POWER NETWORK DESIGN BY THE BRANCH AND BOUND METHOD

By

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1. Introduction

Elaboration of power network development conceptions involved economy calculations for various network types and transformer sizes.

This is followed by the rather complex problem of the actual design of economical network type.

The design of a consistent electric network system is composed of that of subsystems. Such subsystems are e.g. the topological design of the conductor network, the determination of the characteristics of transformer stations, the voltage control.

The design of individual subsystems can be regarded as a suboptimization problem.

In the present paper the exact solution of network design suboptimization problems is discussed where the number of possible and technologically feasible solutions is a very high one (thousands, million). No actual design of evaluation of all possible variants can be realized, even by using a computer. Therefore nowdays only some of the variants are elaborated on the basis of expert knowledge and these are compared. This method can be regarded as a rough approximation.

The Branch and Bound Method to be described in the present paper gives an exact solution algorithm for problems of the type given above, by the help of which all possible variants are evaluated indirectly, without in fact being calculated.

2. The Branch and Bound Method

Before describing the method let us summarize the general definition of a graph.

Consider a countable set $X = \{x_1, x_2, \dots x_n\}$ and a multivalued assignment L to elements of X. The pair $G = (X, L)$ forms a graph of the n-th order. Accordingly, to any of the elements of the set X an arbitrary numbered element of the same set can be assigned. L designates the rule (transformation), expressing these assignments.

For graphical representation the elements of the set X are represented by points, while the assignment rule defined by the transformation L is represented in such a way that each element is connected by a line with that element of the set to which it is assigned. The elements of set X are called the vertices of the graph, while the lines formed by the mapping rule are named the arcs of the graph.

2.1. *Principle, definitions and conditions of the method*

The Branch and Bound Method will be described in terms of the general theory of sets and of the theory of graphs [6, 8]. The method was developed by French mathematicians. It is especially suited to solve problems of a combinatory character.

Let us assume that the set of realizable solutions M is given on the elements of which a real valued function f is defined. The element $k \in M$ of this set is sought for, for which $f(k)$ assumes a minimum (or maximum) value.

First let us define an oriented tree having a reference point (root). Let each node of the oriented tree $x \in X$ correspond to a subset $M(x) \subset M$ of set M.

The complete set M is ordered to the reference point of the tree: $M(x_0) = M$, what is of course only implicitly available, not explicitly. The separation principle (Sz) is the process indicating the way of resolving the various subsets of the set M to further subsets. Resolution according to the separation principle is represented in the tree in such a way that the final points of the arcs issuing from the node x are considered, and these form the set $L(x)$. L is in general a bi-valued or multivalued transformation of set X. The oriented tree is shown in Fig. 1, indicating also the interpretations for x, $M(x)$, Sz, $L(x)$. The tree developed in this way will be called the "solution graph". The separation principle can be formulated in several ways (depending on the type of the problem), but in each case it should satisfy the following conditions.

1. At every separation the union of the arising subsets is equal to the original (separated) set, which is the complete set at the first step, and a subset of the former at the further steps.

2. If $L(x) = 0$ (void set), then set $M(x)$ consists of a single element at the maximum, and if such an element exists, it can be determined unequivocally. Such a vertex x is called a terminating vertex.

If $M(x) = 0$, then vertex x is called a void vertex, containing not a single realizable solution.

3. Let us assume that set X consists of a finite number of points. If M is finite and each separation is a true one, the optimum solution can be obtained by a finite number of steps.

By defining the transitional evaluation function $V(x)$, an evaluation is assigned to each set $M(x)$ (to each node x). The employed transitional evaluation function $V(x)$ should satisfy the following conditions:

1. At each vertex x it minimizes (in the case of a minimum problem) the f values of the solutions (elements) in subset $M(x)$.

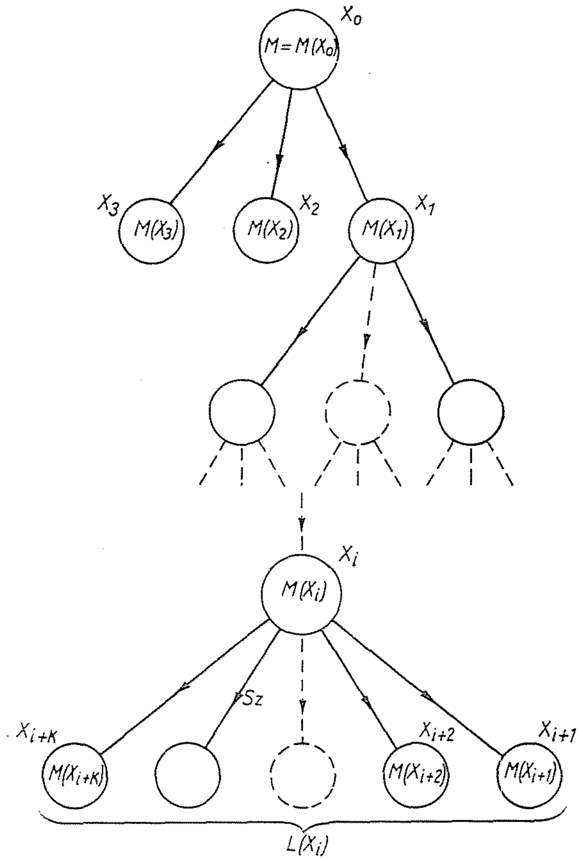

Fig. 1

2. The transitional function value for not void, terminating vertices is exactly equal to the function value f of the solution belonging to the respective vertex, $V(x) = f(k)$.

