

A NEW METHOD FOR THE FAST COMPUTING OF THE FREQUENCY CHARACTERISTIC OF CONTROL SYSTEM

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1. Principle of the approximating frequency analysis

The Computer Process Control gives an ever increasing importance to the fast numerical approximation methods which allow a real-time analysis of the control loop. By this method the computer immediately recognizes the changes in the process, and modifies the process model, and thus it is able to execute in time the compensating or optimizing operation on the process.

Very often, the computer guides the process by the Set Point Control, which generally causes the process to get from a previous steady-state condition into a subsequent steady-state condition. Thus, the step response function belonging to the change of the reference signal of the control system may be available for a computerized analysis.

In the frequency range there are adequate methods given for the analysis and design of the control loops, therefore the corresponding frequency functions must be determined from the time functions by transformation.

If the characteristic function for the system is given either graphically or numerically, the Fourier transformation can be performed only after previous identification or by numerical approximation.

The purpose of our investigations was to find a numerical method by which the middle frequency region of the amplitude-phase function $W(j\omega)$ — the most important part for the design — can be simply determined from the step response $v(t)$ of a linear element. For the design of control systems in practice, first of all, that range is of interest where the phase angle of $W(j\omega)$ is about -180° . The frequency belonging to the phase shift of -180° is the critical frequency ω_k ; the one decade range around this value will be referred to as the middle frequency range mentioned above. Compensation in practice is adjusted on the basis of the middle frequency section of the Bode plot, at low and high frequencies an approximate knowledge of the curve shape is also sufficient.

As it follows from the theory of the Fourier transformation, the knowledge of the whole time function in the range of $0 < t < \infty$ is necessary for the determination of a single point of the frequency function. In practice, however, the values of the time function obtained at different times are weight-

ed very differently in the sum of an infinite series of elements that approximates the Fourier integral. It would be of great interest to find the time range having the greatest influence upon the middle frequency range.

The main object of this work was to develop an approximative formula of the Fourier transformation which, by its simplicity, can be applied as a quick computing method, using the fewest possible points of the step response function for the computation of a single point of the frequency function, and giving a good approximation in the middle frequency range.

The developed new numerical method, which is called middle frequency transformation (henceforth MFT), determines the approximation

$$W_1(j\omega) = R_1(\omega) + jQ_1(\omega)$$

of the frequency function $W(j\omega)$.

As demonstrated in the 2nd part, the expressions of the real and imaginary components are

$$R_1(\omega) = c \left[v \left(\frac{T}{8} \right) + v \left(\frac{3T}{8} \right) - v \left(\frac{5T}{8} \right) - v \left(\frac{7T}{8} \right) + v(T) \right] - \frac{w'(T)}{\omega^2},$$

$$Q_1(\omega) = c \left[v \left(\frac{T}{8} \right) - v \left(\frac{3T}{8} \right) - v \left(\frac{5T}{8} \right) + v \left(\frac{7T}{8} \right) - \frac{w(T)T}{8} \right],$$

where $T = 2\pi/\omega$, i.e. the period belonging to the actual angular frequency, $v(t)$ is the step response function, $w(t)$ the weight function and $w'(t)$ the derivative of the weight function. The value of constant c is 1.13.

Thus, if it is desired to determine the frequency function at an actual frequency ω , the period T belonging to ω has to be divided into 8 parts, and one must know the values of the step response function in the division points 1, 3, 5, 7 and the function value and its first and second derivatives in point 8 (which can numerically be determined from the step response function).

The program of the formulae has been made for a digital computer and the middle frequency transformation performed for elements of different structures and parameters. The approximating plot obtained in this way deviates from the exact curve by less than 10 per cent in the two-octave range around the critical frequency, and this is a sufficient accuracy for the design work.

The results obtained with the new transformation formulae have demonstrated that the step response function has a section decisively influencing the middle frequency range. This so-called "decisive" time interval of the step function can be determined with the help of the MFT. This raises a possibility of performing the analysis of the control system directly in the time domain and of setting the parameters of the compensating element on the basis of the step response function.

In the following chapter the main steps of developing the new transformation formula will be discussed, whereas in the 3rd chapter some results of the computerized examination will be presented and the possibilities of utilizing the MFT considered.

2. Development of the middle frequency transformation formula

2.1. Choice of the upper limit of the integral

The frequency function can be determined from the absolutely integrable weight function of the linear control loop as

$$\int_{-\infty}^{\infty} w(t) e^{-j\omega t} dt = \int_0^{\infty} w(t) e^{-j\omega t} dt = W(j\omega) = R(\omega) = jQ(\omega).$$

Since $w(t)$ is often not absolutely integrable from 0 to ∞ , it is more convenient to write the Fourier transformation of $w'(t)$ or $w''(t)$; the transform of this exists even in the case of a double-integrating element. Here it was considered that $w'(-0) = w(-0) = 0$.

