THE ANALYSIS OF LINEAR NETWORKS CONTAINING TWO-PORTS AND COUPLED TWO-POLES

By

G. Fodor

Department of Theoretical Electricity, Technical University, Budapest

(Received June 14, 1973)

This process is suited for the analysis of networks built up of uncoupled or simple coupled linear two-poles and linear two-ports. The advantage over processes of similar aim is an ease of survey and a suitability for degenerated elements. Its disadvatage is that networks containing multiple coupled two-poles and multi-ports can be calculated only on special conditions which are, however, often satisfied in practice.

1. Stating the problem

Consider a linear time-invariant network. Let network parameters, including source quantities be given (see in detail under item 2). Branch voltages and branch currents are sought for.

Primarily, examination of the stationary state is kept in view, accordingly in the following U, I, V, and J denote the phasors and $s = j\omega$. In principle, the variables may also denote the Laplace transforms, and s is the variable of Laplace transformation. In this case the initial values can be taken into consideration by fictitious sources.

2. Network elements

The examined network may contain the following linear invariant elements:

(a) uncoupled two-poles;

(b) simple coupled two-poles;

(c) multiple coupled two-poles if their impedance or admittance characteristics can be interpreted;

(d) two-ports;

(e) multi-ports if their impedance or admittance characteristics can be interpreted.

From the aspect of the relationship between voltages and currents these five network element types can finally be classified into two groups:

(I) coupled or uncoupled two-poles and multi-ports with immittance characteristics;

(II) two-ports and simple coupled two-poles.

Let us discuss the elements and equations of the two groups.

(I) *Uncoupled two-poles* (a) can always be characterized either by their impedance characteristics, or by their admittance characteristics:

$$U_{zi} = Z_i I_{zi} + V_i, \tag{1}$$

$$I_{yi} = Y_i U_{yi} + J_i, \tag{2}$$

where Z_i and Y_i denote the impedance and admittance of the i-th branch, V_i and J_i the source voltage and source current of the i-th branch, respectively. The two-pole for which description (1) is selected, is denoted as the z-branch, the one described by (2), is denoted as the y-branch (indicated by subscripts z and y, respectively). The selection is arbitrary until $Z_i \neq 0$, and $Y_i \neq 0$, since $Y_i = 1/Z_i$ and $J_i = -Y_i V_i$. A voltage source can be regarded only as a z-branch ($U_{zi} = V_i$), a current source only as a y-branch ($I_{yi} = J_i$). The two-pole may be elementary, though (1) and (2) can also be regarded as characterizing the Thévenin or Norton equivalent of the composed two-pole.

The impedance characteristic of *coupled two-poles* (c) and *multi-ports* (e) with an impedance characteristic is given by

$$U_{zi} = \sum_k Z_{ik} I_{zk} + V_i, \tag{3}$$

and the admittance characteristic by

$$I_{yi} = \sum_k Y_{ik} U_{yk} + J_i. \tag{4}$$

From mathematical aspect, characteristics (1) and (2) are the special cases of characteristics (3) and (4), respectively. The subscripts also indicate that the respective branches are regarded as z- and y-branches, respectively.

Simple coupled two-poles (b) and *two-ports* (d) can also be described by characteristics (3) and (4), if their impedance or admittance characteristics can be interpreted. Since in practice many two-ports are interpreted for which these conditions are not satisfied, it is advisable to choose another description method.

(II) The characteristic for *two-ports* (d) and *simple coupled two-poles* (b) which can be interpreted in every case is the *chain characteristic* expressing primary quantities U_p, I_p by the secondary quantities U_s, I_s.

$$U_{pi} = H_{ik} U_{sk} + K_{ik} I_{sk},$$
$$I_{pi} = M_{ik} U_{sk} + N_{ik} I_{sk}, \quad (5)$$

where U_{pi} and I_{pi} are the voltage and current of the i-th (primary) branch, and U_{sk} and I_{sk} those of the k-th (secondary) branch, respectively. The characteristics can still be completed by members of the type V_i and Z_i, but this is of limited practical significance, and does not mean a restriction of generalization from the theoretical point of view either.

