realization is discussed in particulars in the literature treating the delay lines (e.g. [2], [5]), hence, no detailed discussion will be given here.

Equivalent network in ladder configuration

If an equivalent network in ladder configuration is demanded, the polynomial $h(s)$ cannot be chosen to equal zero. Instead of $h(s)$, now the polynomial $f(s)$ is considered. Its zeros are the transmission zeros, so they must be pure imaginary if ladder is used for the simulation. On the other hand it is most desirable to choose a polynomial $f(s)$ with no pure imaginary zero, because the approximation must be close on the imaginary axis and the trasmission line has no transmission zero there, as anywhere on the complex plane. Accordingly let $f(s)$ be constant and without any restriction $f(s) = 1$. So the transmission zeros of the equivalent ladder network are shifted to infinity, i. e. as far as possible from the low frequencies, where the approximation must be a close one. This is also advantageous, because the phase of the parameter S_{12} of the transmission line increases beyond all bounds with increasing frequency, but the phase of the same parameter of the equivalent network converges to a constant value. If all the transmission zeros of the equivalent network are in infinity, the high frequency components cause a relatively small distortion in spite of the very poor phase characteristic, because they are strongly damped. This is also proved by the inspection of the unit step response in the different cases.

If $f(s) = 1$, the polynomial $h(s)$ must be odd and so (3) assumes the following form:

$$1 - h^2(s) = d(s)d(-s). \tag{5}$$

It follows from comparing (2) with (1a) that $h(s)$ is to be chosen so that $d(s)$ fairly approximates the function $\exp(\tau s)$. Since the polynomial $h(s)$ must be approximately zero in the region of the approximation, there the absolute value of the polynomial $d(s)$ fairly approximates that one of the function $\exp(\tau s)$ according to (5). Special care must be taken to choose $h(s)$ so that $d(s)$ offers a close approximation in phase, too.

The phase of the polynomial $d(s)$ can be expressed by *Bode's* theorem along the imaginary axis as:

$$b(\omega) = \text{arc}\, d(j\omega) = \frac{2\omega}{\pi} \int_0^\infty \frac{\ln |d(jy)| - \ln |d(j\omega)|}{y^2 - \omega^2}\, dy =$$

$$= \frac{\omega}{\pi} \int_0^\infty \frac{\ln [1 - h^2(jy)] - \ln [1 - h^2(j\omega)]}{y^2 - \omega^2}\, dy. \tag{6}$$

The polynomial $h(s)$ must be chosen so that the above expression fairly approximates the linear function $\varphi(\omega) = \tau\omega$ for small values of ω. The relationship (6) cannot be used directly because of its intricacy, this is why it will be only applied to a special case. The polynomial $h(s)$ is supposed to have an m-fold zero at the origin. This assumption is justified by that the polynomial $h(s)$ has to approximate zero in the neighbourhood of the origin. In order to simplify, instead of arc $d(j\omega)$ only some of its first derivatives with respect to ω will be inspected at the point $\omega = 0$. The derivatives of even order equal zero, those of odd order can be expressed after a transformation of (6) as:

$$b^{(k)}(0) = \frac{k!}{\pi} \int_0^\infty \frac{\ln [1 - h^2(jy)]}{y^{k+1}}\, dy, \tag{7}$$

where $k = 1, 3, \ldots 2m - 1$. If $b'(0)$ is required to equal the ideal value τ, the odd polynomial $h(s)$ must satisfy the following condition:

$$\int_0^\infty \frac{\ln [1 - h^2(jy)]}{y^2}\, dy = \pi\tau. \tag{8}$$

The two above relationships clearly show the difficulties and limits of approximation. In order to approximate the scattering parameters in absolute value it is necessary to choose a low absolute value for the polynomial $h(s)$. But it cannot be decreased *ad libitum*, because the integral (8) must yield a defined value. On the other hand the integral (7) must be as low as possible for $k > 1$, but it cannot be decreased *ad libitum* because of (8) if the degree of $h(s)$ is fixed. The last difficulty can be avoided if $h(s)$ has only a simple zero at the origin, but then the approximation in absolute value will be less close. Generally it can be stated that the approximation of either the absolute value or of the phase is better, depending on the applied method. It seems complicated to approximate on the basis of the phase, since because of symmetry, $d(s)$ must satisfy the condition (5) and so it cannot be determined from *Bessel* polynomials, as it is usual in synthetizing delay lines. So here the magnitude characteristic will be approximated, the phase characteristic taken in mind. To this aim, the following relationship, obtained by evaluating the integral in (7), will be used:

