GENERATION OF ABELIAN MATRIX GROUPS FOR USE IN THE SCATTERING ALGEBRA

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(Received September 3, 1971)

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1. Scattering matrix and its transforms — for an anisotropic obstacle

1,1. After World War II, a great development has begun in micro-wave technics. Their *theoretical progress* was also very quick; e.g. the old STOKESIAN difference equations for the optical transfer through discrete glassplates induced the researchers to find and evolve the algebra of obstacles, to introduce the scattering matrix and to find its theory, later to make a generalization for continuous media by functional equations and to use here the recent principle of invariant imbedding, finally to extend the theory to the obstacles with $2n$ contacts and to the HILBERT space and, last but not least, to apply all these to the effect transfer along micro-wave chains etc. In these important investigations, BELLMANN [2], CARLIN [3], GUILLEMIN [6], REDHEFFER [8, 9], REID [10], TWERSKY [11] and many other authors have produced essential results.

Nevertheless, this beautiful progress admits — just because of its quickness — various complements; e.g. common direction of transfer and multiplication, dynamic transform algorithm (DTA)* of scattering matrices,

* Our *dynamic transform algorithm* (DTA) for matrices is amply explained in [14—17] and other works. Its *basic idea* — for n dimensional vectors and without proofs — is the following. A *linear* algebraic vector *equation* has the form

$$u \equiv \mathbf{E}u \equiv u_1 e_1 + \ldots + \sum_{q=1}^{p} u_{k_q} e_{k_q} + \ldots + u_n e_n \equiv u_1 e_1 + \ldots + \mathbf{E}_K \mathbf{u}_K + \ldots + u_n e_n =$$

$$= x_1 a_1 + \ldots + \sum_{q=1}^{p} x_{l_q} a_{l_q} + \ldots + x_m a_m \equiv x_1 a_1 + \ldots + \mathbf{A}_L \boldsymbol{x}_L + \ldots + x_m a_m \equiv \mathbf{A}_0 \boldsymbol{x}. \quad \text{(E)}$$

The bilateral *change* $-\mathbf{E}_K u_K \leftrightarrow -\mathbf{A}_L x_L$ is *feasible*, if pivot submatrix $\mathbf{A}_{KL} = \mathbf{E}_K^{*} \mathbf{A}_L$ is *regular*, that is, $\det(\mathbf{A}_{KL}) \equiv |\mathbf{A}_{KL}| \neq 0$ (Ch). The *arrangement of* (E) corresponding to (Ch) will be (with $|\mathbf{B}_p| = |\mathbf{A}_{KL}| \neq 0$)

$$\mathbf{B}_p \boldsymbol{v}_p \equiv u_1 e_1 + \ldots - \mathbf{A}_L \boldsymbol{x}_L + \ldots + u_n e_n = x_1 a_1 + \ldots - \mathbf{E}_K u_K + \ldots + x_m a_m \equiv \mathbf{A}_p' y_p. \quad \text{(A)}$$

The *transformed* form $\mathbf{A}_p$ of $\mathbf{A}_0$ on the new basis $\mathbf{B}_p$ and with it the solution $v_p = \mathbf{A}_p y_p$ can be produced by our *formulas* (at $|\mathbf{A}_{KL}| \neq 0$, and $a^{(q)}_{k_q l_q} \neq 0$)

$$\boxed{\mathbf{A}_p \equiv \mathbf{B}_p^{-1} \mathbf{A}_p' = \mathbf{A}_0 - (\mathbf{A}_L + \mathbf{E}_K) \mathbf{A}_{KL}^{-1} (\mathbf{A}^K - \mathbf{E}^L) \ldots}$$

investigations and operations with transformed matrices, generation of artificial matrix semigroups and extension to ABELIAN matrix groups etc. Such complements will here be given (and others in a subsequent paper) to elucidate certain problems. The complements must possibly be connected with known facts of the scattering algebra. These latters will be often interpreted in sense of the excellent compilation [13].

1,2. Let us assume an effect (e.g. wave) to permeate a homogeneous obstacle (e.g. network) O_1 under the conditions of constant frequency and wave form. This obstacle can be characterized by the 4 *scattering coefficients* s_{ij}; namely, coefficients $s_{11} \equiv t$, $s_{21} \equiv r$ are complex amplitudes of the trans-

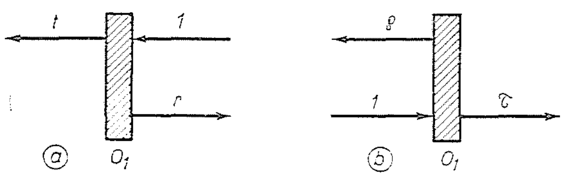

Fig. 1

ferring and reflecting waves at the input unit wave-amplitude on the start (right) side (Fig. 1a); coefficients $s_{12} \equiv \varrho$ (for reflexion), $s_{22} \equiv \tau$ (for transfer) are defined in reasonably analogous manner at the input unit wave-amplitude on the aim (left) side (Fig. 1b). By definition, and under the condition of *affinity* (sum and rate keeping), the relationships between the (input) complex wave-amplitudes z_1 (right), z_4 (left) and the output ones z_2 (right), z_3 (left) can be written as the *input-output equation* (Fig. 2).