If another characteristic was originally given for the two-port, or for the coupled two-pole pair, then the chain parameters defined by (5) can be calculated from the given parameters by the well known method. Formulae are summarized in Table 1 (where Δ denotes the determinant, e.g. $\Delta_z = z_{11}z_{22} - z_{12}z_{21}$). It appears from the Table that for a reciprocal two-port $\Delta = -1$, for a symmetrical two-port $\Delta = -1$ and $H = -N$.

Table 1

Chain	Impedance	Admittance	Hybrid	Inverse hybrid
H_{ik}	z_{ii}/z_{ki}	$-y_{kk}/y_{ki}$	$-\Delta_h/h_{ki}$	$1/k_{ki}$
K_{ik}	$-\Delta_z/z_{ki}$	$1/y_{ki}$	h_{ii}/h_{ki}	$-k_{kk}/k_{ki}$
M_{ik}	$1/z_{ki}$	Δ_y/y_{ki}	$-h_{kk}/h_{ki}$	k_{ii}/k_{ki}
N_{ik}	$-z_{kk}/z_{ki}$	y_{ii}/y_{ki}	$1/h_{ki}$	$-\Delta_k/k_{ki}$
Δ	$-z_{ik}/z_{ki}$	$-y_{ik}/y_{ki}$	h_{ik}/h_{ki}	k_{ik}/k_{ki}

Each two-port has a characteristic type (5). The parameters of frequently occurring idealized two-ports are given in Table 2. For some of them, the primary and secondary port cannot be chosen freely. For instance, in the case of the nullor, the nullator is to be regarded as the primary, and the norator the secondary port; in the case of controlled sources, the control port is primary, the controlled one secondary. This restriction does not mean a reduction of general validity.

As a final result, the branches in group (I) are either z-branches, or y-branches, while half of those in group (II) are p-branches, the other half s-branches.

3. The graph of the network

For writing the constrains imposed upon by the topology of the network, namely the Kirchhoff laws, it is advisable to construct the graph of the network.

Table 2

Denomination	Characteristics	H	K	M	N	Δ
Nullor	$U_p = 0$ $I_p = 0$	0	0	0	0	0
Voltage-controlled voltage source	$U_p = \mu^{-1} U_s$ $I_p = 0$	$\frac{1}{\mu}$	0	0	0	0
Voltage-controlled current source	$U_p = g^{-1} U_s$ $I_p = 0$	0	$\frac{1}{g}$	0	0	0
Current-controlled voltage source	$U_p = 0$ $I_p = r^{-1} U_c$	0	0	$\frac{1}{r}$	0	0
Current-controlled current source	$U_p = 0$ $I_p = \alpha^{-1} I_s$	0	0	0	$\frac{1}{\alpha}$	0
Ideal transformer	$U_p = n^{-1} U_s$ $I_p = -n I_s$	$\frac{1}{n}$	0	0	$-n$	-1
Negative converter	$U_p = k^{-1} U_s$ $I_p = k I_s$	$\frac{1}{k}$	0	0	k	$+1$
Gyrator	$U_p = -r I_s$ $I_p = r^{-1} U_s$	0	$-r$	$\frac{1}{r}$	0	$+1$

Each two-pole represents a single branch in the graph, independently of whether the two-pole is elementary or composed, uncoupled, or coupled with another two-pole. Some examples are shown in Fig. 1.

Each *two-port* represents two branches. The topological relationship of the two branches depends on what basis the four-pole is regarded a two-port. If the two-port character is provided by the internal structure of the four-pole (namely the two-port consists actually of two two-poles connected e.g. by a transformer), then the two branches form a not connected partial graph. (Fig. 2a). If the two-port is actually a three-pole, then the corresponding terminal points of the two branches are common in the graph (Fig. 2b). If nothing is known about the structure of the four-pole, the two-port character is, however, provided by the two-pole terminations, then both substitutions are justiied in the graph (Fig. 2c). In a more general case the four-pole cannot be regarded as a two-port.