If the possibility of defining a separation principle (Sz) and a transitional definition function V satisfying certain conditions is given, a solution algorithm for determining extreme values can be elaborated by the method.

2.2. Solution algorithm

The main steps of the solving algorithm are given here in accordance with the definitions given under 2.1.

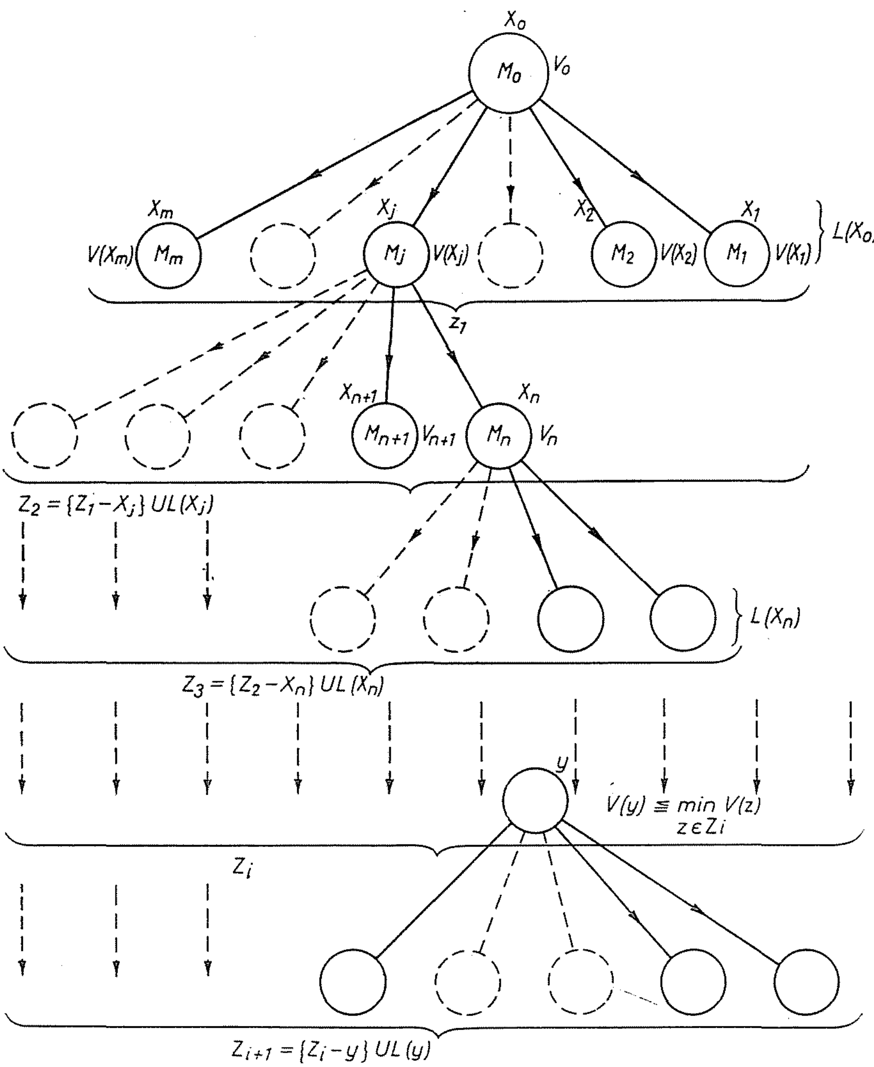

Fig. 2

The value of the transitional evaluation function V_0 is calculated (assigned) to the reference point of the tree, afterwards the values V are divided into two or more groups according to the separation principle, minorizing the values of solutions at the vertices.

This is followed by an iteration process in the course of which the process graph is growing step by step. Z_i designates the set of terminating vertices of the process graph arising at the i-th iteration step. (See Fig. 2).

The steps of the iteration are the following:

1. Among the so far not separated vertices Z_i of the graph that one is selected where the transitional evaluation function is a minimum. Let this be the i-th vertex ($V_i = \min$).

2. The i-th vertex obtained in this way is separated according to the separation principle. If this is a void vertex, the process is continued by iteration step 1, in other cases by step 3.

3. Values V are calculated for the new vertices.

4. Continue the process from the beginning (step 1).

5. Iteration is finished if there is but a single solution at some of the terminating vertices having a minimum V value, $L(x) = 0$ and $M(x) \neq 0$. At the same time this will be the required optimum solution.

If the problem has several optimum solutions these can be determined rapidly by continued iteration. If the maximum is required in the place of the minimum, the algorithm can similarly be employed, but in this case V should majorize solution values.

A further advantage of the method is that also solutions (variants) which are nearly equivalent to optimum can simply be determined.

2.3. *A more simple (heuristic) formulation of the method*

The method will be presented also in our heuristic formulation, the essential subject of the present paper.