$$z_3 = tz_1 + \varrho z_4 \,(\leftarrow)\,, \qquad z_2 = rz_1 + \tau z_4 \,(\rightarrow)\,, \tag{1a, b}$$

or, advantageously, in vector-matrix form:

$$\mathfrak{z}_{\text{out}} \equiv \begin{bmatrix} z_3 \\ z_2 \end{bmatrix} = \begin{bmatrix} t & \varrho \\ r & \tau \end{bmatrix} \begin{bmatrix} z_1 \\ z_4 \end{bmatrix} \equiv \mathrm{S}_0 \, \mathfrak{z}_{in}\,, \tag{1c}$$

. . . in a multiple (p-times) spring (DTA), or

$$\boxed{\mathrm{A}_{q+1} \equiv \mathrm{B}_{q+1}^{-1} \mathrm{A}'_q = \mathrm{A}_q - (a_{l_q}^{(q)} + e_{kq})\, a_{k_q l_q}^{-1}\, (a_{(q)}^{k_q} - e^{l_q}) \ldots}$$

. . . in $q = 0,1,2, \ldots p-1$ single steps (DTA_{q+1}).
If A_0 is of n order and regular, at a possible choice

$$l_q = k_q = \mathrm{k} + 1 \quad \text{there is} \quad \mathrm{B}_n \equiv -\mathrm{A}_0, \mathrm{A}'_n = -\mathrm{E} \text{ and}$$
$$\mathrm{A}_n = (-\mathrm{A}_0^{-1})(-\mathrm{E}) = \mathrm{A}_0^{-1} \text{ (inverse).} \tag{I}$$

where $\mathbf{S}_0$ is the *scattering matrix*, z_{in} and z_{out} are the *input and output vector*, resp. Here it was supposed to exist a close (coupled to the obstacle) working without reflexion ($r_1 = 0$).

1,3. The scalar equation system (1a, b) or the vector equation (1c) can be solved, e.g. by the *right vector* $z_r^* = [z_1, z_2]$ for the *left vector* $z_l^* = [z_3, z_4]$ too, through the change $z_2 \leftrightarrow z_4$ provided $s_{22} \equiv \tau \neq 0$. It goes very elegantly by the help of the *dynamic transform algorithm* (DTA) (given in [14—17] and other works by the author) and at the chosen pivot element $s_{22} \equiv \tau \neq 0$, that is — in accord with the foot-note of item 1,1 —

$$z_l = \mathbf{S}_1 z_r\,,\ \text{where}\ \ \mathbf{S}_1 = \mathbf{S}_0 - \frac{1}{s_{kl}}(\mathbf{s}_l + e_k)(\mathbf{s}^k - e^l) = \mathbf{S}_0 - \mathbf{D}_{kl} =$$

$$= \mathbf{S}_0 - \frac{1}{s_{22}}(\mathbf{s}_2 + e_2)(\mathbf{s}^2 - e^2) = \mathbf{S}_0 - \mathbf{D}_{22} = \qquad (2\text{a}-\text{c})$$

$$= \mathbf{S}_0 = \frac{1}{\tau}\begin{bmatrix} \varrho \\ \tau+1 \end{bmatrix}[r \quad \tau-1] = \begin{bmatrix} \frac{d}{\tau} & \frac{\varrho}{\tau} \\ -\frac{r}{\tau} & \frac{1}{\tau} \end{bmatrix}$$

with the determinant $\qquad d = |\mathbf{S}_0| = t\tau - r\varrho\,,$

or, in the *left-right equation* (with $\tau \neq 0$)

$$z_l \equiv \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} \frac{d}{\tau} & \frac{\varrho}{\tau} \\ -\frac{r}{\tau} & \frac{1}{\tau} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \equiv \mathbf{S}_1 z_r \qquad (2d)$$

with the determinant

$$d_1 \equiv |\mathbf{S}_1| = \frac{1}{\tau^2}(d \cdot 1 + r\varrho) = \frac{t\tau}{\tau^2} = \frac{t}{\tau}\,. \qquad (2\text{e})$$

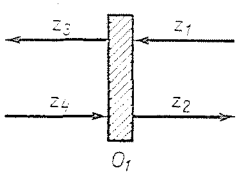

Fig. 2

There are further transformed forms of (1c), $\begin{pmatrix}4\\2\end{pmatrix} = 6$ in all, together with their permutational forms. For instance, with the simple change of the rows and columns of (2d), we get the permutational form

$$\tilde{z}_l \equiv \begin{bmatrix} z_4 \\ z_3 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\tau} & -\dfrac{r}{\tau} \\ \dfrac{\varrho}{\tau} & \dfrac{d}{\tau} \end{bmatrix} \begin{bmatrix} z_2 \\ z_1 \end{bmatrix} \equiv \tilde{S}_1 \tilde{z}_r \tag{2f}$$

with the determinant $d_1 = t/\tau$. Obviously, there are relationships between Eqs (1c), (2d) and (2e), in particular:

$$z_{\text{out}} \leftrightarrow z_l \leftrightarrow \tilde{z}_l\,, \quad S_0 \leftrightarrow S_l \leftrightarrow \tilde{S}_l\,, \quad z_{in} \leftrightarrow z_r \leftrightarrow \tilde{z}_r\,; \tag{3a—c}$$

all three triples z_{out}, S_0, z_{in}; z_l, S_l, z_r; $\tilde{z}_l$, $\tilde{S}_l$, z_r characterize the same scattering circumstances in other forms, which are easily interchangeable. Remark that the left right equations (2d) or (2e) will later enable to find the composition of obstacles — on the basis of the matrix algebra.