In the case of controlled sources often the control current or voltage is at the same time the current or voltage of a two-pole. In such a case we may proceed in two ways. One possibility is to include the two-pole into the two-port and to determine the chain-matrix of the complete two-port. The other possibility is to regard the two-pole and the control branch of the source as separate branches. This is shown in Fig. 3 for the case of a voltage-controlled

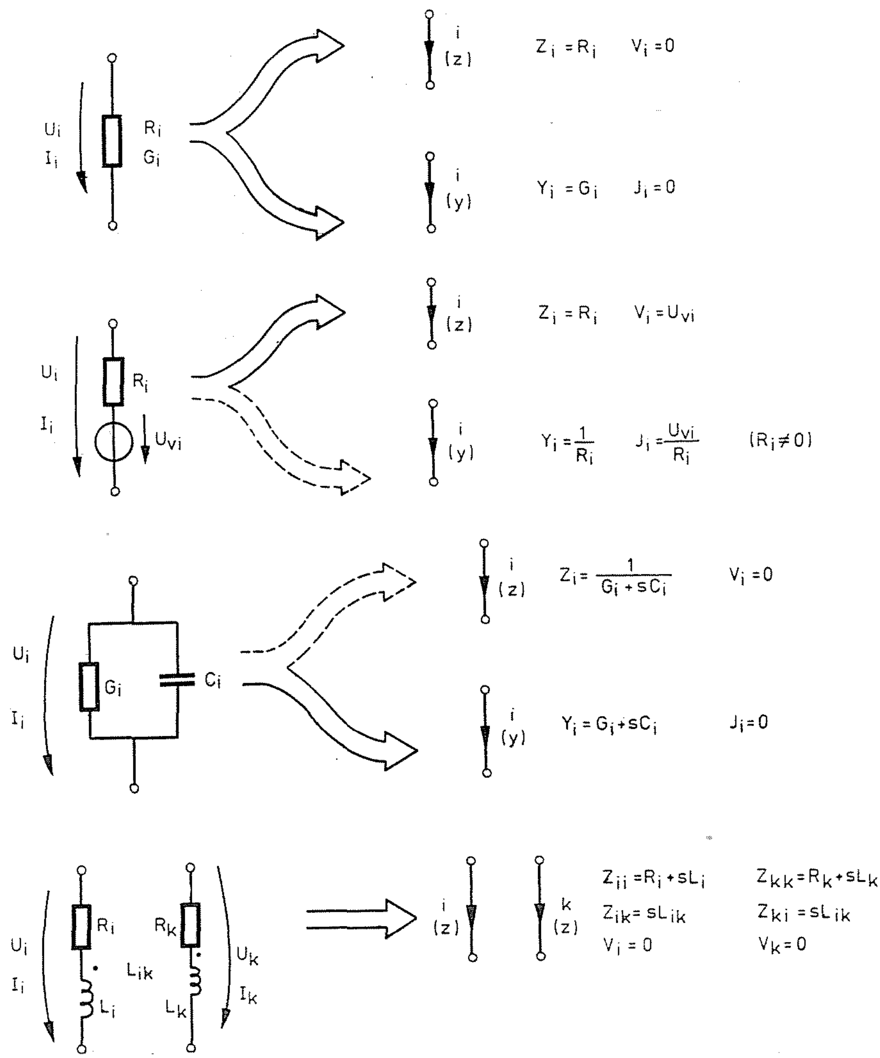

Fig. 1

voltage source. With the other method, though the number of branches is higher, no calculation of the chain matrix of the composed two-port is necessary. The generalized method of the insertion of fictitious parallel and series branches is illustrated in Fig. 4.

Each port of the *multi-ports* is to be regarded as a branch, namely either all of them is a z-branch, or all of them is a y-branch, depending on whether a characteristic of type (3) or (4) is given. Similarly to two-ports, one of the terminals of the branches can be regarded as common (Fig. 5).