$$b^{(k)}(0) = (-1)^{\frac{k+1}{2}} (k-1)! \sum_l \frac{1}{s_l^k}, \tag{9}$$

where s_l denotes the roots of the polynomial $d(s)$ and $k = 1, 3, \ldots 2m-1$. Accordingly (8) assumes the following form:

$$\sum_l \frac{1}{s_l} = -\tau. \tag{10}$$

Maximally flat approximation

First, $h(s)$ is chosen so that it approximates zero in the neighbourhood of the origin with maximal flatness, i. e. let

$$h(s) = (cs)^m, \tag{11}$$

where m is an odd number. Here c is determined so that the first derivative of arc $d(j\omega)$ with respect to ω equals the ideal value τ at the point $\omega = 0$. (5) gives the following values for the m roots of the polynomial $d(s)$:

$$s_l = -\frac{1}{c} e^{jl\frac{\pi}{m}}, \qquad \text{where } l = 0, \pm 1, \pm 2, \ldots \frac{m-1}{2}. \tag{12}$$

Substituting this in (10) yields after transformation that

$$c = \tau \sin \frac{\pi}{2m}. \tag{13}$$

It is clear that c decreases and the approximation gradually improves in absolute value at low frequencies by increasing m.

After transformations the folloeing expression is got from (9) and (12) for the derivatives of $b(\omega) = \text{arc } d(j\omega)$:

$$b^{(k)}(0) = (k-1)!\,\tau^k \frac{\sin^k \dfrac{\pi}{2m}}{\sin \dfrac{k\pi}{2m}}\,. \tag{14}$$

The validity of the above relation can be proven not only for $k = 1, 3, \ldots 2m-1$ but for all odd values of k. The absolute values of all the differential quotiens $b^{(k)}(0)$ but $b'(0)$ decrease with increasing m, and converge to zero when m converges to infinity, i. e. they approximate ever closer the ideal value.

The polynomial $d(s)$ can be expressed with the roots in (12) as:

$$d(s) = \prod_{l=-\frac{m-1}{2}}^{\frac{m-1}{2}} \left(1 + s\tau e^{jl\frac{\pi}{m}} \sin\frac{\pi}{2m}\right) =$$

$$= \left(1 + s\tau \sin\frac{\pi}{2m}\right) \prod_{l=1}^{\frac{m-1}{2}} \left(1 + 2s\tau \cos\frac{l\pi}{m} \sin\frac{\pi}{2m} + s^2\tau^2 \sin^2\frac{\pi}{2m}\right). \tag{15}$$

The polynomial is obtained at less of computation if s' is substituted by $s\tau \sin(\pi/2m)$ in the *Butterworth* polynomials $B_m(s')$ known and tabulated in the theory of the maximally flat low-pass filters. Likewise using results of the theory of filters [5] the coefficients of the polynomial

$$d(s) = 1 + d_1 s + d_2 s^2 + \ldots + d_m s^m \tag{16}$$

can be expressed as:

$$d_k = \tau^k \prod_{i=1}^{k} \frac{1}{1 + \dfrac{\text{tg}\,[(i-1)\,\pi/2m]}{\text{tg}\,(\pi/2m)}} \qquad k = 1, 2, \ldots m. \tag{17}$$