In optimization problems all possible solutions (set M) are supposed to be known, and this is assigned to a node. All solutions are divided into two or three groups according to some well defined principle (Sz, separation principle). Consistent solutions arising in the course of grouping into parts (separation) are represented in individual nodes (assigned). These nodes are connected by oriented arcs to the node from which they originate.

A target function value (f) belongs to each solution. The problem is to find the optimum solution where the value of the target function is e. g. a minimum. A function (transitional evaluation function V) should be found, the value of which, as an index number, is higher (or is equal at the maximum) in each node than the minimum of the target function values among the solutions assigned to the respective node (majorization condition).

This index number is calculated for each node. At the node where the index number is a minimum, the target function value of the included (assigned) solutions is probably lower than at the other vertices, therefore solutions at the node are divided here to further groups on the basis of the separation principle. Thus, further nodes are produced which are similarly connected by oriented arcs to the separated node, and the index number is calculated for each node. The so-called process graph arising in the course of the algorithm will be an oriented tree. Among the terminal vertices of the oriented tree the one having the lowest index number is determined in an iteration step each, and the separation of a part of solutions to further parts is continued here. The index number is calculated for each new node. If in the course of the automatic application of our separation principle a node arises for which no solution is possible (on account of exceeding some of the limiting conditions), then this node is excluded from further examinations, it is qualified as a void vertex.

The iteration algorithm is terminated at the node where the index number is the lowest, and where a single solution occurs as optimum solution.

The function supplying the index number should be constructed in such a way that the value of the index number at the vertex containing the optimum solution is exactly equal to the required minimum target function value, since only this can ensure a really optimum solution, on account of the majorization condition satisfied at the other vertices.

If in connection with an optimization problem having combinatory characteristics we succeed to find, in accordance with the aforesaid, a correct separation principle, and to determine an evaluation function supplying the index numbers, then the described Branch and Bound Method gives an exact algorithm for calculating the optimum solution.

3. A ring (travel) problem and its solution by the Branch and Bound Method

The design of a ring-shaped electric network is analogous to the "Travelling Salesman Problem", since the ring is a round travel. The line starts from a feeder point and returns through n consumer points to the same feeder point. The number of possible rings is very high. The optimum ring involving the minimum total length (or total cost) is to be determined, the restricting condition being that all the n consumers' points should be included [1]. On the analogy of the "round tour" problem known in operational research [1, 8], an exact method for the determination and design of ringed type electric networks is described in the following.

The distances d connecting the individual points, for m a matrix of n-th order if a point corresponds to the crossing of a row and of a column.

$$\mathbf{D} = [d_{ij}]\,.$$

Let K denote the set of subscripts of the pairs of points included in the ring. The length of the ring is given by

$$H = \sum_{i,j \in K} d_{ij} .$$

The ring of the minimum length is to be determined, $H =$ Min!

In the problem of the optimum ring, the content of the mathematical concepts involved in the method described under 2 is the following.

M denotes the set of all possible rings. The real valued function f represents the (geographical, geometrical) total lengths (or costs) belonging to the various rings.

During the procedure an element $k \in M$ (ring) is to be found within the set M for which $f(k)$ is a minimum. The process graph $G(X, L)$ formed step by step in the course of the process will be, — in accordance with those described in point 2, — an oriented tree having a reference point.

Let us now denote geometrical arcs in such a way, that e.g. b denotes the *acceptance* of the respective arc and $\bar{b}$ the rejection of the arc (it cannot be an element of the possible rings $M(x)$ assigned to the vertex).

Set $M(x)$ is divided into two parts in such a way that a decision is made on the use of some arc (b).

a) One of the sets contains those rings which use, accept arc b,

b) while in the other set this arc is not included.

From the aspect of solutions assigned to the individual vertices of the process graph (possible rings) the decision is called acceptance or rejection.

In the present case L is a two-valued transformation. The elements of set $L(x)$ represent in the tree the two subsets which are obtained in the course of the further division of subset $M(x)$.

A transitional evaluation $V(x)$ is assigned to each vertex x of the tree which is the lower limit of the total length of the rings $M(x)$ represented by the respective vertex (minorization).

The introductory steps of the algorithm are [8]:

1. Arrange distances d_{ij} in increasing order.

2. Assign order numbers to the distances arranged in increasing order ($d_{ij} = d_r$).

For the reference point x_0 of the solution tree the value of the transitional evaluation function is calculated by using the formula

$$V(x_0) = \sum_{r=1}^{n} d_r .$$

Upon accepting or rejecting arc No. 1, two new vertices are produced in the tree, namely $x_{1,\,2}$.