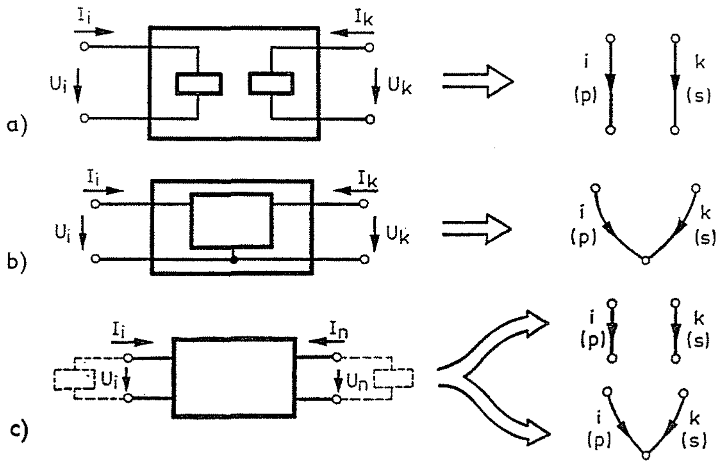

Fig. 2

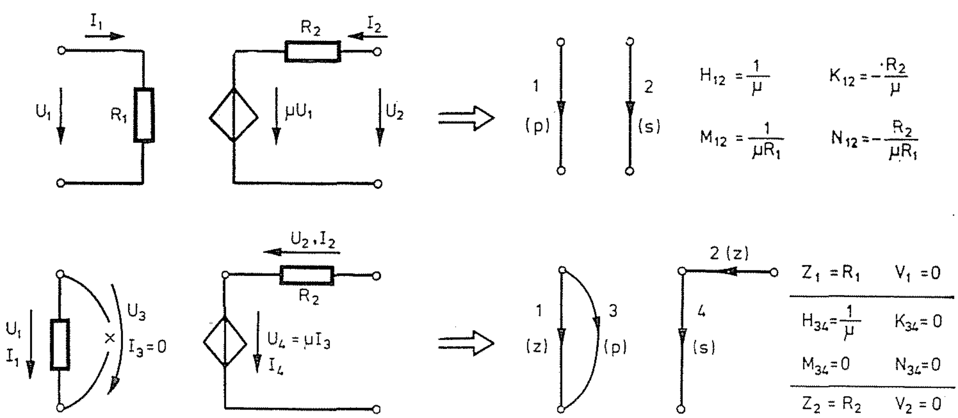

Fig. 3

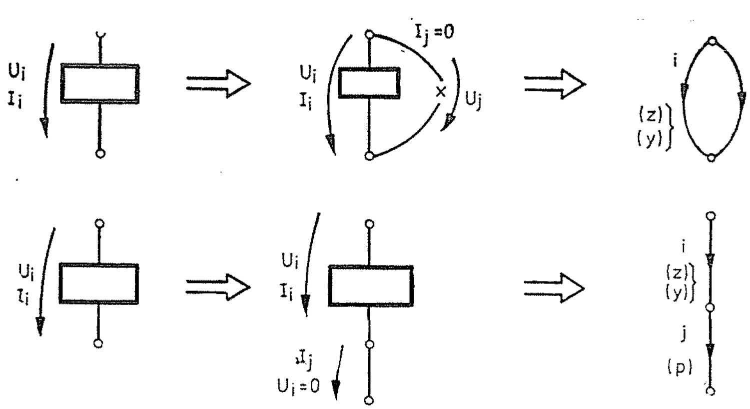

Fig. 4

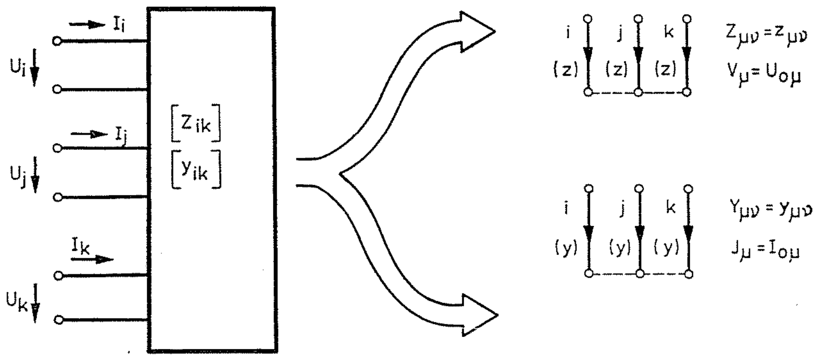

Fig. 5

4. The branch laws

The *first step* in the calculation of the network is to decide which of the two-poles are considered as z-branches and which as y-branches, further which among the coupled two-pole pairs and two-ports is regarded as p-branch, and which as s-branch. The restrictions were seen to be the following: A voltage source can only be a z-branch, a current source only a y-branch, a nullator and the control branch of a controlled source only a p-branch, accordingly the norator and a controlled source only an s-branch. (If there are also internal impedances, these restrictions vanish.) Convenience aspects may be applied to the other branches.

Hereafter branches are numbered in the following order (independently of the fact that the graph of the network may consist of several components):

z-branches; $i = 1, 2, \ldots, z$;
y-branches; $i = z + 1, z + 2, \ldots, z + y$;
p-branches; $i = z + y + 1, z + y + 2, \ldots, z + y + s$;
s-branches; $i = z + y + s + 1, z + y + s + 2, \ldots, z + y + s + p$.

Of course $p = s$. The total number of branches is equal to

$$b = z + y + p + s = z + y + 2p.$$

Branch voltages and branch currents are given subscripts in the given order. Let us form column matrices from these and partition according to the type of the branches

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_z \\ \mathbf{U}_y \\ \mathbf{U}_p \\ \mathbf{U}_s \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} \mathbf{I}_z \\ \mathbf{I}_y \\ \mathbf{I}_p \\ \mathbf{I}_s \end{bmatrix}. \tag{6}$$