As a first approximation, let the upper limit of the integral be a finite value. This approximation can be demonstrated to be acceptable if this upper limit is a multiple of the period $T = 2\pi/\omega$ corresponding to the actual frequency (as demonstrated in Appendix 1).

Let $R_K(\omega)$ be the symbol for the real part of the approximating frequency function computed with the integral to the upper limit kT , and $Q_K(\omega)$ for its imaginary part

$$\int_0^{kT} w''(t) \cos \omega t dt = -\omega^2 R_K(\omega) \quad (1)$$

$$\int_0^{kT} w''(t) \sin \omega t dt = -\omega^2 Q_K(\omega). \quad (2)$$

After integration by parts of the above expressions and using the relationships $w'(0) = 0$, $\cos kT = 1$ and $\sin kT = 0$, the following equations will be obtained:

$$w'(kT) + \omega \int_0^{kT} w'(t) \sin \omega t dt = -\omega^2 R_K(\omega) \quad (3)$$

$$\omega \int_0^{kT} w'(t) \cos \omega t dt = -\omega^2 Q_K(\omega). \quad (4)$$

The problem is, how to choose the upper limit of the integration for the sake of a good approximation.

Figs 1a and b show the derivative of the weight function of a multiple-storage proportional element and that of a double-integrating multiple-storage

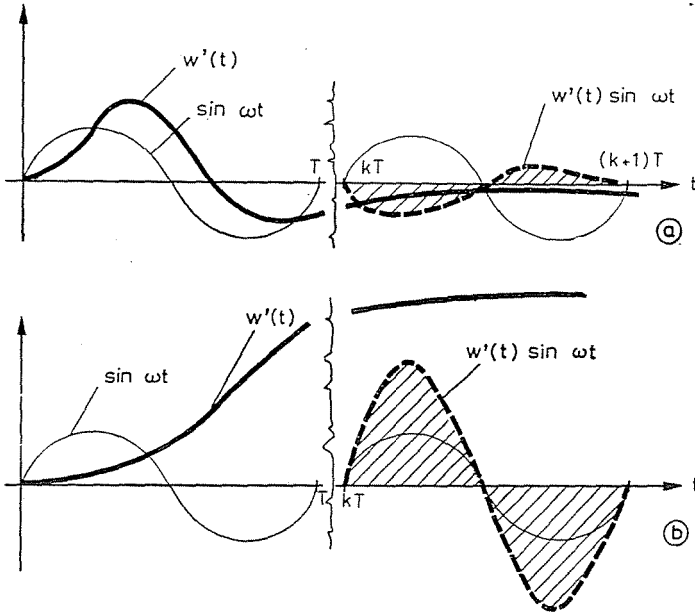


Fig. 1

element, resp., as well as the sine wave with period T and angular frequency $\omega = 2\pi/T$.

If k has a value where $w'(t)$ has sufficiently approached its steady-state value for $t > kT = k \frac{2\pi}{\omega}$, then the integral of $w'(t) \sin t$ for the $(k + 1)$ -th and further periods will be small, and thus the integral taken up to kT will provide a good approximation; the same applies to the integral of $w'(t) \cos t$.

It is clear that the value of k chosen under such conditions depends on the actual value of ω . The higher the angular frequency for which $R(\omega)$ and $Q(\omega)$ are to be determined, the greater k has to be chosen in order that kT lies in the steady-state portion of the curve. If T is sufficiently great, a reasonable approximation will be obtained also for $k = 1$. This has also been proved by a computer method. The analysis of transient behaviour according to Appendix 1 will demonstrate why this approximation is satisfactory also for $k = 1$.

With the choice of $k = 1$, the period $T = 2\pi/\omega$ will be taken in the following as the upper limit of the integral. Accordingly:

$$\frac{w'(T)}{\omega} + \int_0^T w'(t) \sin \omega t \, dt = -\omega R_1(\omega) \tag{5}$$