1.4. Eqs (1c) and (2d) or (2e) and other transformed forms are known in micro-wave technics as *four-pole (-terminal)* ones for different given parameters under different names; e.g. for $z_1 \equiv I_1$, $z_2 \equiv I_2$, and $z_3 \equiv U_1$, $z_4 \equiv U_2$ there are *impedance* parameters $s_{ij} \equiv z_{ij}$; for $z_1 \equiv U_1$, $z_2 \equiv U_2$ and $z_3 \equiv I_1$, $z_4 \equiv I_2$ there are *admittance* parameters $s_{ij} \equiv y_{ij}$ etc.; evidently, there are $\begin{pmatrix}4\\2\end{pmatrix} = \dfrac{4.3}{1.2} = 6$ forms in all.

The *power* of an obstacle is — as for the four-pole one — proportional to the square absolute amplitude of the effect. Accordingly, the effect propagates through the obstacle *without loss* or the obstacle is neutral (in both directions) provided:

$$|t|^2 + |r|^2 = 1 \quad \text{and} \quad |\tau|^2 + |\varrho|^2 = 1\,; \tag{4a, b}$$

it remains so at the change of the dipoles with (t, r), and (ϱ, τ) too. On the contrary, the obstacle works *with loss* or it is passive (in both directions) for

$$|t|^2 + |r|^2 \leqq 1 \quad \text{and} \quad |\tau|^2 + |\varrho|^2 \leqq 1\,. \tag{5a, b}$$

The neutral and passive obstacles (four-poles) will come up again in the further discussion.

2. Composed scattering matrix got by transform for two or more anisotropic obstacles

2,1. To study the composition of the obstacles, we shall treat here the scattering of effects through *two obstacles* O_1, O_2 directly connected (at zero spacing) and having the characteristical matrices $\mathbf{S}_0'$, $\mathbf{S}_1'$, $\tilde{\mathbf{S}}_1'$, . . ., and $\mathbf{S}_0''$, $\mathbf{S}_1''$, $\tilde{\mathbf{S}}_1''$, . . . For sake of simplicity, we choose the left-right equations of type (2d) with the left, middle, and right vectors

$$\boldsymbol{z}_l^* \equiv \boldsymbol{z}_l''^* = [z_5, z_6]; \quad \boldsymbol{z}_m^* \equiv \boldsymbol{z}_l'^* \equiv \boldsymbol{z}_r''^* = [z_3, z_4], \quad \boldsymbol{z}_r \equiv \boldsymbol{z}_r'^* = [z_1, z_2]$$

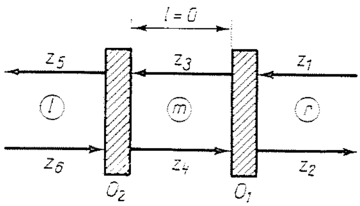

Fig. 3

(Fig. 3) and with the intermediary and product matrices $\mathbf{S}_1'$, $\mathbf{S}_1''$, $\mathbf{S}_1 \equiv \mathbf{S}_1'' \mathbf{S}_1'$, respectively, and write in the *composite left-right vector equation* (with $\tau' \neq 0$, $\tau'' \neq 0$)

$$\boldsymbol{z}_l \equiv \begin{bmatrix} z_5 \\ z_6 \end{bmatrix} = \mathbf{S}_1'' \boldsymbol{z}_m = \mathbf{S}_1'' \mathbf{S}_1' \boldsymbol{z}_r = \begin{bmatrix} \dfrac{d''}{\tau''} & \dfrac{\varrho''}{\tau''} \\ -\dfrac{r''}{\tau''} & \dfrac{1}{\tau''} \end{bmatrix} \begin{bmatrix} \dfrac{d'}{\tau'} & \dfrac{\varrho'}{\tau'} \\ -\dfrac{r'}{\tau'} & \dfrac{1}{\tau'} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{d'\,d''}{\tau'\tau''} - \dfrac{\varrho''\,r'}{\tau'\tau''} & \dfrac{d''\,\varrho'}{\tau'\tau''} + \dfrac{\varrho''}{\tau'\,\tau''} \\ -\dfrac{r''d'}{\tau'\tau''} - \dfrac{r'}{\tau'\tau''} & -\dfrac{\varrho'r''}{\tau'\tau''} + \dfrac{1}{\tau'\tau''} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \equiv \mathbf{S}_1\, \boldsymbol{z}_r\,. \tag{6}$$

From this, we get again by our dynamic transform algorithm (DTA) at the pivot element $s_{22} \equiv (1 - \varrho'\,r'')/\tau'\,\tau'' \neq 0$ (that is, for $\varrho'\,r'' \neq 1$, because of physical circumstances for $|\,\varrho'\,r''\,| < 1$) the *composite input-output vector equation*

$$\boldsymbol{z}_{\text{out}} = \mathbf{S}_O\, \boldsymbol{z}_{\text{in}}, \quad \text{where} \quad \mathbf{S}_O = \mathbf{S}_\text{I} - \frac{1}{s_{22}^{(\text{I})}}\,(s_2^{(\text{I})} + e_2)\,(s^2_{(\text{I})} - e^2) =$$

$$= \mathbf{S}_\text{I} - \mathbf{D}_{22}^{(\text{I})} = \frac{1}{\tau'\,\tau''} \begin{bmatrix} d'\,d'' - \varrho''\,r' & d''\,\varrho' + \varrho'' \\ -r''\,d' - r' & 1 - \varrho' r'' \end{bmatrix} - \tag{7a.b}$$