It can easily be proved that all the coefficients d_k but d_1 increase monotonously with increasing m and

$$\lim_{m\to\infty} d_k = \frac{\tau^k}{k!}\,. \tag{18}$$

Table 1

m	d_1/τ	d_2/τ^2	d_3/τ^3	d_4/τ^4	d_5/τ^5	d_6/τ^6	d_7/τ^7	d_8/τ^8	d_9/τ^9
1	1								
3	1	0.5	0.12500						
5	1	0.5	0.15451	0.029509	2.8178×10^{-3}				
7	1	0.5	0.16078	0.035776	5.5091×10^{-3}	0.5456×10^{-3}	0.2695×10^{-4}		
9	1	0.5	0.16318	0.038176	6.6292×10^{-3}	0.8544×10^{-3}	0.7894×10^{-4}	0.4761×10^{-5}	0.1436×10^{-6}
	1	0.5	0.16667	0.041667	8.3333×10^{-3}	1.3889×10^{-3}	1.9841×10^{-4}	2.4802×10^{-5}	2.7557×10^{-6}

Thus the limit values of the coefficients d_k equal the coefficients of the *McLaurin* expansion of the function exp (τs), and so a deliberately close approximation can be obtained in a given frequency band.

The coefficients d_k are given in Table 1 for the values $m = 1, 3, 5, 7$ and 9. The last row of the table contains the corresponding coefficients of the *McLaurin* expansion of the function exp (τs) for the sake of comparison.

Equiripple approximation

The maximally flat approximation is known to be little advantageous if zero is to be approximated by a polynomial in a given interval. From this point of view the use of *Chebyshev* polynomial gives a better approximation. Accordingly $h(s)$ is now chosen so that

$$\frac{h(j\omega)}{j} = \varepsilon T_m\left(\frac{\omega}{\omega_0}\right), \tag{19}$$

where T_m is the *Chebyshev* polynomial of degree m (here m is an odd number). For such a choice

$$|h(j\omega)| \leq \varepsilon, \quad \text{if} \quad 0 \leq \omega \leq \omega_0, \tag{20}$$

i. e. the accuracy requirements of the approximation determine the value of ε and the approximation is valid in the frequency band $[0; \omega_0]$. But ω_0 must be chosen so that arc $d(j\omega)$ approximates the function $\tau\omega$ in the interval $[0; \omega_0]$.

It is known from the theory of the *Chebysev* filters that if $h(s)$ is chosen according to (19) the zeros of the polynomials $d(s)$ are as follows:

$$s_k = \omega_0\,(-\text{sh}\,\varrho\cos\varphi_k + j\text{ch}\,\varrho\sin\varphi_k), \tag{21}$$

where $\varrho = \frac{1}{m}\,\text{arsh}\,\frac{1}{\varepsilon}, \quad \varphi_k = \frac{k\pi}{m}$ and $k = 0, 1, 2, \ldots \frac{m-1}{2}$.

ω_0 can be chosen so that $d\text{arc}d(j\omega)/d\omega$ equals τ at the point $\omega = 0$ similarly as in the maximally flat approximation. Then, on the basis of (10) and (21), ω_0 has the value

$$\omega_{0a} = \frac{1}{\tau}\left[\frac{1}{\text{sh}\,\varrho} + 2\text{sh}\,\varrho\sum_{k=1}^{\frac{m-1}{2}}\frac{\cos\varphi_k}{\text{sh}^2\varrho + \sin^2\varphi_k}\right]. \tag{22}$$

At such a choice the approximation of the phase characteristic is very close in the neighbourhood of the zero frequency, but it is rather poor in the neighbourhood of ω_0. Therefore it is more reasonable to choose ω_0 so that arc $d(j\omega)$ equals the ideal value at the end points of the interval $[0; \omega_0]$, i. e. let arc $d(j\omega_0) = \tau\omega_0$. In this case ω_0 has the following value:

$$\omega_{0b} = \frac{1}{\tau}\left[\operatorname{arc\,tg}\frac{1}{\operatorname{sh}\varrho} + 2\sum_{k=1}^{\frac{m-1}{2}} \operatorname{arc\,tg}\frac{\cos\varphi_k}{\operatorname{sh}\varrho}\right]. \tag{23}$$