$$\int_0^T w'(t) \cos \omega t \, dt = -\omega Q_1(\omega). \tag{6}$$

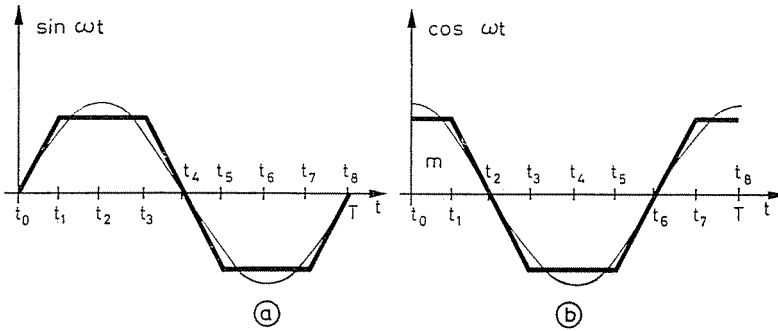


Fig. 2

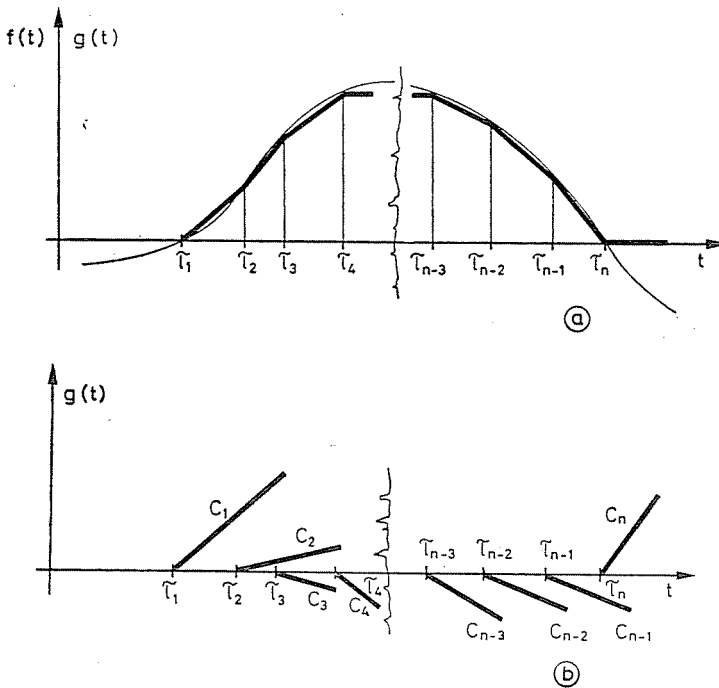


Fig. 3

2.2. Numerical approximation of the integrals

For the numerical evaluation of the integrals in expressions (5) and (6) let the sine and cosine functions be approximated with straight lines.

Replace the sine and cosine function in the interval $[0, T]$ by the trapezoids of Figs 2a and b respectively.

For practical reasons it is convenient to divide the period T into 8 equal parts, i.e. $t_8 = T, t_1 = T/8; t_3 = 3T/8; t_5 = 5T/8$, etc.

Express the slope of the side of the trapezoid as $c \cdot \omega$, i.e. the product of the derivative of the sine function in the starting point by a constant (c).

In this case the height of the trapezoid will be

$$m = c \cdot \omega \cdot \frac{T}{8} = c \cdot \frac{\pi}{4}. \quad (7)$$

The error in the approximation of the trapezoid can be estimated by writing the Fourier series for the trapezoidal function:

$$g(t) = \frac{8\sqrt{2}}{\pi^2} \left[\sin \omega t + \frac{1}{9} \sin 3\omega t - \frac{1}{25} \sin 5\omega t - \dots \right].$$

The amplitude of the fundamental harmonic of the trapezoidal function will be 1, if $m = 0.88$ is chosen as the height of the trapezoid. For this a value of $c = 1.12$ will be needed according to formula (7).

In the trapezoidal function the 3rd harmonic is seen to be present with a weight factor $1/9$, and the 5th harmonic with a weight factor $-1/25$. Thus the error produced by the harmonics is small.

Approximating the sine and cosine functions with the trapezoids of the geometry shown in Fig. 2, the following approximation of the integrals will be obtained (after substitution into the formula (F 20) of Appendix 2):

$$\int_0^T w'(t) \sin \omega t \, dt \approx c \cdot \omega [-v(t_1) - v(t_3) + v(t_5) + v(t_7) - v(t_8)]$$

$$\int_0^T w'(t) \cos \omega t \, dt \approx c \cdot \omega \left[-v(t_1) + v(t_3) + v(t_5) - v(t_7) + \frac{w(t_8)t_8}{8} \right].$$

Their substitution into formulae (5) and (6) will lead to the middle frequency transformation formulae:

$$R_1(\omega) = c \left[v\left(\frac{T}{8}\right) + v\left(\frac{3T}{8}\right) - v\left(\frac{5T}{8}\right) - v\left(\frac{7T}{8}\right) + v(T) \right] - \frac{w'(T)}{\omega^2} \quad (8)$$

$$Q_1(\omega) = c \left[v\left(\frac{T}{8}\right) - v\left(\frac{3T}{8}\right) - v\left(\frac{5T}{8}\right) + v\left(\frac{7T}{8}\right) - \frac{w(T) \cdot T}{8} \right]. \quad (9)$$

3. Tests of the middle-frequency transformation

The middle frequency transformation formulae (8) and (9) were programmed for a digital computer. By means of this program the frequency function of elements of various types were tabulated and compared with the exact curve.