$$X(x_1) = V(x_0)$$

and

$$V(x_2) = V(x_0) - d_1 + d_{n+1} .$$

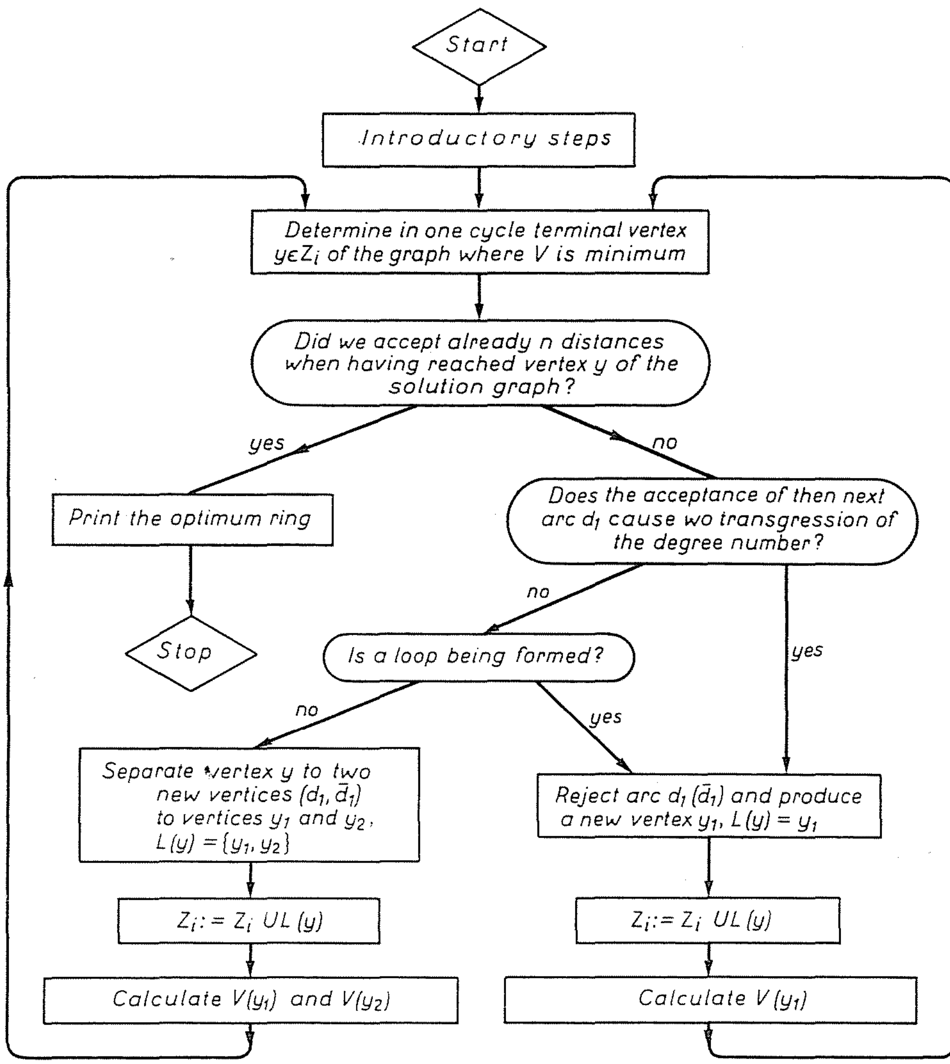

Fig. 3. Scheme of the exact algorithm

Z_i denotes the set of the terminal vertex of the solution graph obtained in the i-th step, e.g.

$$Z_1 = \{x_1, x_2\}$$

After these introductory steps, the optimum ring can be obtained by employing the iteration process shown in Fig. 3.

The value of $V(x)$ is calculated as follows.

To each vertex of the tree a numerical value (transitional evaluation) is assigned, which is the lower limit of the total length of rings $M(x)$ represented by the respective vertex (H_K).

$$V(x) \leqq \min_{K \in M(x)} H_K,$$

where

$$H_K = \sum_{(x,y) \in K} d(x, y) .$$

The program of the algorithm has also been prepared in the ALGOL 60 programming language.

4. Designing the layout of several rings

The problem of several rings is discussed as the generalization of the case described in Chapter 3. In addition to paper [8] and beyond the application to power networks, in the present paper the problem of terminating iteration is discussed and a solution method is proposed.

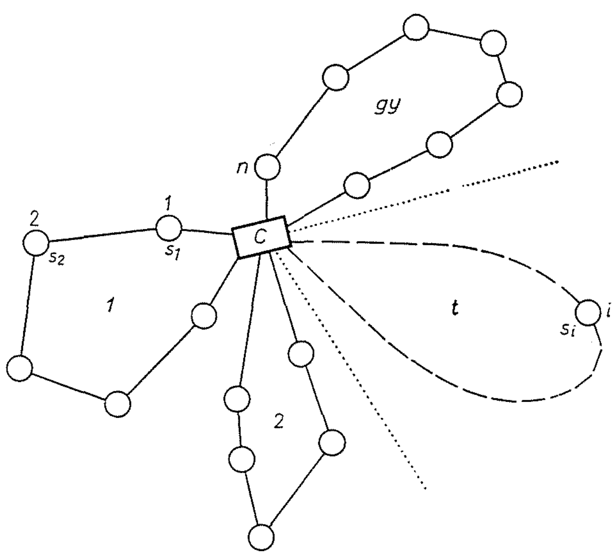

Fig. 4. Several rings

In the "one ring" problem the ring is in general closed at a starting (center, feeder) point. If the n consumer points can be connected by conductors in such a way that more than one ring is formed starting from the feeder point, we speak of the problem of "several rings", see Fig. 4. The number of rings is denoted by gy.

