SIMPLIFIED REGRESSION PROCEDURE FOR THE FORMS OF RATIONAL FRACTIONAL FUNCTIONS

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Introduction

In the case where the measurements are appreciated by real rational fractional function, the least-squares method is usually a nonlinear estimation method that has no explicit solution. Such problems can be solved by the generalized-least-squares method. The principle of this method was applied by LEVY [2] for identifying complex rational fractional functions (NYQUIST diagram); this method was improved by SANATHAN—KOERNER [3] and STROBEL [5], it was developed by KALMAN [1] for the identification of impulse response. This latter method was improved by STEIGLITZ [4] and CLARKE [6].

Further on the principle of generalized-least-squares method (GLS) will be shown for the identification of real rational fractional function.

Identification of real rational fractional function

Let us consider the approximation of a real function $y = y(x)$ in the form of

$$\hat{y} = \frac{\sum_{j=0}^{m} a_j h_j(x)}{\sum_{j=0}^{m} b_j g_j(x)} \frac{A(x)}{B(x)} \tag{1}$$

which is based on the values of x and y, respectively $(x_i, y_i;\ i = 1, \ldots, N)$. (It is assumed that x is measured without error and y is measured with an error of permanent variance and with a mean value 0 independent of x.)

According to the principle of least-squares method, parameters a_j, b_j can

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by obtained by minimization

$$Q = \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{N} \left(y_i - \frac{A(x_i)}{B(x_i)} \right)^2 \tag{2}$$

by parameters nonlinear with respect to the parameters a_j, b_j.

In the generalized-least-squares method the error $\varepsilon_i = y_i - \hat{y}_i$ is weighted by the denominator of $\hat{y}_i$, $B(x_i)$, minimizing

$$Q' = \sum_{i=1}^{N} [\varepsilon_i B(x_i)]^2 = \sum_{i=1}^{N} [y_i B(x_i) - A(x_i)]^2 \tag{3}$$

composed with the generalized error $\varepsilon_i' = \varepsilon_i B(x_i)$. Thus — as it is seen from Eq. (3) — in the GLS method the solution is attributed to a linear method (i.e. Q' is quadratic, so its gradient is a linear function of the parameters).

Taking $b_0 g_0(x) \equiv 1$ — without loss of generality — and introducing the notations:

$$\mathbf{z} = [a_0, a_1, \ldots, a_n, b_1, b_2, \ldots, b_m]^{\mathrm{T}} \tag{4}$$

$$\mathbf{y} = [y_1, y_2, \ldots, y_N] \; ; \qquad \mathbf{Y} = \mathbf{diag}\,[y_1, y_2, \ldots, y_N] \tag{5}$$

$$\mathbf{G} = \begin{bmatrix} g_1(x_1) & \ldots & g_m(x_1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ g_1(x_N) & \ldots & g_m(x_N) \end{bmatrix}; \qquad \mathbf{H} = \begin{bmatrix} h_0(x_1) & \ldots & h_n(x_1) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ h_0(x_N) & \ldots & h_n(x_N) \end{bmatrix} \tag{6}$$

the minimum of (3) is found at

$$\mathbf{z} = (\mathbf{CD})^{-1}\,\mathbf{Cy} \tag{7}$$

as it can be deduced easily [7]. Here

$$\mathbf{C} = \begin{bmatrix} (\mathbf{YG})^{\mathrm{T}} \\ \mathbf{H}^{\mathrm{T}} \end{bmatrix} \quad \text{and} \quad \mathbf{D} = [-\mathbf{YG}\ \mathbf{H}]\,. \tag{8}$$

The solution given by (7) minimizes the generalized error but the generalized error can not be easily interpreted phisically. The error weighted by $B(x)$ favours arbitrarily domains where the error is higher weighted, so there is also a better accuracy than elsewhere.

The principle of GLS method offers an iterative estimation method for eliminating the undesirable effects of weighting. Results obtained from (7) are to be improved gradually, namely in the k-th step weighting $q_k\, B_k(x)/B_{k-1}(x)$ is applied where $B_{k-1}(x)$ is to be computed with the coefficients obtained in the former step and parameters b_j of $B(x)$ are to be determined. Obviously, in the first step we choose $B_0(x) \equiv 1$.

Introducing

$$\mathbf{B}_{k-1}^{-1} = \mathbf{diag}\left[B_{k-1}(x_1), \ldots, B_{k-1}(x_N)\right] \tag{9}$$

a simple deduction [7] — similar to the previous one — shows that in the k-th step

$$\mathbf{z}_k = (\mathbf{C}_k\, \mathbf{D}_k)^{-1}\, \mathbf{C}_k\, \mathbf{y} \tag{10}$$

where

$$\mathbf{C}_k = \mathbf{C}_{k-1}\, \mathbf{B}_{k-1} \quad \text{and} \quad \mathbf{D}_k = \mathbf{B}_{k-1}\, \mathbf{D}_{k-1}. \tag{11}$$

The iterative procedure is to be continued until $\| \mathbf{z}_k - \mathbf{z}_{k-1} \|$ is sufficiently small.