The step response functions were generated by a program. The structures analyzed were as follows (also the identification letters are given in brackets):

1. Two-storage integrating element (KI)
2. Three-storage proportional element (HT)
3. Three-storage integrating element (HI)
4. Four-storage proportional element (NT)
5. Two-storage integrating element, with two time constants in its numerator (KID).

The time constants were in each case adjusted to give the product 1, and their ratio was chosen according to the powers of 2 (1, 2, 4, 8) with each possible combination. The gain of the loop was set to the critical value, thus the exact Nyquist diagrams always crossed the point -1 of the real axis (therefore at this frequency the amplitude curve of the Bode diagram intersected the 0 dB axis).

The approximation formulae were used for tabulating the Nyquist and Bode diagrams of the elements and determining the calculated value of the critical frequency. Because of the approximation the intersection of the Nyquist plot does not cross the -1 point, i.e. the value of the amplitude curve of the Bode diagram differs from 0 dB at the critical frequency. So one possible measure of the goodness of the approximation is the deviation of the calculated value (ω_{ksz}) of the critical frequency from its exact value (ω_{kp}) as shown by the error column h :

$$h_{\omega} = \frac{\omega_{ksz} - \omega_{kp}}{\omega_{kp}} \cdot 100\%$$

The error of the calculated value of the amplitude at the critical frequency is

$$h_a = (R(\omega_{ksz}) + 1) \cdot 100\%$$

since the exact value of the amplitude ratio would be -1 .

As demonstrated in Table 1, the error of the critical frequency is below 5 per cent in each element, but for most elements the error does not exceed 3 per cent. The error of the axis segments is below 10 per cent even in the few most outstanding cases, and in the overwhelming majority it is below 5 per cent. This means that by the approximative formulae the critical frequency of any member together with the amplitude value at this frequency can be determined at a sufficient accuracy, and thus it is possible to set the parameters of a compensating element approximatively by some known method. Thus, the MFT is suitable for a very fast coarse compensation which then can be refined with other methods; such starting values are well used in any adaptive optimizing procedure. From the viewpoint of computerized real-time process

Table 1

Proportional elements							Integrating elements								
structure	time constants				error		structure	time constants					error		
	T_1	T_2	T_3	T_4	h_e %	h_a %		T_1	T_2	T_3	τ_1	τ_2	h_e %	h_a %	
HT	1	1	1		3.52	2.3	KI	1	1				0.8	- 2.3	
	1	1	2		3.03	1.2		1	2				0.4	- 3.2	
	1	1	4		1.96	-1.1		1	4				- 0.8	- 6.1	
	1	1	8		1.16	-2.5		1	8				- 3.3	-12.0	
	1	2	2		2.99	1.0		HI	1	1	1			1.7	0.8
	1	2	4		2.1	-1.1			1	1	2			1.8	0.6
	1	2	8		1.0	-3.3			1	1	4			1.5	- 0.5
	1	4	4		1.3	-3.2			1	1	8			0.4	- 3.2
	1	4	8		0.1	-5.6			1	2	2			1.6	0.6
	1	8	8		-1.5	-9.6			1	2	4			1.5	0.1
NT	1	1	1	1	4.3	8.0	1		2	8			1.0	- 1.7	
	1	1	1	2	4.1	6.9	1		4	4			1.8	0.2	
	1	1	1	4	3.4	4.6	1		4	8			1.3	- 0.7	
	1	1	1	8	2.5	2.3	1		8	8			12.8	17.2	
	1	1	2	2	4.1	6.8	KID	1	1		0.043	0.043	- 0.1	- 4.5	
	1	1	2	4	3.7	5.1		1	1		0.061	0.061	- 0.8	- 6.3	
	1	1	2	8	2.9	2.7		1	1		0.086	0.086	- 1.9	- 9.0	
	1	1	4	4	3.5	4.0		1	1		0.121	0.121	- 4.4	-16.4	
	1	1	4	8	2.9	2.1		1	1		0.172	0.172	-30.1	-	
	1	1	8	8	2.3	0.0									
	1	2	2	2	4.2	7.3									
	1	2	2	4	3.9	5.9									
	1	2	2	8	3.2	3.7									
	1	2	4	4	4.0	5.6									
	1	2	4	8	3.5	3.9									
	1	2	8	8	3.2	2.4									
	1	4	4	4	4.3	6.3									
	1	4	4	8	3.8	4.9									
	1	4	8	8	4.0	4.4									
	1	8	8	8	4.2	5.2									

control, too, the method of quick compensation setting is useful, even though it does not give the optimum result.

