REFLECTION OF PRESSURE WAVES AT FLOW IN ELASTIC TUBES

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1. In the thrombotic or degeneratic occlusion of arterias of medium size, new, so-called collateral arteriolas are opened, which do not participate in normal circulation. This phenomenon is assumed partly to be caused by the systolic pressure wave reflected at the thrombus placed in the lumen of the arteria, so that a standing wave develops. A pressure pick by this standing wave is formed proximally at the thrombus, which is significantly higher than the physiological one, and so the collaterals are opened.

To analyse the effect of this phenomenon, the following mathematical model has been established to inform of the quantitative picture. The blood is a practically incompressible fluid. This fluid flows in a very elastic tube, namely in the arteria to be analysed. The viscosity of the blood is too high to be ignored. We assume in the following that the pressure and the velocity of the flow are constant in a cross-section, and depend only on time and on length coordinate. So the effect of viscosity is determined by the mean velocity in each cross-section and by the viscosity coefficient. The effective diameter of the cross-section is, however, a function of the pressure at the moment, because of the elasticity of the tube-wall. This effect has also to be considered. Accordingly, the partial differential equations of flow [1] are:

$$p'_x = \varrho \left(v'_t - v \cdot v'_x + \lambda f(p\,;d) \cdot \frac{v^2}{2} \right) \tag{1}$$

$$\varrho c^2 v'_x = p'_t - v p'_x\,, \tag{2}$$

where x is the length coordinate, t is the time, $p(x, t)$ is the (absolute) pressure and $v(x, t)$ is the velocity; ϱ denotes the density, λ is the coefficient of viscosity and c denotes the local velocity of sound (notice that the meaning of c is here "the low-frequency" velocity of sound; the high-frequency velocity of sound — in a strict sense — belongs to a much higher domain, because in expansion, high-frequency pressure variations affect only the compressibility of the fluid,

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rather than the elasticity of the wall. The expansion of pulse wave-fronts is characterized by the low-frequency speed of sound, and normally ranges from 180 cm/sec to 500 cm/sec, whereas the strict velocity of sound is about 100.000 cm/sec.) c is assumed to be a constant of about 200—500 cm/sec, which is approximately true at frequencies not higher than 100 to 200 Hz. $f(p, d)$ denotes the coefficient of the effective diameter; in first approximation it is $1/d$, in second approximation we take it into consideration according to the expression

$$f(p, d) = \frac{1}{d}(1 - d_1 p), \quad (d_1 > 0) \tag{3}$$

though, theoretically, $f(p, d)$ may be an analytical function of p, without affecting the following mathematical investigations.

Let us assume that in the initial cross-section ($x = 0$) the normal pulse wave can be described by

$$p(0, t) = p_0 + \sum_{k=1}^{K} (p_k \cos k\omega t + r_k \sin k\omega t) = I(t) \tag{4}$$

where the higher frequencies ($k > K$) are neglected as stated before for the high frequencies. But if c can be taken as a constant for any frequency, we suppose that $p(0, t)$ is a continuously differentiable function, periodic with a period $2\pi/\omega$.

For the final cross-section ($x = x_L$) we assume that the tube is entirely closed:

$$v(x_L, t) \equiv 0 \tag{5a}$$

or it is nearly closed, i.e. the speed of outflow is proportional to the momentaneous pressure

$$v(x_L, t) \equiv \alpha p(x_L, t) \, . \tag{5b}$$

(5a) and (5b) can be summarized in the final condition

$$v(x_L, t) \equiv \alpha \cdot p(x_L, t) \tag{5}$$

where α is O or a very small value.

The system (1)—(2) is hyperbolic, and so it has a unique solution in the segment $0 \leqq x \leqq x_L$ for any pair of initial conditions $p(0, t)$, $v(0, t)$; if they are continuously differentiable [2, 3]. We consider, however, the system with the pair of initial-final conditions (4) and (5), resp. For this case we do not know anything about the uniqueness or existence of the solution. In the following it will be proved that the problem (1)—(2)—(4)—(5) has a solution, $p(x, t)$,

$v(x, t)$ periodic in t for all fixed x in $0 \leqq x \leqq x_L$, if λ is small and c great enough, and has a single solution in the class of functions, periodic in t for all fixed x in $0 \leqq x \leqq x_L$. To prove this fact, we use an approximative algorithm, giving an effective approximation for $p(x, t)$ and $v(x, t)$ as well. Let us consider first the method of approximation of the periodic solution.