$$- \frac{1}{(1 - \varrho'\,r'')\,\tau'\,\tau''} \begin{bmatrix} d''\,\varrho' + \varrho' \\ 1 - \varrho' r'' + \tau'\tau'' \end{bmatrix} [-r''\,d' - r',\, 1 - \varrho' r'' - \tau'\tau''],$$

that is (7c)

$$z_{\text{out}} \equiv \begin{bmatrix} z_5 \\ z_2 \end{bmatrix} = \begin{bmatrix} \dfrac{t'\,t''}{1-\varrho'\,r''} & \varrho'' + \dfrac{t''\,\varrho'\,\tau''}{1-\varrho' r''} \\ r' + \dfrac{t'\tau' r''}{1-\varrho' r''} & \dfrac{\tau'\,\tau''}{1-\varrho' r''} \end{bmatrix} \begin{bmatrix} z_1 \\ z_6 \end{bmatrix} \equiv S_O\, z_{in} \equiv (S_0'' * S_0')\, z_{in}$$

where S_O is the *composite scattering matrix of both obstacles* (marked by ′ and ″), then e.g. $z_{20} \equiv r' + t'\,\tau'\,r'' \cdot (1 - \varrho'\,r'')^{-1}$ is the *left composite reflexion* at $z_1 = 1$ and $z_2 = 0$ (and zero spacing).

2,2. Obviously, an *isomorphism* can be established between two special matrix algebras; the *first* one has the matrices $S_1', S_1'', S_1''', \ldots$ (the dynamical transformed of $S_0', S_0'', S_0''', \ldots$) as elements, the ordinary matrix multiplication $S_I \equiv S_1''\, S_1'$ as the single operation (composition) with the associativity $S_1'''(S_1''\, S_1') = (S_1'''\, S_1'')S_1'$ (and without the commutativity $S_1''\, S_1' = S_1'\, S_1''$); the *second* one has the scattering matrices $S_0', S_0'', S_0''', \ldots$ as elements, the artificial matrix multiplication $S_O \equiv S_0'' * S_0'$ (the dynamical transformed of S_I) as the single operation (composition) with the associativity $S_0''' * (S_0'' * S_0') = (S_0''' * S_0'') * S_0'$ (and without the commutativity $S_0'' * S_0' = S_0' * S_0''$). Hence, both isomorph algebras are *semigroups* with the correspondences

$$S_0 \equiv \begin{bmatrix} t & \varrho \\ r & \tau \end{bmatrix} \underset{(\tau \neq 0)}{\longleftrightarrow} \frac{1}{\tau} \begin{bmatrix} d & \varrho \\ -r & 1 \end{bmatrix} \equiv S_1\,, \tag{8a}$$

$$S_O \equiv S_0'' * S_0' = \frac{1}{1-\varrho' r''} \begin{bmatrix} t'\,t'' & \varrho''(1-\varrho'\,r'') + t''\,\tau''\,\varrho' \\ r'(1-\varrho'\,r'') + t'\tau' r'' & \tau'\,\tau'' \end{bmatrix} \longleftrightarrow$$

$$\underset{\substack{1-\varrho' r'' \neq 0 \\ \tau'\tau'' \neq 0}}{\longleftrightarrow} \frac{1}{\tau'\,\tau''} \begin{bmatrix} d'd'' - r'\varrho'' & \varrho' d'' + \varrho'' \\ -d' r'' - r' & -\varrho' r'' + 1 \end{bmatrix} = S_1''\, S_1' \equiv S_I\,, \tag{8b}$$

$$S_0''' * (S_0'' * S_1') = (S_0''' * S_0'') * S_0' \leftrightarrow S_1'''\,(S_1''\,S_1') = (S_1'''\,S_1'')\,S_1'. \tag{8c}$$

It must be remarked that we *generate* from the given matrix set $\mathfrak{S}_0 = \{S_0', S_0'', S_0''', \ldots\}$, through DTA (2b) the isomorphic matrix set $\mathfrak{S}_1 = \{S_1', S_1'', S_1''', \ldots\}$, them from its (ordinary) production set $\mathfrak{S}_I = \{S_1''\, S_1', \ldots\} \subseteq \mathfrak{S}_1$ again through DTA (7b) the isomorphic (artificial) composition set sought for. $\mathfrak{S}_O = \{S_0' * S_0'', \ldots\} \subseteq \mathfrak{S}_0$. Hence, DTA *can be considered as a generator of isomorph matrix semigroups*, namely here at $s_{22}^{(i)} \neq 0$ and also at all possible pivot elements $s_{kl}^{(i)} \neq 0$.