The MFT was used to record the Bode diagrams of the elements under analysis and then they were compared with the exact Bode diagrams. As shown in Figs 4 to 8, the computed diagram is very much like the exact curves in the middle frequency range. In the 2-octave interval around the critical frequency the deviation of the amplitude curves is 1–2 dB and that of the phase curves is below 5 degrees.

The curves allow the conclusion that the MFT gives very good results in the middle frequency range.

In the range of higher frequencies, however, the approximation involves greater errors. This becomes clear by the consideration that during the short period belonging to high frequencies the transients are not damped and the steady state is still far.

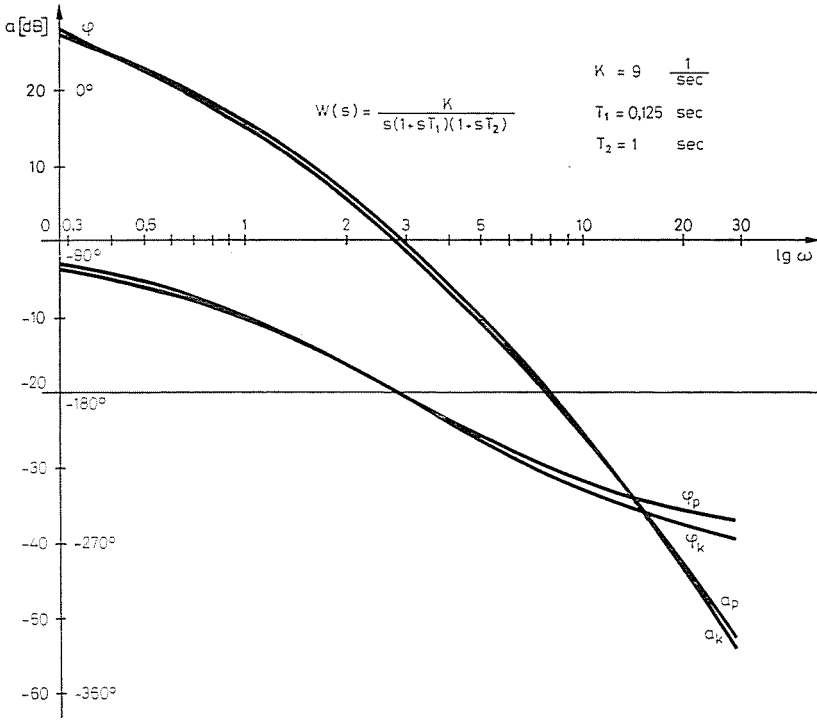


Fig. 4

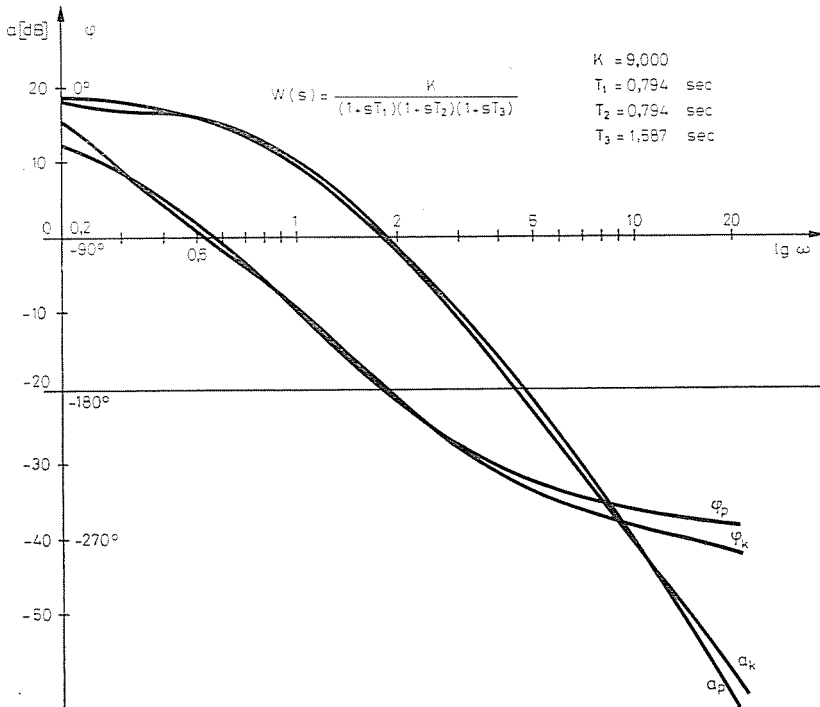


Fig. 5

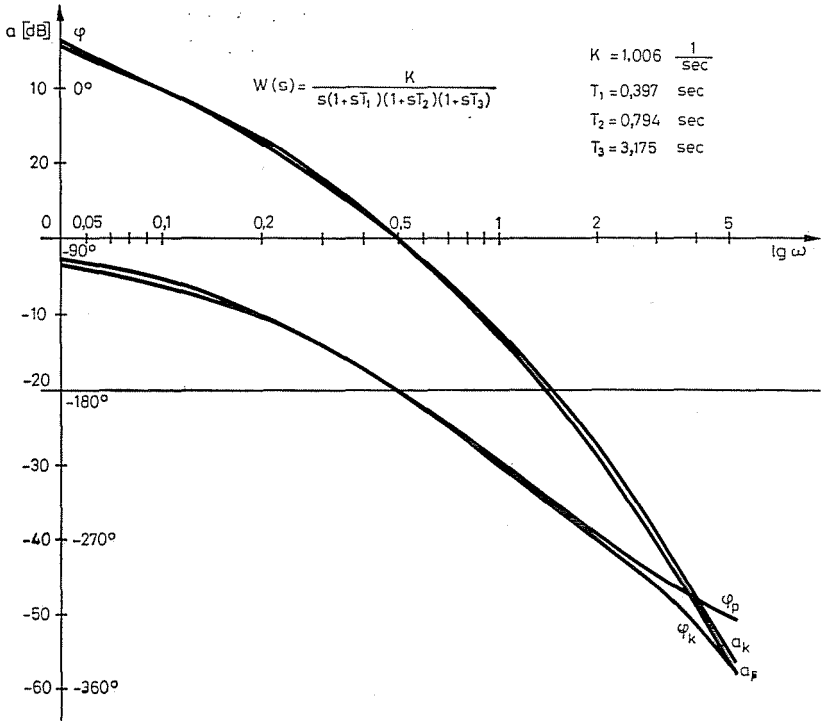


Fig. 6

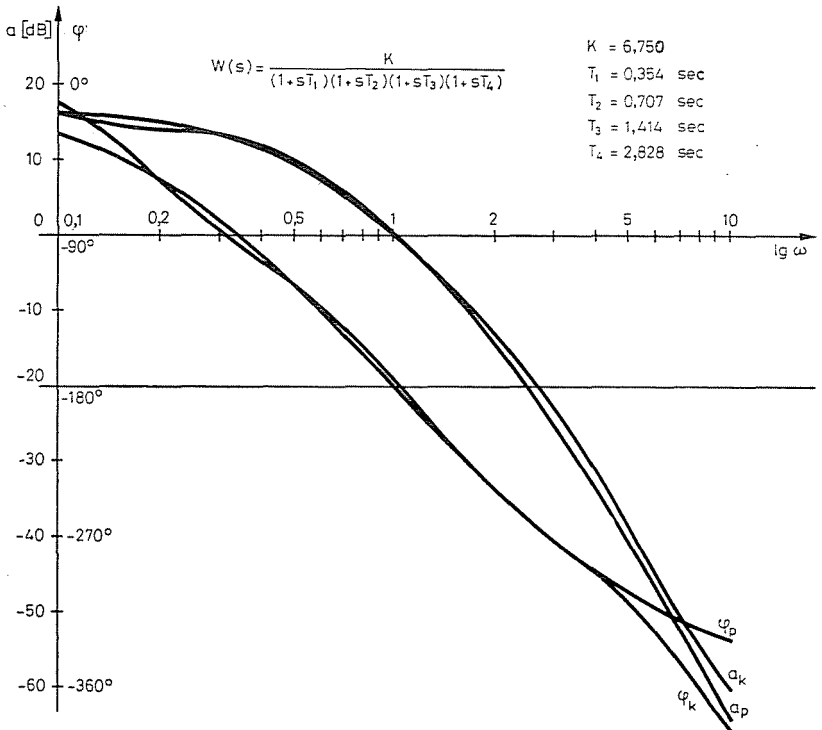


Fig. 7

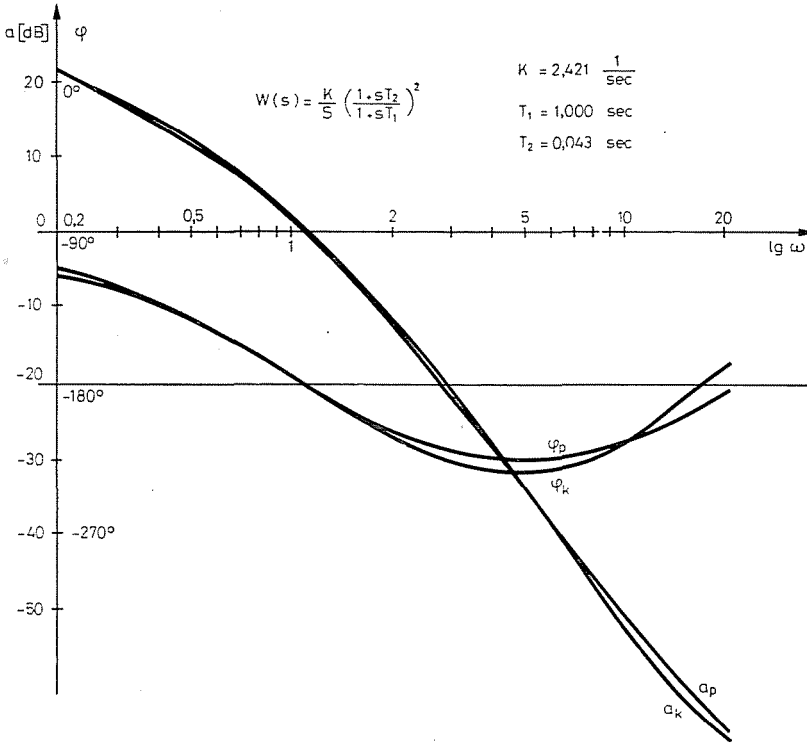


Fig. 8

The error increasing towards the lower frequencies is due to the trapezoidal approximation of the sine and cosine functions; the points sampled from the step response function get more and more distant from each other, giving less and less information on the intermediate course of the curve.

Since the approximative Bode diagram has a satisfactory accuracy in the middle frequency range essential for design work, a possibility is given to determine the Bode diagram from the step response function of the control loop by means of the MFT and, on this ground, to compensate the system with the proved engineering methods.