At every consumer point a demand $s_i \geqslant 0$ (load) arises, to be satisfied by the ring passing the point. The load of the ring (T) is defined as the sum of the loads of the passed points:

$$T = \sum_{i \in P} s_i$$

where P denotes the set of the order numbers of points traversed by the ring, the demands of which points are satisfied by the ring.

The capacity of the ring (Q) is defined as the maximum load which the ring can still support.

$$T \leqq Q .$$

In the case of the problem of several rings the given data are the feeder point (center), n consumer points each having a demand (load) $s_i > 0$. $i = 1, 2, \ldots, n$. The mutual distances of the consumer points are also known:

$$\mathbf{D} = [d_{ij}].$$

The capacity of the rings to be designed is Q, the number of the rings (gy) should be determined in the course of the design process. Each possible solution, as a network with several rings, is characterized, as a graph, by the consumer points of 2nd, degree, while the degree of the feeder point is $2\,gy$.

Restricting conditions:

1. The load on any ring cannot exceed the capacity of that ring.
2. The voltage drop along any ring cannot exceed the permitted value.

The set of possible solutions satisfying the above conditions has a very high number of elements. Among these that one (those) should be determined to which the minimum total length belongs.

Total length ($\ddot{O}H$) can be calculated in the case of a possible solution by using the following relationship.

$$\ddot{O}H = \sum_{t=1}^{gy} \sum_{(i,j) \in k_t} d_{ij} .$$

The problem is solved by the Branch and Bound Method on the analogy of the single ring. The separation principle Sz is identical, with the difference that the possibility of using an arc is excluded not only by the formation of loops, when more than two arcs are joining a vertex, but also in the case when in selecting an arc a path section arises on which the sum of the demand of vertices (total consumption) exceeds Q.

The first task is to determine the minimum number of rings S.

At least S rings should pass the feeder point P_0, this means that S is the lower limit of the number of necessary rings.

$$gy_{\min} = S.$$

If there are gy independent loops (round tours, rings) in a graph, the number of vertices is $(n + 1)$, and the graph itself is connected $(p = 1)$, the relationship for the cyclomatic number (zeroity) will be

$$gy = E - (n + 1) + 1 = E - n$$

where E denotes the number of arcs.

Accordingly the number of the arcs in a multi-ringed network graph is found to be $E = gy + n$. The minimum number of arcs in the network is

$$E_{\min} = gy_{\min} + n = S + n.$$

By the Branch and Bound Method at least $n + s$ arcs should be accepted in the course of the process. Thus the minorization condition of the transitional evaluation function can be satisfied, calculated. The following evaluation is assigned to the root of the tree.

$$V(x_0) = \sum_{r=1}^{E_{\min}} d_r$$

where d_r is the r-th distance among those arranged in increasing order ($r = 1, 2, \ldots$).

The algorithm is identical with that given in Chapter 3, with the above complements. The question of finishing the iteration should be examined separately.

In the case of the one-ring problem this was a simple question: The procedure should be finished at the vertex having a minimum V value, where already exactly n arcs occurred.

In the present case we know only that at least S rings should be formed and at least $n + S$ arcs are to be accepted. Hence we do not know in advance how many arcs and rings are contained in the optimum solution. All we know is that their difference is constant, $E - gy = n$. The calculation of $V(x)$ should be carried out accordingly. The problem types possible theoretically and in practice, together with their solving algorithms, were described in details in [10].

In designing electrical (distribution) networks, existing networks can be taken into consideration so (in the case of a program calculating with lengths), that the value $d_{ij} = 0$ is assigned to the distance of any points i and j between which the conductor already exists. Connections forbidden on geographical or other reasons are similarly taken into consideration in the algorithm by substituting ∞ at these places of the distance matrix (in computer programs a very high number).

The method is best suited for designing a 10 kV ringed cable network. Here the cross section of the conductor is constant, voltage drop is low (in general much lower than the permitted value), and it is very simple to check the loadability.

5. Calculation of the optimum feeding zone of a transformer station (feeder point)

If the location of feeder and consumer points is known, the practical question arises, which consumers should be supplied from the individual feeder points.

Various stations are provided for supplying the consumers. Each consumer has a given load, differing for each one. Station capacity is limited, i.e. there is a certain installed capacity. The optimum supply area of the stations should be determined so as not to surpass the capacity limit. Several approximation methods have already been tried but only the Branch and Bound Method permitted to elaborate an exact algorithm.

5.1. Mathematical formulation of the optimum feeding zone

The problem is formulated not only for electrical networks, but also for consumer-producer type problems. m consumer points are given geographically (x, y)

$$S_j = X_j k + Y_j l$$

each having a demand (weight) $P_j > 0$, where $j = 1, 2, \ldots, m$. n feeder points (source, remitter, loader serve for supplying the consumers, determined by

$$Z_i = X_{z_i} k + Y_{z_i} l$$

where $i = 1, 2, \ldots, n$).

All the demands of a single consumer should be satisfied by a single feeder point. The set of points supplied by a producer is named area. Thus, each feeder point will have an area, i.e. n pieces in all. The optimum area of feeder points should be determined in such a way that the value of the target function

$$\sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} d_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} |Z_i - S_j| = \text{Min}$$

is a minimum. The i-th feeder point from the j-th consumer point is at a distance d_{ij} (see Fig. 5).