2,3. Composing an obstacle of $S_0' \equiv S_0 = [t, \varrho; r, \tau]$ with its *mirror image* of $S_0'' \equiv S_{\square} = [\tau, r; \varrho, t]$ *at 0 spacing*, we get the *composite scattering* *matrix* after (7c)

$$S_O \equiv S_{\Xi} * S_0 + \frac{1}{2-\varrho^2}\begin{bmatrix} t\tau & r(1-\varrho^2)+\tau t\varrho \\ r(1-\varrho^2)+t\tau\varrho & \tau t \end{bmatrix}; \tag{9a}$$

this is obviously symmetric. The double has the *composite transfer maximal* in absolute value at the case (neutral) being *without loss* after (4a, b), namely

$$(1 \geqq) \max |t_O| = \frac{|t|\,|\tau|}{1-|\varrho|^2} = \frac{|t|}{|\tau|} = \frac{|\tau|\,|t|}{1-|r|^2} = \frac{|\tau|}{|t|} = 1 \tag{9b}$$
$$(|t| = |\tau|)\,.$$

For getting $\vartheta_t \equiv \text{arc}\,(t) = \text{arc}\,(\tau) \equiv \vartheta_\tau$, consequently the *reciprocity* $t = \tau$, it must look at the double of spacing x; the *modified composite transfer* has its maximum in absolute value

$$\max |t_O| \equiv \max \frac{|t\tau e^{jx}|}{|1-\varrho^2 e^{2jx}|} = \max \frac{|t|\,|\tau|}{|1-|\varrho|^2 e^{2j(\delta_\varrho+x)}|} = \frac{|t|\,|\tau|}{1-|\varrho|^2} = 1 \tag{9c}$$

at $x = -\delta_\varrho \equiv -\text{arc}\,(\varrho)$. Hence, the *modified composite reflexion* must equal zero, that is:

$$r_O \equiv r + \left.\frac{t\tau\varrho\, e^{2jx}}{1-\varrho^2 e^{2jx}}\right|_{\delta-\varrho=x} = |r|\, e^{j\delta r} + \frac{|t|\,|\tau|\,|\varrho|}{1-|\varrho|^2}\, e^{j\,(\vartheta_t+\vartheta_\tau-\delta_\varrho)} = 0$$

and from it $\qquad \varphi_0 \equiv \vartheta_t + \vartheta_\tau - (\delta_r + \delta_\varrho) = (2n+1)\pi.$

This phase relation (based on energy situation) is valid also for $\vartheta_t \neq \vartheta_\tau$.

2.4. For the double (passive) being *with loss*, the conditions (5a, b) are not necessary (but sufficient) for the composition $S_C \equiv S_0'' * S_0'$. To complete them, we can write the inequalities

$$|z_2|^2 + |z_3|^2 \leqq |z_1|^2 + |z_4|^2, \quad |z_4|^2 + |z_5|^2 \leqq |z_1|^2 + |z_6|^2, \tag{10a,b}$$

(because the output power cannot be higher than the input one), or by their addition, the necessary condition

$$|z_2|^2 + |z_5|^2 \leqq |z_1|^2 + |z_6|^2\,. \tag{10c}$$

Such an other one can be obtained from (1c) as the inequality

$$|r\bar{\tau} + t\bar{\varrho}|^2 \leqq (1 - |t|^2 - |r|^2)\,(1 - |\tau|^2 - |\varrho|^2) \equiv a^2\,\alpha^2 \tag{11a}$$

(where a^2, $\varkappa^2$ are the power-absorption coefficients) and from it the inequalities

$$\left|\left|\frac{r}{t}\right| - \left|\frac{\varrho}{\tau}\right|\right| \leqq \frac{a\varkappa}{|t\tau|}, \quad \left|\cos\frac{\varphi_0}{2}\right| \leqq \frac{a\varkappa}{2\sqrt{|t\tau r\varrho|}}, \qquad (11\text{b,c})$$

where for $a\varkappa = 0$, $|t| = |\tau|$ and (9e).

2,5. Take into (7c) now $S_0' = S_0$, then $r'' = z$ (the close reflexion as complex variable) and $z_1 = 1$, $z_6 = 0$, so for the left-side composite reflexion the function can be written:

$$w_r \equiv R(z) = r + \frac{t\tau z}{1-\varrho z} = \frac{r+dz}{1-\varrho z} \quad (d \equiv t\tau - r\varrho, \quad z \neq 1/\varrho). \qquad (12\text{a})$$

In reasonably similar way, for $S_0'' = S_0$, $\varrho' = z$, $z_1 = 0$ and $z_6 = 1$, we get the analogous function of the right-side composite reflexion

$$w_\varrho \equiv P(z) = \varrho + \frac{\tau t z}{1-rz} = \frac{\varrho+dz}{1-rz} \quad (d \equiv t\tau - r, \quad z \neq 1/r). \qquad (13\text{a})$$

Both are *linear fractional function of complex variable*, which are — as well known — circle keeping. $R(z)$ map on the circle line $|z| = a < 1/|\varrho|$ to another one with centre and radius

$$c_a = r + \frac{a^2\, t\tau\bar{\varrho}}{1-a^2|\varrho|^2}, \quad \text{and} \quad R_a = \frac{a\,|t\tau|}{1-a^2|\varrho|^2}, \quad \text{resp.,} \qquad (12\text{b,c})$$

where arc

$$[(c_a - r)/r] = \text{arc}\,(t\tau) - \text{arc}\,(r\varrho) = \vartheta_t + \vartheta_\tau - \delta_r - \delta_\varrho \equiv \varphi\,. \qquad (12\text{d})$$

These circular points give the answer to the close reflexion $z = ae^{i\varphi}$ (with constant modulus and variable phase) and the centre c_a to the $z_a = a^2\bar{\varrho}$.