The approximative Bode diagram allows conclusions on the structure of the system, on the values of the time constants, and thus the system can approximately be identified later.

Beyond the practical results in checking stability and in setting compensation, the MFT has justified also the theoretical consideration that a step response function has a so-called decisive time range decisively affecting the narrow environment of the frequency function around the critical frequency. This means that the stability of a system can be checked on the basis of the decisive time-interval of the step-response function.

Appendix 1

Examination of the MFT by transient phenomena

The output signal $y(t)$ of a linear element can be expressed from its input signal $x(t)$ and the weight function $w(t)$ by means of the convolution integral:

$$y(t) = \int_0^{\infty} w(\tau)x(t - \tau) d\tau. \quad (\text{A1})$$

Let a sinusoidal signal of frequency ω and amplitude x_m appear at the input of the element:

$$x(t) = x_m \cdot \sin \omega t. \quad (\text{A2})$$

This signal has been connected to the system at the time $t = -\infty$, thus, by the positive time the transients are certainly over and a steady or quasi-stationary state has been established. Thus, as an effect of the input signal (F2), the output signal of the element at $t > 0$ will be

$$y(t) = y_m \cdot \sin (\omega t + \varphi). \quad (\text{A3})$$

If the signal is imposed to the input at $t = 0$, then the output signal

$$y(t) = \int_0^t w(\tau)x(t - \tau) d\tau \quad (\text{A4})$$