The branch laws can be formulated in the following form:

$$\begin{aligned} \mathbf{U}_z &= \boldsymbol{Z}\mathbf{I}_z + \mathbf{V}, \\ \mathbf{I}_y &= \boldsymbol{Y}\mathbf{U}_y + \mathbf{J} \end{aligned} \tag{7}$$

$$\begin{bmatrix} \mathbf{U}_p \\ \mathbf{I}_p \end{bmatrix} = \begin{bmatrix} \boldsymbol{H} & \boldsymbol{K} \\ \boldsymbol{M} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \mathbf{U}_p \\ \mathbf{I}_p \end{bmatrix}.$$

The first matrix equation sums up Eqs (1) and (3) (of a number z), the second matrix equation Eqs (2) and (4) (of a number y), while the third matrix equation Eqs (5) (of a number $2p = 2s$).

If the network does not contain multiple coupled two-poles and if the simple coupled two-pole pairs are regarded as two-ports, then $\boldsymbol{Z}$ and $\boldsymbol{Y}$ are diagonal. Blocks $\boldsymbol{H}$, $\boldsymbol{K}$, $\boldsymbol{M}$ and $\boldsymbol{N}$ are always diagonal.

5. The complete system of equations

The *Kirchhoff* laws for voltages and currents can be written by the help of loop matrix $\boldsymbol{B}$ and of cut-set matrix $\boldsymbol{Q}$

$$\boldsymbol{B}\mathbf{U} = \mathbf{0}, \qquad \boldsymbol{Q}\mathbf{I} = \mathbf{0} \cdot \qquad (8$$

Matrix $\boldsymbol{B}$ has m rows, matrix $\boldsymbol{Q}$ has $r = n - c = b - m$ rows, while both have b columns, where n is the number of nodes, m of the independent loops, c of the components, r of the independent cut-sets. Partition the matrices with respect to branches, i.e. columns:

$$\begin{aligned} \boldsymbol{B} &= [\boldsymbol{B}_z \quad \boldsymbol{B}_y \quad \boldsymbol{B}_p \quad \boldsymbol{B}_s], \\ \boldsymbol{Q} &= [\boldsymbol{Q}_z \quad \boldsymbol{Q}_y \quad \boldsymbol{Q}_p \quad \boldsymbol{Q}_s]. \end{aligned} \tag{9}$$