Identification of power density spectra of stochastic signals

In linear, time-invariant, minimum-phase, self-adjusting systems power density spectra of stochastic signals can be approached by the following form:

$$\hat{\Phi}(\omega) = \frac{\sum_{j=0}^{n} a_j\, \omega^{2j}}{1 + \sum_{j=1}^{m} b_j\, \omega^{2j}} = \frac{A(\omega)}{B(\omega)}. \tag{12}$$

Here ω is the circular-frequency (or frequency), and $n \leqq m$ in the case of physically realizable systems.

The relationships obtained in the previous chapter according to the principle of GLS method can now be used with the following substitutions:

$$\hat{y} = \hat{\Phi}; \qquad x = \omega; \qquad b_0 g_0 \equiv 1 \tag{13}$$

$$h_j(x) = \omega^{2j} \quad (j = 0, 1, \ldots, n); \qquad g_j(x) = \omega^{2j} \quad (j = 1, 2, \ldots, m).$$

Identification of the transfer function

The approximate forms $\hat{\Phi}_{\text{inp}}(\omega) = A_1(\omega)/B_1(\omega)$ and $\hat{\Phi}_{\text{outp}}(\omega) = A_2(\omega)/B_2(\omega)$ of the power density spectra of the input and output signals can be determined

by means of (10) and (11) based on the measurements. Then the frequency function $K(j\omega)$ and the transfer function $K(s)$ of a linear plant can be specified by the factorization implemented on the basis of the well-known relationship:

$$K(-s)\,K(s)\Big|_{s=j\omega} = \frac{\hat{\Phi}_{\text{outp}}(\omega)}{\hat{\Phi}_{\text{inp}}(\omega)} = \frac{A_2(\omega)\,B_1(\omega)}{A_1(\omega)\,B_2(\omega)} = \frac{P(\omega)}{R(\omega)}\,. \tag{14}$$

The factorization can be realized according to the following algorithm. Find the numerator of transfer function $K(s)$ in the form $F(s) = \sum_{j=0}^{k} f_j\, s^j$ while $P(\omega) = \sum_{j=0}^{k} p_j\, \omega^{2j}$.

Let us introduce the vectors

$$\mathbf{f} = [f_0, f_1, \ldots, f_k]^{\mathrm{T}}; \quad \mathbf{p} = [p_0, 0, -p_1, 0, p_2, 0, \ldots, (-1)^k p_k]^{\mathrm{T}} \tag{15}$$

and the matrix:

$$\mathbf{F} = \begin{bmatrix} f_0 & 0 & \ldots\ldots & 0 \\ -f_1 & f_0 & \ldots\ldots & 0 \\ f_2 & -f_1 & \ldots\ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ (-1)^k f_k & \cdot & \ldots\ldots & f_0 \\ 0 & (-1)^k f_k & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & (-1)^k f_k \end{bmatrix} \tag{16}$$

The vector $\mathbf{f}$ containing the requested unknown quantities can be determined taking an initial value $\mathbf{f}_0$ using either of the following two iterative algorithms [7]:

$$\mathbf{f}_u = (\mathbf{F}_{u-1}^{\mathrm{T}}\,\mathbf{F}_{u-1})^{-1}\,\mathbf{F}_{u-1}^{\mathrm{T}}\,\mathbf{p} \tag{17}$$

or

$$\mathbf{f}_u = \frac{1}{2}\left[\mathbf{f}_{u-1} + (\mathbf{F}_{u-1}^{\mathrm{T}}\,\mathbf{F}_{u-1})^{-1}\,\mathbf{F}_{u-1}^{\mathrm{T}}\,\mathbf{p}\right] \tag{18}$$

$\mathbf{F}_{u-1}$ is derived from $\mathbf{f}_{u-1}$ as per Eq. (16). The iteration is continued until $||\mathbf{f}_u - \mathbf{f}_{u-1}||$ is sufficiently small. After this the denominator is factorized as well, by the same algorithm and so we obtain the coefficients of the whole transfer function $K(s)$.

The initial value $\mathbf{f}_0$ of vector $\mathbf{f}$ is given by coefficients of the polynom

$$F(s) = \sqrt{(-1)^k p_k} \prod_{j=1}^{k} \left(s - \sqrt{x_j}\right) \tag{19}$$

after factorizing $P(x)$ according to the

$$P(x) = \sum_{j=0}^{k} (-1)^j p_j x^j = (-1)^k p_k \prod_{j=1}^{k} (x - x_j). \tag{20}$$

If the roots x_j are sufficiently exact, iteration cycles (17) and (18) may also be omitted.

(Because of rational limits here we did not deal with the problems of structure estimation and of choosing $\mathbf{f}_0$, in addition we did not prove the algorithms but only showed the results. Similarly we also disregarded from presenting practical examples.)

Conclusions

In our paper a simple algorithm is given for the determination of approximations of rational fractional function form. It is based on the GLS method and can be programmed easily on a computer.

The algorithm is very useful in processing measurement data, in identifying amplitude-density spectra, magnetization curves, load characteristics, and so on. In this paper, a method, combined with an iterative algorithm for the numerical solution of factorization is offered for the identification of linear plants based on power density spectra.

Summary

In this paper a simple regression method is considered for the approximation of measurements by means of real rational fractional functions. This numerical procedure can be applied on several fields of engineering practice. A procedure is given for the identification of linear systems with concentrated parameters by power density functions.

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