will still contain the transients. With high values of t (as compared with the damping time of the transients), these transients will however be superposed on the steady-state or quasi-stationary state only with small amplitudes. Therefore, if the signal (A2) is applied to the input at $t = 0$, the output signal equals (A3) only approximately. Writing these in (A4) gives:

$$y_m \sin (\omega t + \varphi) \cong \int_0^t w(\tau)x_m \sin [\omega(t - \tau)] d\tau.$$

Determine the output signal at the point of $t = kT$, i.e. after k periods of the sine wave,

$$y_m \sin \varphi \cong - \int_0^{kT} w(\tau) x_m \sin \omega \tau d\tau \quad (\text{A5})$$

where the relationships

$$y_m \sin (\omega t + \varphi)|_{t=kT} = y_m \sin \varphi \quad (\text{A6})$$

and

$$x_m \sin (\omega t + \varphi)|_{t=kT} = -x_m \sin \omega \tau \quad (\text{A7})$$

have been used.

Dividing formula (A5) by x_m we obtain

$$\frac{y_m}{x_m} \sin \varphi \cong - \int_0^{kT} w(\tau) \sin \omega \tau \, d\tau. \quad (\text{A8})$$

In this formula y_m/x_m is the amplitude ratio of the quasi-stationary signals which equals the absolute value of the frequency function. Accordingly:

$$\frac{y_m}{x_m} \sin \varphi = a(\omega) \sin \varphi = Q(\omega) \quad (\text{A9})$$

which is exactly the imaginary part of the frequency function. It follows from (A8) and (A9) that

$$\int_0^{kT} w(\tau) \sin \omega \tau \, d\tau \cong -Q(\omega). \quad (\text{A10})$$

Thus the approximation formula of the Fourier transformation has been obtained without using the original Fourier integral. In a similar way, also the expression of the real part can be derived if the cosine input function is considered as starting point.