If not only the end points but the whole interval $[0; \omega_0]$ is to be considered ω_0 can be choosen so that the quantity

$$H(x) = \int_0^{\omega_0} [\operatorname{arc} d(j\omega) - x\omega]^2 \, d\omega \tag{24}$$

very characteristic of the error of the approximation as a function of x has a minimum at $x = \tau$. A somewhat lengthy calculation gives the following value for ω_0:

$$\omega_{0c} = \frac{3}{2\tau}\left\{\operatorname{ch}^2\varrho \operatorname{arc\,tg}\frac{1}{\operatorname{sh}\varrho} - \frac{\operatorname{sh}\varrho}{\sin\frac{\pi}{2m}} + \right.$$

$$\left. + \sum_{k=1}^{\frac{m-1}{2}} \left[(1 + \operatorname{ch} 2\varrho \cos 2\varphi_k) \operatorname{arc\,tg}\frac{\cos\varphi_k}{\operatorname{sh}\varrho} + \operatorname{sh} 2\varrho \sin 2\varphi_k \operatorname{arth}\frac{\sin\varphi_k}{\operatorname{ch}\varrho}\right]\right\}. \tag{25}$$

The last choice of ω_0 seems to be the more justified among the three different ones shown so far. For reasonable, i.e. not too high values of ε, $\omega_{0b} < \omega_{0c} < \omega_{0a}$. If ε is low, ω_{0a} and ω_{0b} differ hardly. With increasing ε the difference increases, too, which reflects that arc $d(j\omega)$ approximates worse and worse a linear function in the interval $[0; \omega_0]$.

If ε is so low that exp $\varrho \gg 1$, then all the three formulae [(22), (23) and (25)] give practically the following value for ω_0:

$$\omega_0 \approx \frac{2}{\tau}\left(\frac{\varepsilon}{2}\right)^{\frac{1}{m}} \frac{1}{\sin\frac{\pi}{2m}}. \tag{26}$$

Substituting this in (21) and using exp $\varrho \gg 1$, the same values are yielded for the zeros of the polynomial $d(s)$ as at the maximally flat approximation, i. e. when ε converges to zero, the equiripple approximation turns into the maximally flat one.

Comparison of the maximally flat and the equiripple approximation

The accuracy requirements concerning the simulation of transmission lines by lumped networks can be formulated by giving the frequency band in which the difference between the original and the equivalent network must remain under a given bound. This difference can only be characterized by more than one data, but here a single one, probably the most characteristic, the value of $|h(j\omega)|$ will be considered. It must be lower than a given value ε in a frequency band $[0; \Omega_0]$.

It follows from (11) and (13) that for the maximally flat approximation the condition $|h(j\omega)| \leq \varepsilon$ is satisfied in the frequency band $[0; \omega_0]$, where

$$\omega_0 = \frac{\varepsilon^{\frac{1}{m}}}{\tau \sin \frac{\pi}{2m}} . \tag{27}$$

For the equiripple approximation this quantity ω_0 equals that one represented in (19), given by one of the formulae (22), (23) and (25). If ε and Ω_0 are given, m must be chosen so that $\omega_0 \geq \Omega_0$.

As m cannot be simply expressed in terms of ε and Ω_0 either for the maximally flat or the equiripple approximation, the two different values of ω_0 will be compared supposing that ε and m are given. For sake of simplicity it is also supposed, that ε is low, and so the values of ω_0 for the equiripple approximation and for the maximally flat one (denoted by ω_{0e} and ω_{0m}, resp.) are given by (26) and (27), resp. Then

$$\frac{\omega_{0e}}{\omega_{0m}} = 2^{1-\frac{1}{m}} \tag{28}$$

i. e. if ε is low and m is high, the equiripple approximation is acceptable in a frequency band twice as wide.

Obviously, if the transmission line must be simulated in given frequency band at a given accuracy, it is more advantageous to choose the equiripple approximation. The *Chebyshev* polynomials increase very fast outside the interval $[-1; 1]$ and so the error of the equiripple approximation begins to increase much more rapidly above the frequency ω_0 than that one of the maximally flat approximation. It may also be a disadvantage of the equiripple approximation that the value of the error oscillates in the frequency band $[0; \omega_0]$, whereas the error of the maximally flat approximation increases monotonously with increasing frequency. Consequently, if the transmission line is to be simulated at low frequencies in principle, but the upper limit of the frequency band of the approximation is not unambiguously defined and at higher frequencies a greater error is allowable, then the maximally flat approximation is more advantageous to be used.