Subscripts denote at the same time the number of columns. The partitioned form of the Kirchhoff laws is the following:

$$\begin{aligned} \boldsymbol{B}_z\mathbf{U}_z + \boldsymbol{B}_y\mathbf{U}_y + \boldsymbol{B}_p\mathbf{U}_p + \boldsymbol{B}_s\mathbf{U}_s &= \mathbf{0}, \\ \boldsymbol{Q}_z\mathbf{I}_z + \boldsymbol{Q}_y\mathbf{I}_y + \boldsymbol{Q}_p\mathbf{I}_p + \boldsymbol{Q}_s\mathbf{I}_s &= \mathbf{0}. \end{aligned} \tag{8a}$$

Eliminate variables $\mathbf{U}_z$, $\mathbf{I}_y$, $\mathbf{U}_p$, $\mathbf{I}_p$ by using the branch laws under (7). For the remaining variables we obtain the equation:

$$\boldsymbol{W} \begin{bmatrix} \mathbf{I}_z \\ \mathbf{U}_y \\ \mathbf{U}_y \\ \mathbf{I}_s \end{bmatrix} = - \begin{bmatrix} \boldsymbol{B}_z\mathbf{V} \\ \boldsymbol{Q}_y\mathbf{J} \end{bmatrix}. \tag{10}$$

The partitioned expression for matrix $\boldsymbol{W}$ is found to be

$$\boldsymbol{W} = \begin{bmatrix} \boldsymbol{W}_m \\ \boldsymbol{W}_r \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_z\boldsymbol{Z} & \boldsymbol{B}_y & \boldsymbol{B}_p\boldsymbol{H} + \boldsymbol{B}_s & \boldsymbol{B}_p\boldsymbol{K} \\ \boldsymbol{Q}_z & \boldsymbol{Q}_y\boldsymbol{Y} & \boldsymbol{Q}_p\boldsymbol{M} & \boldsymbol{Q}_p\boldsymbol{N} + \boldsymbol{Q}_s \end{bmatrix}. \tag{11}$$

The partial matrices $\mathbf{W}_m$ and $\mathbf{W}_r$ with m and r rows, respectively, and $b = m + r$ columns, can be expressed by means of even more elementary matrices:

$$\mathbf{W}_m = \mathbf{B}\begin{bmatrix} \mathbf{Z} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H} & \mathbf{K} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}.$$

$$\mathbf{W}_r = \mathbf{Q}\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M} & \mathbf{N} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}. \tag{12}$$

For solving the system of Eqs (10), the b-dimensional matrix $\mathbf{W}$ is to be inverted. With the knowledge of $\mathbf{I}_z$, $\mathbf{U}_y$, $\mathbf{U}_s$, $\mathbf{I}_s$ the other variables can be calculated on the basis of (7).

The process can further be simplified by selecting a special tree (e.g. each z-branch is a twig, each y-branch a link, the p- and s-branches are either twigs, or links). In such a case no special calculation of matrices $\mathbf{B}$ and $\mathbf{Q}$ is necessary since any of the fundamental matrices determines the other. The relevant formalism is not described here because it can be deduced from the preceding in the known way.

6. Illustrative example

To illustrate the method, consider the network shown in Fig. 6. Choose the branches as follows:

z-branches	y-branches	p-branches	s-branches
1. a	6. c	7. e'	10. α
2. e		8. IT right	11. IT left
3. b		9. Gyr. left	12. Gyr. right
4. d			
5. f			

There are several other possibilities of grouping the two-poles, e.g. branch "c" could be a z-branch too. The controlled source was considered as a separate two-port requiring the insertion of a redundant short-circuit (7th branch). The two branches of IT and of the gyrator can be chosen in the opposite way too. The numeral of the branches is indicated in parantheses in the figure.