2. In the following it will be assumed that $|\lambda|$ is small enough and for such λ the solution of (1)—(2)—(4)—(5) is an analytic function of λ, i.e.

$$p(x, t; \lambda) = p_0(x, t) + \lambda p_1(x, t) + \lambda^2 p_2(x, t) + \dots \tag{6}$$

and

$$v(x, t; \lambda) = v_0(x, t) + \lambda v_1(x, t) + \lambda^2 v_2(x, t) + \dots \tag{7}$$

are convergent in a circle $|\lambda| < \lambda_0$ for all t and $0 \leqq x \leqq x_L$, and are periodic solutions of (1)—(2)—(4)—(5). In this case the pairs of functions $\{p_n(x, t), v_n(x, t)\}$ are continuously differentiable and periodic in t for all x in $0 \leqq x \leqq x_L$, and they must satisfy the boundary conditions

$$\begin{gathered} p_0(0, t) \equiv p(0, t);\ v_0(x_L, t) \equiv \alpha p_0(x_L, t) \\ p_n(0, t) \equiv 0;\ v_n(x_L, t) \equiv \alpha p_n(x_L, t);\ n = 1, 2, \dots \end{gathered} \tag{8}$$

Setting now (6)—(7) in (1)—(2), we get the system of equations for the coefficients of the corresponding powers of λ:

$$p'_{0_x} = \varrho(v'_{0_t} - v_0 \cdot v'_{0_x}) \tag{9}$$

$$v'_{0_x} = \frac{1}{\varrho c^2}(p'_{0_t} - v_0 \cdot p'_{0_x}) \tag{10}$$

$$p'_{1_x} = \varrho \left(v'_{1_t} - v_0 \cdot v'_{1_x} - v_1 \cdot v'_{0_x} + \frac{1}{d}(1 - d_1 p_0)\frac{v_0^2}{2} \right) \tag{11}$$

$$v'_{1_x} = \frac{1}{\varrho c^2}(p'_{1_t} - v_0 p'_{1_x} - v_1 \cdot p'_{0_x}) \tag{12}$$

and in general

$$\begin{aligned} p'_{n_x} = \varrho \big((v'_{n_t} - v_0 v'_{n_x} - v_n \cdot v'_{0_x} + \varphi_n(v_0, v_1 \dots, \\ \dots, v_{n-1}, p_0, \dots, p_{n-1}) \big) \end{aligned} \tag{13}$$

$$\begin{aligned} v'_{n_x} = \frac{1}{\varrho c^2} \big(p'_{n_t} - v_0 p'_{n_x} - v_n \cdot p'_{0_x} + \\ + \psi_n(v_0, v_1, \dots, v_{n-1}, p_0, \dots, p_{n-1}) \big) \end{aligned} \tag{14}$$

where

$$\varphi_n = - v_1 \cdot v'_{n-1_x} - v_2 v'_{n-2_x} - \dots - v_{n-1} v'_{1_x} + \\ + \frac{1-d_1 p_0}{2d} (2v_0 v_{n-1} + 2 v_1 v_{n-2} + \dots) - \\ - \frac{d_1}{2d} p_1 (2v_0 v_{n-2} + 2v_1 v_{n-3} + \dots) - \dots - \frac{d_1}{2d} p_{n-1} v_0^2 \tag{15}$$

and

$$\Psi_n = - v_1 p'_{n-1_x} - v_2 p'_{n-2_x} - \dots - v_{n-1} p'_{1_x}. \tag{16}$$

The systems of Eqs (11)—(12) and (13)—(14) are now linear for the unknown functions p_1, v_1, and p_n, v_n, resp., no direct solution, satisfying also the boundary conditions (8) in the form of Fourier-expansions can, however, be found because there are unknown terms with variable coefficients, too. Let us therefore consider the system (9)—(10) first, setting v'_{0_x} in (9) through (10), and solve the equations for p'_{0_x} and v'_{0_x}, resp.:

$$p'_{0_x} = \frac{1}{1 - \frac{v_0^2}{c^2}} \left(\varrho v'_{0_t} - \frac{v_0}{c^2} p'_{0_t} \right)$$

$$v'_{0_x} = \frac{1}{\varrho c^2} (p'_{0_t} - v_0 p'_{0_x}).$$

Now, if $|v_0| < |c|$ is true for all t and $0 \leq x \leq x_L$, and if v_0 and p_0 are analytic functions of $\varepsilon = 1/c^2$ for $|\varepsilon|$ small enough, i.e. if

$$p_0(x, t; \varepsilon) = P_0^{(0)}(x, t) + \varepsilon P_1^{(0)}(x, t) + \varepsilon^2 P_2^{(0)}(x, t) + \dots \tag{17}$$

and

$$v_0(x, t; \varepsilon) = V_0^{(0)}(x, t) + \varepsilon V_1^{(0)}(x, t) + \varepsilon^2 V_2^{(0)}(x, t) + \dots \tag{18}$$

are solutions of (8)—(9)—(10) in the circle $|\varepsilon| < \varepsilon_0$ for all t and $0 \leq x \leq x_L$, and the expansion