The realization of the equivalent network

Let us examine now how to realize the scattering matrix already chosen. This problem can be solved in the same manner as for the synthesis of filters.

The driving-point impedance of the two-port terminated in a resistance Z_0 can be used as starting-point. It is expressed as:

$$Z_{\text{in}} = Z_0 \frac{1 + S_{11}}{1 - S_{11}} = Z_0 \frac{d(s) + h(s)}{d(s) - h(s)} . \tag{29}$$

This impedance must be realized. It follows from (5) that two polynomials $h(s)$, differing only by sign belong to a polynomial $d(s)$. In one case Z_{in} has a pole at infinity, in the other a zero. As all the transmission zeros are at infinity, the two equivalent networks in Fig. 1 are got in the two respective cases. The two networks are dual to each other with respect to the value Z_0. It is probably simpler to express the open-circuit driving-point impedance and to realize it:

$$z_{11} = \frac{d_1(s)}{d_2(s) - h(s)} Z_0 , \tag{30}$$

where $d_1(s)$ and $d_2(s)$ denote the even and odd part of the polynomial $d(s)$, resp.

The realization can be done by expanding in continued fraction, but this is in general superfluous, because of the availability of tables and explicit formulae of the synthesis of filters. Tables are suitable where the internal resistance of the generator and the load resistance are equal, because the scattering matrix has been normalized to the same value (the characteristic impedance) at both sides of the two-port.

As the tabulated values are normalized to unit load and unit cutoff frequency, they are to be multiplied by Z_0/ω_1 for the inductances and by $1/(Z_0\,\omega_1)$ for the capacitances. At maximally flat approximation ω_1 is the reciprocal value of c occurring in relationship (11), i.e.

$$\omega_1 = \frac{1}{\tau \sin \dfrac{\pi}{2m}} . \tag{31}$$

In equiripple approximation ω_1 is the frequency ω_0 from (19), i. e. its value is given by one of the formulae (22), (23) and (25).

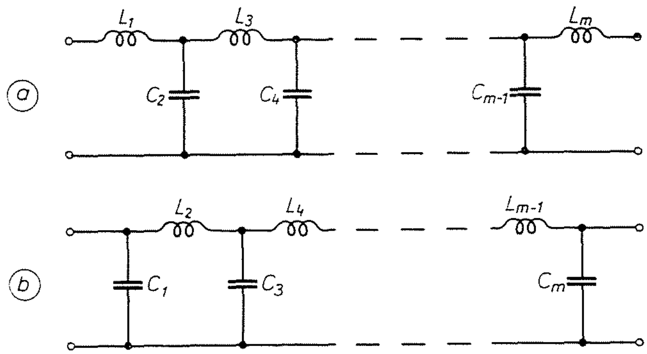

Fig. 1

In the theories of *Butterworth* and *Chebyshev* filters, formulae are known which give directly the values of the network elements occurring in the filter. In case of maximally flat approximation these formulae give the following results for both networks in Fig. 1:

$$\begin{aligned} L_k &= 2L \sin\frac{\pi}{2m} \sin\frac{(2k-1)\pi}{2m} \\ C_k &= 2C \sin\frac{\pi}{2m} \sin\frac{(2k-1)\pi}{2m}, \end{aligned} \tag{32}$$

where L and C denote the total inductance and capacitance of the transmission line, resp. It can be proved that the sum of the inductances equals L, and the sum of the capacitances equals C (excepted for $m = 1$).