In the target function p_{ij} denotes the product quantity supplied from the i-th feeder point to the j-th consumer point (weight, load). Accordingly, for the points arranged in the i-th area $p_{ij} = pj$, for the other points (not supplied from the i-th feeder point) $p_{ij} = 0$.

Each feeder point has a given capacity. Optimum areas should be determined so as not to exceed capacity limits.

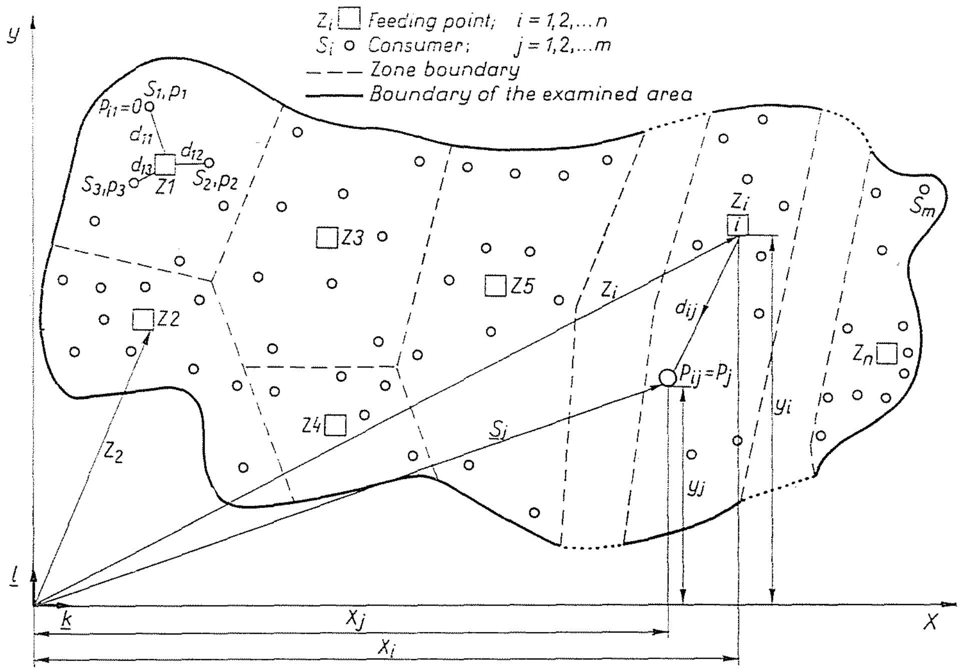

Fig. 5. Optimum supply zone

5.2. *Solution by the Branch and Bound Method*

By utilizing the combinatoric character of the problem, the solution algorithm was elaborated by the Branch and Bound Method.

Meanings of the mathematical symbols for the optimum supply area problem are:

M = set of all the possible (realizable) solutions, having here the meaning of possible supply combinations.

$f = \Sigma\Sigma\, p_{ij} \cdot d_{ij}$ real valued function.

The solution is to be determined for which f is a minimum. Each vertex $x \in X$ of the solution graph corresponds to a subset in the set M. The d_{ij} values are arranged in increasing order and decision is made on the acceptance of the individual arcs, one after another. The separation principle Sz denotes here that some (the just following) arc $i - j$ representing the value d_{ij} is accepted or rejected. If arc $i - j$ is accepted, this means that the j-th consumer is supplied from the i-th feeder point. Accordingly when accepting an arc, a decision has already been made on the "destiny" of a consumer. Since there are m consumers, the possible solutions require the acceptance of m arcs.

An evaluation $V(x)$ is ordered to each subset $M(x)$.

The value

$$V(x_0) = \sum_{k=1}^{m} d_k$$

is assigned to the root of the tree, where k denotes the ordinal number of the distances arranged in increasing order, i.e. the first m values are summarized. At intermediate vertices, if the arc is accepted, then the value of $V(x)$ remains unchanged; if it is rejected, then the length of this arc is substracted and the length of the next arc is added, together with which exactly m arcs are included in $V(x)$. Separation is always carried out at the terminal vertex of the lowest $V(x)$ value. The limiting condition is always checked for transgression.

The iteration steps given in Chapter 2 result in composing the process graph and determining optimum supply areas.

6. Design of radial networks
Determination of a tree of minimum total value

The problem of a tree of minimum total value arises in electrical network design in the case of radial networks.

The radial network as a graph corresponds to a tree. If the nodes of the network are known and connections can only be realized between these nodes (i.e. no new nodes may be produced), then the determination of the radial network of minimum total length (or cost) is equivalent to that of the minimum tree in the complete graph of n nodes.

The minimum tree can be determined also by using the Kruskal algorithm. It consists essentially in accepting minimum length sections which do not form a loop with the preceding ones. If there are several optimum solutions, they can be determined only by a high number of repeated calculations if using the Kruskal algorithm.

The author has elaborated an exact algorithm for determining the minimum tree by using the Branch and Bound Method.

By employing the Branch and Bound Method, radial network problems with several optimum solutions can also be determined rapidly by the algorithm described in connection with the calculation of ringed networks, with the difference that $n - 1$ arcs should be accepted here and no loops arise.