As it could be seen, formula (A10) is only be obtained for $t = kT$, i.e. the transformation formula provides a good result only if a multiple of the period of the sine wave is assumed to be the upper limit of the integral. Since $T = 2\pi/\omega$, the upper limit of the integral always depends on the actual frequency.

On the basis of the above discussion a physical background of the accuracy of the approximation can be presented. The errors are due to transients not completely damped; since, however, they damp exponentially, formula (A10) can also be expected to give a satisfactory approximation with $k = 1$.

Appendix 2

Numerical approximation of special integrals

The integrals in expressions (5) and (6) had the following form:

$$I = \int_0^T w'(t) f(t) \, dt. \quad (\text{A11})$$

The question arises how the integral of a product of functions can be computed if one of the factors is approximated by straight lines.

Let the function $f(t)$ be approximated by the function $g(t)$ composed of straight lines in the interval between two zero crossings of τ_1 and τ_n . The function $g(t)$ is time-limited.

$$g(t) = \begin{cases} 0 & \text{if } t \leq \tau_1 \\ \approx f(t) & \text{if } \tau_1 \leq t \leq \tau_n \\ 0 & \text{if } t \geq \tau_n. \end{cases} \quad (\text{A12})$$

Their amplitude is very small as compared with the quasi-stationary part of the output signal even after one single period supposed that the corresponding frequency is in a certain frequency range. Fig. 3a shows the function $f(t)$ and the approximate function $g(t)$. In Fig. 3b only the function $g(t)$ has been plotted, decomposed into half-lines of slope c_i entering at time τ_i .

Considering this, the function $g(t)$ can be written as

$$g(t) = \sum_{i=1}^n (t - \tau_i) c_i l(t - \tau_i) \quad (\text{A13})$$

where $l(t - \tau_i)$ is the unit-step function entering at time τ_i .

It will be easy to understand that

$$\sum_{i=1}^n c_i = 0 \quad (\text{A14})$$

since the function $g(t)$ equals zero at any time $t > \tau_n$.

If the approximate function $g(t)$ is substituted for the function $f(t)$ in integral (A11), the approximation J_1 of the integral J will be obtained. Its integration by parts gives

$$J_1 = \int_{\tau_1}^{\tau_n} g(t) w'(t) dt = [g(t) w(t)]_{\tau_1}^{\tau_n} - \int_{\tau_1}^{\tau_n} g'(t) w(t) dt. \quad (\text{A15})$$

The first member is zero, since $g(\tau_n) = g(\tau_1) = 0$ (see (A12)). The derivative of the approximate function will be

$$g'(t) = \frac{d}{dt} \sum_{i=1}^n (t - \tau_i) c_i l(t - \tau_i) = \sum_{i=1}^n c_i l(t - \tau_i). \quad (\text{A16})$$

Putting the result into formula (A15) we obtain

$$I_1 = - \int_{\tau_1}^{\tau_n} w(t) \sum_{i=1}^n c_i \cdot l(t - \tau_i) dt. \quad (\text{A17})$$

After changing the sequence of integration and summing:

$$I_1 = - \sum_{i=1}^n c_i \int_{\tau_1}^{\tau_n} w(t) l(t - \tau_i) dt = - \sum_{i=1}^n c_i \int_{\tau_i}^{\tau_n} w(t) dt. \quad (\text{A18})$$

The integral of the function $w(t)$ is $v(t)$, i.e. the integral of the weight function is the step-response function. Thus

$$\int_{\tau_i}^{\tau_n} w(t) dt = v(\tau_n) - v(\tau_i).$$

With this, expression (16) will have the form

$$I_1 = - \sum_{i=1}^n c_i v(\tau_n) + \sum_{i=1}^n c_i v(\tau_i). \quad (\text{A19})$$

However, because of relationship (A14), the expression

$$\sum_{i=1}^n c_i v(\tau_n) = v(\tau_n) \sum_{i=1}^n c_i$$

as a consequence has a value of zero.

$$I_1 = \sum_{i=1}^n c_i v(\tau_i). \quad (\text{A20})$$

Thus, integral (A11) is very simple to evaluate by means of formula (A20) if one knows the function

$$v(t) = \int_0^t \int_0^t w'(t) dt$$

which, in the present case, is the step-response function, and the design has been performed by using this function.

Summary

The new computing method determines the approximate frequency function from the step response function, using the fewest possible points of the time function to get one point of the frequency function. The transformation formulae give the best approximation in the so-called middle frequency range belonging to the environment of the -180° phase shift, which is the most important part for the design. With the help of this fast computing method the real-time stability-checking of the control system can be performed.

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