In the case of equiripple approximation the values of the network element can be computed with a recursion formula if a quantity K_r is introduced, which can denote both products $L_r C_{r+1}$ and $C_r L_{r+1}$. It is defined as:

$$K_r = \frac{4 \sin\dfrac{(2r-1)\pi}{2m} \sin\dfrac{(2r+1)}{2m}}{\omega_0^2 \left(p^2 + \sin\dfrac{2r\pi}{m}\right)} \tag{33}$$

where $\quad p = \mathrm{sh}\left(\frac{1}{m} \mathrm{arsh}\frac{1}{\varepsilon}\right).$

The values of the first elements in the two networks in Fig. 1 are:

$$L_1 = \frac{2Z_0}{\omega_0 p} \sin\frac{\pi}{2m} \quad \text{and} \quad C_1 = \frac{2}{\omega_0 Z_0 p} \sin\frac{\pi}{2m}. \tag{34}$$

Though it is more convenient to compute with the aid of the recursion formula, the values of the elements can be expressed directly; e. g. in the network in Fig. 1a:

$$L_k = \frac{2Z_0}{\omega_0 p} \sin \frac{(2k-1)\pi}{2m} \prod_{i=1}^{\frac{k-1}{2}} \frac{p^2 + \sin^2 \frac{(2i-1)\pi}{m}}{p^2 + \sin^2 \frac{2i\pi}{m}}$$

$$C_k = \frac{2}{\omega_0 Z_0 p} \cdot \sin \frac{(2k-1)\pi}{2m} \prod_{i=0}^{\frac{k}{2}-1} \frac{p^2 + \sin^2 \frac{2i\pi}{m}}{p^2 + \sin^2 \frac{(2i+1)\pi}{m}} \qquad (35)$$

Finally notice that rather than m, only $(m+1)/2$ values have to be computed for symmetry reasons.

Simulation of lossy transmission lines

It results from the foregoing that the self-inductances of coils and the capacitances of condensers in an equivalent network of a transmission line can be expressed as $L_k = u_k L$ and $C_k = v_k C$, resp., where u_k and v_k are independent of L and C. So the equivalent network of a lossy transmission line can be deduced from that of a lossless transmission line the following way: a resistance $u_k R$ must be connected in series with every coil of self-inductance $u_k L$ and a resistance $1/(v_k G)$ must be connected in parallel with every condenser of capacitance $v_k C$, where R and G denote the total resistance and leakage conductance of the transmission line, resp. Namely the equations describing a lossy transmission line in the domain of *Laplace* transforms are obtained by substituting $sL + R$ for sL and $sC + G$ for sC in the equations of a lossless transmission line. The new equivalent network is acceptable at frequencies where $j\omega + (R/L)$ and $j\omega + (G/C)$ fall in the domain of the s-plane where the approximation of the original, lossless transmission line is valid.

In case of maximally flat approximation the absolute value of the polynomial $h(s)$ is constant in the s-plane along circles concentrical about the origin. The approximation may be accepted valid inside of a circle of a radius given by ω_0 previously defined. So the simulation of the lossy transmission line is acceptable in the frequency band $[0; \omega_0']$, where

$$\omega_0' = \sqrt{\omega_0^2 - \max\left(\frac{R^2}{L^2}; \frac{G^2}{C^2}\right)} = \sqrt{\frac{\varepsilon^{\frac{2}{m}}}{LC\sin^2 \frac{\pi}{2m}} - \max\left(\frac{R^2}{L^2}; \frac{G^2}{C^2}\right)} . \qquad (36)$$

Here ε is the quantity characteristic of the accuracy of the approximation, as seen previously. If a negative number stands under the square root in the relationship (36), the approximation has not the prescribed accuracy with the chosen m even for zero frequency. Then the value of m must be increased.

In case of equiripple approximation the absolute value of the polynomial $h(s)$ increases much more rapidly along the real axis than the imaginary one, especially for great m, and so the absolute value of the error is constant along rather flat ovals adhering closely to the imaginary axis. This is a disadvantage of the equiripple approximation, inhibiting it to be used for simulating transmission lines other than of very low loss.