The number of possible trees is very high. Among these the one having the minimum total length (cost) should be selected.

$$H_F = \sum_{d\,(i,j) \in F} d(i, j) = \text{Min}!$$

where F denotes the set of the arcs of a possible tree.

According to the Branch and Bound Method, M denotes here the set of possible trees. At each vertex of the process tree $V(x)$ is the lower limit of the total length of network trees $M(x)$ represented by the respective vertex.

Separation is carried out in the process graph in such a way that the just examined k-th arc is accepted in the one branch (k), and rejected in the other branch $(\bar{k})$.

Introductory steps

1. Arrange distances d_{ij} in increasing order and assigne them ordinal numbers $(k = 1, 2, \ldots)$.

2. Assign the set of all possible network trees to the reference vertex (vertex x_0) of the solution tree, $M(x_0) = M$.

Calculate the transitional evaluation function to vertex x_0 by using the relationship

$$V(x_0) = \sum_{k=1}^{n=1} d_k$$

to vertex x_0.

Since there are $(n - 1)$ arcs in each tree, the value $V(x_0)$ satisfies the minorization requirement.

3. According to the separation principle divide set $M(X_0)$ into two parts. One part accepts the arc of ordinal number 1, while the other does not. Let x_1 and x_2 designate the two vertices arising in the solution graph in this way. Z_i designates the set of terminal vertices of the solution graph arising in the i-th step. In the first step $Z_1 = \{x_1, x_2\}$.

Calculate the values $V(x_1)$, $V(x_2)$.

$$V(x_1) = V(x_0)$$
$$V(x_2) = V(x_0) = d_i + d_n.$$

Each value $V(x)$ will be the sum of $(n - 1)$ distances.

Iteration steps

4. Select from set Z_i the element y with the lowest transitional evaluation. Vertex y in the solution graph was produced by "deciding" on the $(l - 1)$th length.

5. Examine the next following l-th distance, whether it can be accepted or not. Does it form a loop with previously accepted arcs, or is the degree of the feeder point two or not?

6. If it would produce a loop or cause a transgression of the permitted degree, reject this distance and form a new vertex in the solution graph. Calculate the transitional evaluation for the new vertex, modify Z_i, and continue by step 4.

7. If the l-th distance just examined can be accepted, then employ the separation principle for subset $M(y)$. Two new vertices are obtained in the solution graph. One part contains the l-th distance, while the other not.

8. Calculate the values of the transitional evaluation function for the two new vertices.

9. Modify the set of the terminal vertices of the solution graph (Z_i) and continue by step 4.

10. Terminate iteration at vertex y at the lowest evaluation, where $(n - 1)$ distances had already been accepted. This gives an optimum network tree.

The Branch and Bound Method checks loadability and voltage drop by enlarging the limiting conditions. By a further development of the method it is also suited for solving problems where connections with different cross sections (costs) are possible between different nodes of the network. Thus the Branch and Bound Method is suited also for designing radial low voltage overhead line networks, the determination of the minimum tree in an "n graph".

7. Determination of optimum short-run plans

In determining optimum plans for one year (or for other periods) the problem is that only a part of projected investments can be realized during the period on account of limited financial possibilities. It is reasonable to utilize available financial means so that profits be a maximum, or to realize those projects first where necessity is of the highest degree. In the present chapter this type of problems is discussed, and a practical design application has been elaborated.

7.1. Formulation of the problem

As a result of previous decisions, various ready projects are available, among which the investment program for the given period should be determined, taking into consideration the existing financial limitations. Two characteristics are determined for each project.

a) Investment costs (*Ft*),

b) Weighting (importance) factor, e.g. profits.

The problem is to determine which projects should be taken into the investment program, in order to obtaining maximum importance (profit), without transgressing total investment costs.

The given quantities are

1) n (pieces) projects
2) $s_j\,(Ft)$ investment demands $(j = 1, 2, \ldots, n)$ and
3) c_j importance (necessity, weighting) factor for the j-th project
4) $S\,(Ft)$ the total investment cost available for all the realized (programmed) projects.

Let us introduce a bi-valued variable X_j.
$X_j = 1$, if the j-th project is included in the program,
$X_j = 0$, if not.
Condition is that

$$\sum_{j=1}^{n} s_j x_j \leq S ,$$

i.e. the total investment cost for all the realized (programmed) projects is lower, or at the maximum equal to S.

The target function is

$$\sum_{j=1}^{n} c_j x_j = \text{Max}\,!$$

This means that those projects capable of providing maximum importance, necessity should be utilized. The x_j values should be determined $(j = 1, 2, \ldots, n)$, namely $x_1 x_2 \ldots x_n$. These can have the value 1 or 0. It should be noted that the factors c_j can be given any meaning, just the one, which is required to reach a maximum, e.g. profit, importance, etc.

The above problem is analogous with the so-called "knapsack" problem in operation research.

7.2. Application of the Branch and Bound Method

On account of the combinatoric character of the problem, also here the Branch and Bound Method is employed [8]. The meaning of the mathematical symbols in this problem is the following.

M is the set of all possible solutions satisfying the limiting conditions, $f = \Sigma\, c_j x_j$ real valued function.

In the course of the process, element $k \in M$ should be found in set M for which $f(k)$ is a maximum (optimum solution). The meaning of the separation principle is here that two decisions are made on the following project: Accepted $(x_i = 1)$, or rejected $(x_i = 0)$. The separation principle satisfies the conditions.