Obviously, the mappings on (12a), (13a) have *two fixed points* ζ and ζ_* (possibly coincident) written as $\zeta = R(\zeta)$, and $\zeta_* = P(\zeta_*)$, resp., or in detail, by the quadratic equations (with $2\delta \equiv d - 1 = t\tau - r\varrho - 1$)

$$Q_r(\zeta) \equiv \zeta^2 + \frac{2\delta}{\varrho}\zeta + \frac{r}{\varrho} = 0\,, \quad R_\varrho(\zeta_*) \equiv \zeta_*^2 + \frac{2\delta}{r}\zeta_* + \frac{\varrho}{r} = 0\,. \quad (13\text{a,b})$$

Their zeros or fixed points are:

$$\zeta_{1,2} = -\frac{\delta}{\varrho} \pm \sqrt{\frac{\delta^2}{\varrho^2} - \frac{r}{\varrho}}, \quad \zeta_{1,2}^* = -\frac{\delta}{r} \pm \sqrt{\frac{\delta^2}{r^2} - \frac{\varrho}{r}}, \qquad (13\text{c,d})$$

where

$$\zeta_1 \zeta_2^* = \zeta_2 \zeta_1^* = \frac{\delta^2}{r\varrho} + \delta\left(\sqrt{\frac{\delta^2}{r^2\varrho^2} - \frac{1}{r\varrho}} - \sqrt{\frac{\delta^2}{r^2\varrho^2} - \frac{1}{r\varrho}}\right) - \left(\frac{\delta^2}{r\varrho} - 1\right) = 1\,, \tag{13e,f}$$

hence ζ_1 and ζ_2^*, then ζ_2 and ζ_1^* are reciprocal values. The special close reflexions ζ_1 and ζ_2 become — without change in the obstacle — into left-side composite reflexion.

The fixed points ζ_1, ζ_2 are defined after (13a, c) by two parameters (coefficients) $2b \equiv 2\delta/\varrho$, $c \equiv r/\varrho$ and also inversely, as

$$2b \equiv \frac{2\delta}{\varrho} = \frac{t\tau - r\varrho - 1}{\varrho} = -(\zeta_1 + \zeta_2),\ \ c \equiv \frac{r}{\varrho} = \zeta_1\zeta_2. \tag{13g,h}$$

If two obstacle with $\mathbf{S}_0' = [t', \tau'; r', \varrho']$, and $\mathbf{S}_0'' = [t'', \varrho''; r'', \tau'']$ have equal parameters $2b$ and c, namely

$$2b' \equiv \frac{t'\tau' - r'\varrho' - 1}{\varrho'} = \frac{t''\tau'' - r''\varrho'' - 1}{\varrho''} \equiv 2b''\,, \tag{14a}$$

$$c' \equiv \frac{r'}{\varrho'} = \frac{r''}{\varrho''} \equiv c'', \tag{14b}$$

then α) their fixed points are *common*:

$$\zeta_1' = \zeta_1'',\ \zeta_2' = \zeta_2''; \tag{14c}$$

β) the composition (artificial multiplication) of their scattering matrices is *commutative*:

$$\mathbf{S}_0'' * \mathbf{S}_0' = \mathbf{S}_0' * \mathbf{S}_0'' \quad \text{and vice versa.} \tag{14d}$$

The first fact follows from (13c); the second one flows from (7c) as:

$$\frac{t'\,t''}{1 - \varrho'\,r''} = \frac{t''\,t'}{1 - \varrho''\,r'} \Rightarrow \varrho'\,r'' = \varrho''\,r'(=C) \Rightarrow \frac{r'}{\varrho'} = \frac{r''}{\varrho''}(=c)\,;$$

$$r' + \frac{t'\tau'r''}{1 - \varrho'r''} = r'' + \frac{t''\tau''r'}{1 - \varrho''r'}$$

$$\Rightarrow r' + (t'\tau' - r'\varrho) = r'' = r'' + (t''\tau'' - r''\varrho'')\,r'\ |:C$$

$$\Rightarrow \frac{1}{\varrho''} + \frac{t'\tau' - r'\varrho'}{\varrho'} = \frac{1}{\varrho'} + \frac{t''\tau'' - r''\varrho''}{\varrho''}\left(= 2b + \frac{1}{\varrho'} + \frac{1}{\varrho''}\right),\ \text{q.e.d.} \tag{14e}$$

On the basis of the former, it can be stated: the (artificial) matrix, semi-group $\mathfrak{S}_0$ defined by the formulas (8a, b, c) will be a *commutative matrix semi-*

group, if all its matrices $S_0 \in \mathfrak{S}_0$ have common parameters $2b$ and c after (13g, h), consequently common fixed points ζ_1 and ζ_2 after (13c, d).

2,6. A commutative matrix semigroup $\mathfrak{S}_0$ turns into a commutative or ABELIAN *matrix group*, if any matrix $S_0 \in \mathfrak{S}_0$ is invertible by the artificial multiplication $*$ into its reciprocal (inverse) $^*S_0^{-1}$ after the definition

$$^*S_0^{-1} * S_0 = S_0 * {}^*S_0^{-1} = \mathbf{E}. \tag{15a}$$

Now, the condition and the formula of this inversion are to be found.