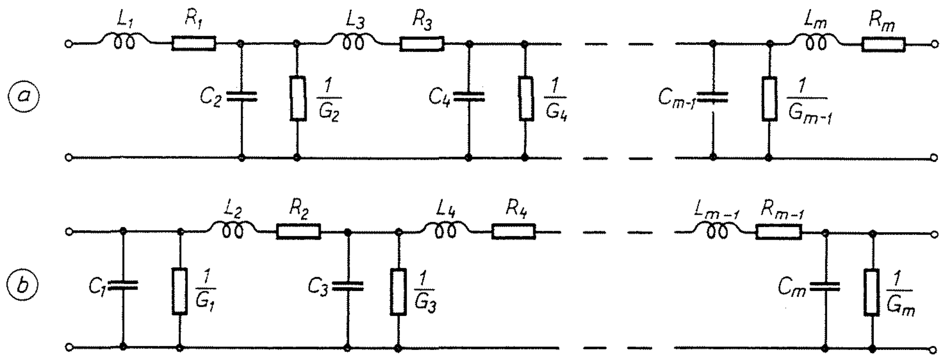

Fig. 2

Here the results concerning equivalent networks of lossy transmission lines are summarized in case of maximally flat approximation. A uniform transmission line can be simulated at low frequencies by a lumped network according to either Fig. 2a or Fig. 2b. The values of the elements in both networks are given by the following formulae:

$$L_k = q_k L \quad C_k = q_k C \quad R_k = q_k R \quad G_k = q_k G, \tag{37}$$

where L, C, R and G are the total self-inductance, capacitance, resistance and leakage conductance of the transmission line, resp., and

$$q_k = 2 \sin \frac{\pi}{2m} \sin \frac{(2k-1)\pi}{2m} . \tag{38}$$

The error of the approximation can be decreased *ad libitum* by increasing m, i. e. the frequency band where the approximation is acceptable can be increased *ad libitum* by increasing m.

Finally an illustrative interpretation of these results is given. The transmission line of length l is divided into $m-1$ parts; the successive lengths of these parts are:

$$\Delta l_k = \operatorname{tg}\frac{\pi}{2m}\sin\frac{k\pi}{m}, \qquad k = 1, 2, \ldots m-1. \tag{39}$$

It is easy to prove that

$$\sum_{k=1}^{m-1} \Delta l_k = l. \tag{40}$$

The distributed self-inductance, capacitance, resistance and leakage conductance of such a part of length Δl_k are substituted by lumped elements. The whole taken as a two-port can be substituted by a lumped network in two

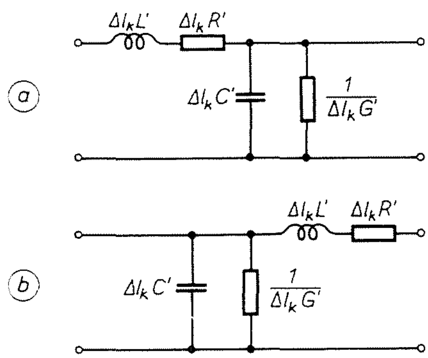

Fig. 3

different ways shown in Fig. 3. The parts of the transmission line are substituted one by one alternately by these two networks and where it is possible, the elements in series and parallel are contracted. Depending on whether the first part is substituted by the network in Fig. 3a or 3b the network in Fig. 2a or 2b is got. It is worth mentioning that the partition is much more finer near the two ends of the transmission line than in the middle.

Summary

Lumped equivalent networks of lossless and lossy transmission lines have been treated. In case of lossless transmission line first its scattering matrix is to be approximated by a matrix which is realizable by a reactive lumped network. Two different methods have been dealt with, namely the maximally flat and the equiripple approximation. The approximating scattering matrix was realized by a ladder network. The equivalent network of a lossy transmission line can be determined from that of a lossless transmission line.

References

1. BELEVITCH, V.: Classical Network Theory. Holden-Day, San Francisco, 1968.
2. GÉHER, K.: Lineáris hálózatok. Műszaki Könyvkiadó, Budapest, 1968.
3. MAGOS, A.: Simulation of Transmission Lines by Lumped Networks. Per. Pol. El. Eng. Vol. 16. No. 3. 1972.
4. NEWCOMB, R. W.: Linear Multiport Synthesis. McGraw-Hill, New York, 1966.
5. WEINBERG, L.: Network Analysis and Synthesis. McGraw-Hill, New York, 1962.

Dr. András MAGOS, 1502. Budapest, P.O.B. 91. Hungary