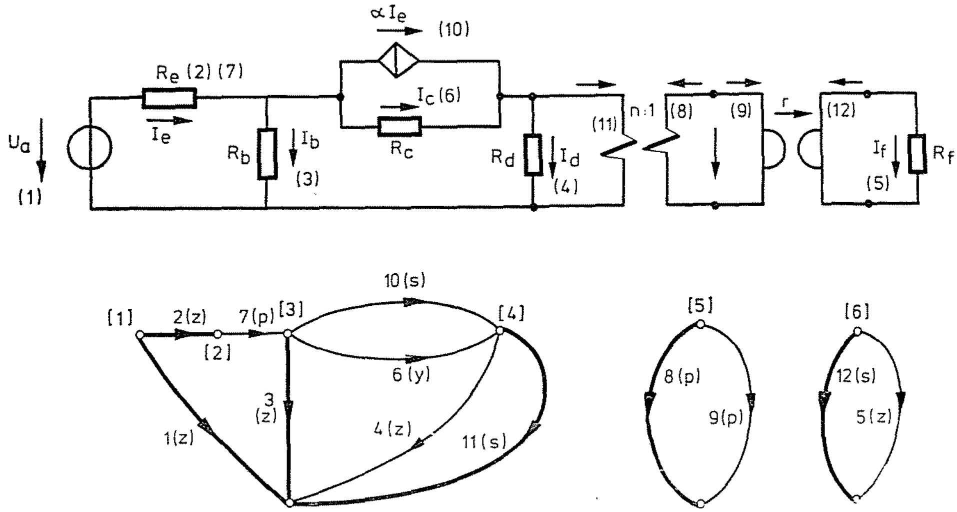

Fig. 6

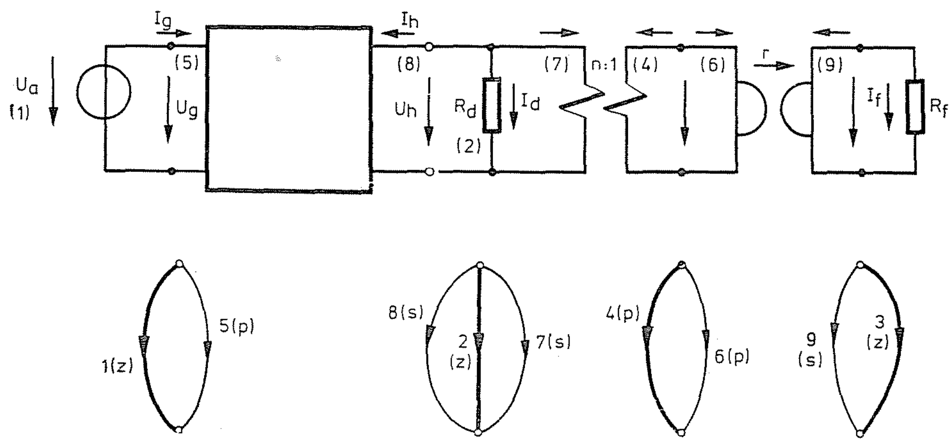

Fig. 7

The characteristic matrices are the following:

$$Z = \langle 0, R_e, R_b, R_d, R_f \rangle, \quad V = [U_a, 0, 0, 0, 0]^{+};$$
$$Y = \langle G_c \rangle, \quad J = [0];$$
$$H = \langle 0, n^{-1}, 0 \rangle, \quad K = \langle 0, 0, -r \rangle;$$
$$M = \langle 0, 0, r^{-1} \rangle, \quad N = \langle \alpha^{-1}, -n, 0 \rangle.$$

Topological matrices can be written conveniently on the basis of the graph in Fig. 6. The loop matrix was constructed on the basis of the tree drawn in thick lines, while the cut-set matrix on the basis of the nodes. No special aspects governed the choice of the tree.