According to the relationship $S_0 \leftrightarrow S_1$ after (3b) [realized from S_0 by the DTA for $s_{22} \neq 0$ after (2b, c)], the inverse $^*S_0^{-1}$ can be *interpreted* through the relationship $S_1^{-1} \leftrightarrow {}^*S_0^{-1}$ [realized from $S_1^{-1} \equiv \mathbf{R}_1$ by the DTA for $r_{22} \neq 0$, reasonably after (2b, c)]. Now, look at the *single DTA-steps* to produce sequentially the matricial transfers $S_0 \to S_1 \to S_1^{-1} \to {}^*S_0^{-1}$, then *unit* into *a multiple DTA-spring* [14] to produce directly the matricial transfer $S_0 \to {}^*S_0^{-1}$. It can be written that

$$\begin{aligned} ^*S_0^{-1} \equiv \mathbf{B}_4^{-1} S_{04}' \equiv & [\mathbf{E} - (e_2 + s_2)e^2 | -(e_1 + s_1)e^1 + (s_2 + e_2)e^2 | -(e_2 + s_2)e^2]^{-1} \cdot \\ & \cdot [S_0 - (s_2 + e_2)e^2 | -(s_1 + e_1)e^1 + (e_2 + s_2)e^2 | -(s_2 + e_2)e^2] = \\ = & [\mathbf{E} - (e_1 + s_1)e^1 - (e_2 + s_2)e^2]^{-1} \cdot \\ & \cdot [S_0 - (s_1 + e_1)e^1 - (s_2 + e_2)e^2] = S_0^{-1}\mathbf{E} = S_0^{-1}, \\ & \text{where } S_0^{-1} S_0 = S_0 S_0^{-1} = \mathbf{E}. \end{aligned} \tag{15b}$$

After this result, the inverse $^*S_0^{-1}$ concerning the artificial multiplication and the inverse S_0^{-1} concerning the ordinary one are *identic*; their *common formula* and *condition* are the following:

$$S_0^{-1} \equiv {}^*S_0^{-1} = \frac{1}{d} \begin{bmatrix} \tau & -\varrho \\ -r & t \end{bmatrix} \quad (d \equiv t\tau - r\varrho \neq 0). \tag{15c}$$

We can *control* after the product rules (6) and (7c), that in fact, $S_0^{-1} S_0 = {}^*S_0^{-1} * S_0 = \mathbf{E}$. Our DTA can be considered as *a generator of* ABELIAN *matrix group*, too.

2,7. A chain O_N of the obstacles O_k (coupled in series) and given by their scattering matrices $S_0^{(k)} = [t^{(k)}, \varrho^{(k)}; r^{(k)}, \tau^{(k)}]$ can be characterized by its *composite scattering matrix* (*Fig.* 4)

$$\begin{aligned} S_{0n} \equiv \begin{bmatrix} t_n & \varrho_n \\ r_n & \tau_n \end{bmatrix} = & S_0^{(n)} * S_0^{(n-1)} * \ldots * S_0^{(2)} * S_0^{(1)} \equiv \\ & \equiv {}^*\prod_{k=1}^{n} S_0^{(k)} \equiv {}^*\prod_{k=1}^{n} \begin{bmatrix} t^{(k)} & \varrho^{(k)} \\ r^{(k)} & \tau^{(k)} \end{bmatrix}. \end{aligned} \tag{15}$$

Instead of generalizing, it will do here to make *some special remarks*:

α) As for $n = 1, 2$ there are t/τ, $t'\,t''\,(1 - \varrho'\,r'')^{-1}/\tau'\,\tau''\,(1 - \varrho'\,r'')^{-1} = t'\,t''/\tau'\,\tau''$, in general the ratio

$$\frac{t_n}{\tau_n} = \prod_{k=1}^{n} t^{(k)} : \prod_{k=1}^{n} \tau^{(k)} = \prod_{k=1}^{n} \frac{t^{(k)}}{\tau^{(k)}} \,. \tag{16a}$$

prevails.

β) If two obstacles are connected by a *line* of length x propagation coefficients being k and $\varkappa$, they can be considered an obstacle (without reflexions) with the scattering matrix $S_0 = [e^{jkx}, 0; 0, e^{j\varkappa x}]$.

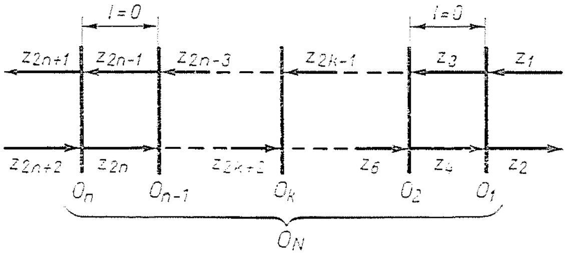

Fig. 4

γ) If the chain O_n of S_{O_n} has all its obstacles O_k *without loss*, it is itself without loss and the conditions are valid again:

$$|t_n|^2 + |r_n|^2 = |\tau_n|^2 + |\varrho_n|^2 = 1\,, \quad |t_n| = |\tau_n|\,, \tag{18}$$
$$\varphi_0^{(n)} \equiv \vartheta_t^{(n)} + \vartheta_\tau^{(n)} - (\delta_r^{(n)} + \delta_\varrho^{(n)}) = \pi\,.$$

δ) If all the obstacles O_k (their $S_0^{(k)}$) are *commutative*, their chain O_N (its S_{O_n}) is commutative again, with the specialities

$$\frac{t_n}{\varrho_n} = c = \zeta_1\,\zeta_2\,, \quad \frac{t_n\,\tau_n}{\varrho_n^2} = c\,\frac{\tau_n}{\varrho_n} = \varrho_n^{-2} + 2b\varrho_n^{-1} + c\,; \tag{19a,b}$$

if $\tau_n \to 0$ (dissipation), so $\varrho_n^{-2} + 2b\,\varrho_n^{-1} + c \to 0$, $\varrho_n^{-1} \to \zeta_2$ and $t_n \to \zeta_1$, which are the common fixed points. (19c)