An evaluation $V(x)$ is assigned to each subset $M(x)$.

The value

$$V(x_0) = \sum_{j=1}^{n} c_j$$

is assigned to the root of the tree. Hereafter the iteration process is in principle identical with the preceeding, and the optimum short-run plan can be obtained by constructing the process graph.

8. Solution of the assignment problem by the Branch and Bound Method

In division of labour problems the assignment problem arises.

8.1. The mathematical model

A quadratic matrix of the n-th order is given

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \hline c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix}.$$

The problem is to select n elements from the above matrix in such a way that only one element is taken from each row and column and the sum of elements selected should be a maximum or minimum among all the possible selections.

It is easy to conceive that this is essentially a special transportation problem. A solution matrix

$$\mathbf{X} = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ x_{n1} & \dots & x_{nn} \end{bmatrix}$$

is to be found to the general transportation problem where only one element in each row and column equals 1, all the other elements being zero, and for which the sum

$$f = \sum \sum C_{ij} X_{ij}$$

is minimum among all the matrices of this kind.

The problem has a combinatoric character, the number of all possible selections is $n!$

8.2. The practical model

Labour division problems should be dealt with as assignment problems.

Let us have n workers to be assigned for n jobs. The i-th worker carries out the j-th job with a cost (or time) c_{ij}. Each worker can perform only a single job. Let x_{ij} denote the element in the i-th row and j-th column of the graph where the sum of each row and column is 1, and it may contain only the numbers 1 or 0. The problem is to assign the n jobs to the n workers in such a way that the total cost (total time) for all the jobs should be a minimum. The problem consists in minimizing the sum of all the products

$$\sum \sum x_{ij} \cdot C_{ij}.$$

The target function is

$$\sum \sum x_{ij} c_{ij} = \min ! \qquad c_{ij} \leqq 0 .$$

The restricting condition are

$$\sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1$$

$$i, j = 1, 2, 3 \ldots n$$

where $x_{ij}^2 = x_{ij}$ $[x_{ij} = 0$, or $1]$.

The problem can be examined with indices as well, from the aspect of the computer algorithm this is even more advantageous. The assignment of the i-th worker to the j-th job represents the acceptance of the pair of indices $i - j$. In each possible solution a job can only be assigned to a single worker (or inversely). If the pairs of indices of the accepted elements are represented in the solution as a column, then this condition has the meaning that each index should occur in each column only once, but at least once.

The concise though general formulation of the problem is the following: n pieces of indivisible production factors (men, machines, tools, etc.) should be divided (assigned, designated) to n working places in such a manner that the resulting total profit should be a maximum, with given C_{ij} specific profits. This points out obviously that there is a very wide field of practical applications.

8.3. Solution algorithm using the Branch and Bound Method

The process is described here by uniting the methods to be found in [5, 8].

The meaning of the mathematical symbols in the assignment problem.

M is the set of all possible (realizable) assignments,

$f = \Sigma \, \Sigma \, C_{ij} \cdot x_{ij}$ a real valued function,

total cost, total length, time.

The solution, where f is minimum, is to be determined.

The elements C_{ij} are arranged in increasing order and decision is made on their acceptance, one after another: The i-th "matter" is either assigned to the j-th, or not. The separation principle Sz has here the meaning that the pair of indices $i - j$ (assignment) representing the just following value c_{ij} is accepted or not. Each solution should contain n assignments in all.

As known, by using the Branch and Bound Method each vertex $x \in X$ of the process graph to be constructed step by step corresponds to a subset of set M. An evaluation $V(x)$ is assigned to each subset $M(x)$. The value

$$V(X_0) = \sum_{k=1}^{n} c_k$$

is assigned to the root of the tree, where k denotes the ordinal number of costs (times, distances) arranged in increasing order, c_k the value of the element having the ordinal number k. At an arbitrary intermediate vertex, if C_{ij} is accepted, the value $V(x)$ remains unchanged. If it is not accepted, the value c_{ij} is substracted from $V(x)$. Together with c_{ij}, n elements occur in $V(x)$. Separation is always performed at the terminal vertex having the lowest $V(x)$ value.

Iteration is finished if at some vertex n elements (assignments) have already been accepted by considering the restricting condition. This will be the optimum solution.

Summary

The number of set elements of the possible solution is very high in the case of integer-valued or combinatory optimization problems. Complete counting algorithms cannot be realized even by employing a computer. For problems of this character, the possibility of elaborating the exact solution process is given by the Branch and Bound Method, without concretely enumerating all possible solutions.

The paper gives the elaborated processes for solving the following practical problems (especially from the field of electrical network design) by using the Branch and Bound Method known from the literature of operations research.

Design of one-ringed or multi-ringed type networks.

Calculation of the optimum supply area of transformer stations.

Design of radial networks.

Preparation of optimum yearly programs.

Soution of assignment problems.

Calculation methods are described in connection with the elaborated computer iteration algorithms.

Beside the optimum solution the alternative optima and solutions which are nearly equivalent are also given by the method which is of great significance from the aspect of the economic and realizability evaluation of concrete design problems.

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