$$\boldsymbol{B} = \left[\begin{array}{ccccc|c|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 9 \\ 10 \end{array}$$

$$\underbrace{1 \quad 2 \quad 3 \quad 4 \quad 5}_{z} \qquad \underbrace{6}_{y} \qquad \underbrace{7 \quad 8 \quad 9}_{p} \qquad \underbrace{10 \quad 11 \quad 12}_{s}$$

$$\overbrace{1 \quad 2 \quad 3 \quad 4 \quad 5} \qquad \overbrace{6} \qquad \overbrace{7 \quad 8 \quad 9} \qquad \overbrace{10 \quad 11 \quad 12}$$

$$\underset{\sim}{Q} = \left[\begin{array}{ccccc|c|ccc|ccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} [1] \\ [2] \\ [3] \\ [4] \\ [5] \\ [6] \end{array}$$

Hereafter the calculation of matrix $\boldsymbol{W}$ on the basis of (11) or (12) is a routine task. By inverting, the quantities

$$\mathbf{I}_z = [I_1 \ I_2 \ I_3 \ I_4 \ I_5]^+, \qquad \mathbf{U}_y = [U_6],$$
$$\mathbf{U}_s = [U_{10} \ U_{11} \ U_{12}]^+, \qquad \mathbf{I}_s = [I_{10} \ I_{11} \ I_{12}]^+$$

can be calculated on the basis of (10).

Finally, by using (7), the still unknown quantities $\boldsymbol{U}_z, \boldsymbol{I}_y, \boldsymbol{U}_p, \boldsymbol{I}_p$ can be determined.

The problem can also be solved so that branches e, b, c and the controlled source are considered as a single two-port. The circuit diagram and the graph are shown in Fig. 7. Now the branches are numbered in such a way that the first four branches are twigs, while the last five branches links.

The characteristic matrices are the following:

$$\boldsymbol{Z} = \langle 0, R_d, R_f \rangle, \qquad \mathbf{V} = [U_a, 0, 0]^+$$
$$\boldsymbol{Y} = \boldsymbol{0} \qquad \mathbf{J} = 0 .$$

After some calculation we obtain:

$$\boldsymbol{H} = \left\langle n^{-1}, \frac{R_e + R_b}{R_b + \alpha R_c}, 0 \right\rangle, \quad \boldsymbol{K} = \left\langle 0, R_b - \frac{(R_b + R_c)\ (R_b + R_e)}{R_b + \alpha R_c}, -r \right\rangle$$

$$\boldsymbol{M} = \left\langle 0, \frac{1}{R_b + \alpha R_c}, r^{-1} \right\rangle, \quad \boldsymbol{N} = \left\langle -n, -\frac{R_b + R_c}{R_b + \alpha R_c}, 0 \right\rangle .$$

The topological matrices are

$$\boldsymbol{B} = [\boldsymbol{F}\ \boldsymbol{1}_m], \qquad \underset{\sim}{\boldsymbol{Q}} = [\boldsymbol{1}_r\ -\boldsymbol{F}^{+}].$$

$$\boldsymbol{F} = \left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right\} p_2 \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} s \end{array}$$

$$\underbrace{\qquad\qquad\qquad}_{z} \quad \underbrace{\quad}_{p_1}$$

Hereafter the course of calculation is completely mechanical.

By comparing the two methods of discussion, their advantages and disadvantages are evident. The reduction of the number of branches (9 in place of 12) offset by the preliminary calculation and more complex structure of the coupling matrix.

Summary

A method is given for the systematic analysis of linear networks containing two-poles, simple coupled two-pole pairs and two-ports (including also degenerated cases, e.g. the nullor), as well as multiple coupled two-poles and multi-ports for which either the impedance characteristics or the admittance characteristics are interpreted. The process is suited for the examination of the stationary state (phasor representation). further, by employing the Laplace transformation, also for not stationary processes.

References

1. BALABANIAN, N.—BICKART, T. A.: Electrical Network Theory. Wiley, New York, 1969.
2. KIM, W. H.—MEADOWS, H. E.: Modern Network Analysis. Wiley, New York, 1971.
3. REZA, M.—SEELY, I.: Modern Network Analysis. McGraw-Hill, New York, 1959.
4. SIGORSKY, V. P.—PETRENKO, A. I.: Algoritmi analiza electronnyh skhem. Technica, Kiev, 1970.

Prof. Dr. György FODOR, 1502 Budapest, P. O. B. 91, Hungary