2,8. Let us take the chain O_N of n *identic obstacles* $O_k \equiv O$ (coupled in series) given by their common scattering matrix $S_0^{(k)} = S_0 = [t, \varrho; r, \tau]$. The composite scattering matrix is now obviously

$$\begin{bmatrix} t_n & \varrho_n \\ r_n & \tau_n \end{bmatrix} \equiv S_{O_n} = \underbrace{S_0}_{n} * \underbrace{S_0}_{n-1} * \ldots * \underbrace{S_0}_{2} * \underbrace{S_0}_{1} \equiv {}^*\prod_{k=1}^{n} \underbrace{S_0}_{k} \equiv S_0^n \equiv \begin{bmatrix} t & \varrho \\ r & \tau \end{bmatrix}^n . \tag{20}$$

The commutativity is valid again (because $\mathbf{S}_0^{(k)} * \mathbf{S}_0^{(l)} = \mathbf{S}_0^2$). The associativity is also valid (as ever in the semigroups), consequently

$$\mathbf{S}_0^{m+n} \equiv \mathbf{S}_{O,m+n} \equiv \begin{bmatrix} t_{m+n} & \varrho_{m+n} \\ r_{m+n} & \tau_{m+n} \end{bmatrix} = \begin{bmatrix} t_m & \varrho_m \\ r_m & \tau_m \end{bmatrix} * \begin{bmatrix} t_n & \varrho_n \\ r_n & \tau_n \end{bmatrix} \equiv \mathbf{S}_{Om} * \mathbf{S}_{On} \equiv \mathbf{S}_0^m * \mathbf{S}_0^n . \tag{21a}$$

On the model of (8a), we can write for t_{m+n} and r_{m+n} (and by analogy, for τ_{m+n} and ϱ_{m+n}) the *difference equations*

$$t_{m+n} = \frac{t_m\, t_n}{1-\varrho_m\, r_n}\,, \quad r_{m+n} = r_m + \frac{t_m\, \tau_m\, r_n}{1-\varrho_m r_n}\,. \tag{21b,c}$$

For $n = 1$ the same becomes into *recoursing formulas* by which we can count all the t_{m+1}, r_{m+1} (and τ_{m+1}, ϱ_{m+1}) starting from their *initial values*

$$t_1 = t\,, \quad r_1 = r \quad (\varrho_1 = \varrho\,, \quad \tau_1 = \tau)\,. \tag{21d}$$

In this way, we get e.g.

$$t_1 = t, \ t_2 = \frac{t^2}{1-\varrho r} = t\left(\frac{t}{\tau}\right)^{1/2} \cdot \frac{sh\,\delta}{sh\,2\delta - T\,sh\,\delta}\,, \ldots, \tag{22a}$$

$$t_n = \left(\frac{t}{\tau}\right)^{\frac{n-1}{2}} \cdot \frac{sh\,\delta}{shn\delta - T\,sh(n-1)\,\delta}\,;$$

$$r_1 = r, \quad r_2 = r + \frac{t\tau r}{1-\varrho r} = \frac{r\,sh\,2\delta}{sh\,2\delta - T\,sh\delta}\,, \ldots, \tag{22b}$$

$$r_n = \frac{r\,sh\,n\delta}{sh\,n\delta - T\,sh(n-1)\,\delta}\,,$$

notations being

$$2\hat{d} \equiv d+1 \equiv t\tau - r\varrho + 1, \quad T \equiv \sqrt{t\tau}\,;$$

$$ch\,\delta = \frac{\hat{d}}{T}\,(\geq 1)\,, \qquad sh\,\delta = \frac{\sqrt{\hat{d}^2 - T^2}}{T}\,. \tag{22c}$$

2,9. In the former treatment, homogeneous and anisotropic (i.e. with different scattering properties in the opposite directions) discrete (thin) obstacles have been considered throughout, further, their anisotropic discrete chain of inhomogeneous or homogeneous composition (by different or identic, homogeneous discrete obstacles). The former contributions — based on our dynamical transform algorithm (DTA) as generator of various matrix algebraic structures, at last one of ABELIAN matrix groups — were given for the algebra of such obstacles and their chains. They could perhaps raise some ideas to set out certain problems in this domain.

In a subsequent paper, investigations of discrete obstacles with $2n$ contacts (where our DTA in hypermatrix form is essentially more advantageous), of the continuous media with $2 \cdot 2$ and $2n$ contacts (treated by matrix- and hypermatrix-functional equations) and other specialities will be treated.

Summary

After World War II, *quick development* began in the theory of micro-wave technics. Among others, the algebra of obstacles or the scattering algebra was evolved for discrete obstacles, then for continuous media, with the application of various mathematical methods, and though the works written by CARLIN, REDHEFFER, TWESKY and by many others (see References [1—13]).

Nevertheless, this beautiful progress admits—just because its quickness—various contributions. Such complements will be here given (and others in a subsequent paper) based on our *dynamic transform algorithm (DTA)* of matrices and hypermatrices [14—17]. It can be considered as *a generator of certain matrix semigroups and groups*, e.g. the ABELIANS ones. Our contributions are connected possibly with known facts of scattering algebra; these latters are often interpreted in sense of [13].

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