$$\frac{1}{1 - \frac{v_0^2}{c^2}} = 1 + \varepsilon v_0^2 + \varepsilon^2 v_0^4 + \dots = 1 + \varepsilon V_0^{(0)^2} + \varepsilon^2 (V_0^{(0)^4} + 2 V_0^{(0)} V_1^{(0)}) + \dots + \\ + \varepsilon^n (V_0^{(0)^{2n}} + 2 V_0^{(0)^{2n-3}} V_1^{(0)} + \dots) + \dots$$

is also convergent in this circle, then, setting the expansions above in (9)—(10) and comparing the coefficients of the corresponding powers of ε results in the following system of equations:

$$P_{0x}^{(0)'} = \varrho \cdot V_{0t}^{(0)} \tag{19}$$

$$V_{0x}^{(0)\prime} = 0 \tag{20}$$

$$P_{1x}^{(0)\prime} = \varrho V_{1t}^{(0)\prime} + \varrho V_0^{(0)2} \cdot V_{0t}^{(0)\prime} - V_0^{(0)} \cdot P_{0t}^{(0)\prime} \tag{21}$$

$$V_{1x}^{(0)\prime} = \frac{1}{\varrho} P_{0t}^{(0)\prime} - \frac{1}{\varrho} V_0^{(0)} \cdot P_{0x}^{(0)\prime} \tag{22}$$

$$P_{2x}^{(0)\prime} = \varrho V_{2t}^{(0)\prime} + \varrho V_0^{(0)2} V_{1t}^{(0)\prime} + \varrho \cdot (V_0^{(0)4} + 2 V_0^{(0)} V_1^{(0)}) V_{0t}^{(0)\prime} - \\ - V_0^{(0)} \cdot P_{1t}^{(0)\prime} - V_1^{(0)} \cdot P_{0t}^{(0)\prime} - V_0^{(0)3} P_{0t}^{(0)\prime}; \tag{23}$$

$$V_{2x}^{(0)\prime} = \frac{1}{\varrho} P_{1t}^{(0)\prime} - \frac{1}{\varrho} V_0^{(0)} \cdot P_{1x}^{(0)\prime} - \frac{1}{\varrho} V_1^{(0)} P_{0x}^{(0)\prime}; \tag{24}$$

and in general

$$P_{kx}^{(0)\prime} = \varrho \cdot V_{kt}^{(0)\prime} + \alpha_k^{(0)}(P_0, P_1 \ldots, P_{k-1}, V_0, \ldots, V_{k-1}) \tag{25}$$

$$V_{kx}^{(0)\prime} = \beta_k^{(0)}(P_0, P_1, \ldots, P_{k-1}, V_0, V_1, \ldots, V_{k-1}) \,. \tag{26}$$

Now in the circle $|\varepsilon| < |\varepsilon_0|$ the pairs of functions $P_k^{(0)}$, $V_k^{(0)}$ are continuously differentiable, periodic in t for all $0 \leq x \leq x_L$ and satisfy the boundary conditions

$$P_0^{(0)}(0, t) \equiv p(0, t);\; V_0^{(0)}(x_L, t) \equiv \alpha \cdot P_0^{(0)}(x_L, t) \\ P_k^{(0)}(0, t) \equiv 0;\; V_k^{(0)}(x_L, t) \equiv \alpha \cdot P_k^{(0)}(x_L, t);\;\; k=1,2,\ldots \tag{27}$$

Now, applying the same procedure for (11)—(12), we get the system

$$p'_{1_x} = \frac{1}{1 - \frac{v_0^2}{c^2}} \left(\varrho \cdot v'_{1_t} - \frac{v_0}{c^2} p'_{1_t} + \frac{2 v_0 v_1}{c^2} p'_{0_x} - \frac{v_1}{c^2} p'_{0_t} + \right. \\ \left. + \frac{\varrho}{d} (1 - d_1 p_0) \frac{v_0^2}{2} \right);$$

$$v'_{1_x} = \frac{1}{c^2} (p'_{1_t} - v_0 p'_{1_x} - v_1 p'_{0_x}) \,.$$

In the circle $|\varepsilon| < |\varepsilon_1|$, we use the expansions

$$p_1(x, t; \varepsilon) = P_0^{(1)}(x, t) + \varepsilon P_1^{(1)}(x, t) + \ldots \\ v_1(x, t; \varepsilon) = V_0^{(1)}(x, t) + \varepsilon \cdot V_1^{(1)}(x, t) + \ldots \tag{28}$$

and after substitution we get the system

$$P_{0x}^{(1)\prime} = \varrho \cdot V_{0t}^{(1)\prime} + \frac{\varrho}{d} (1 - d_1 p_0) \frac{v_0^2}{2} \tag{29}$$

$$V_{0x}^{(1)\prime} = 0 \tag{30}$$

$$P_{1x}^{(1)\prime} = \varrho \cdot V_{1t}^{(1)\prime} + \varrho v_0^2 V_{0t}^{(0)\prime} - v_0 P_{0t}^{(1)\prime} + 2 v_0 V_0^{(1)} p'_{0x} -$$

$$- V_0^{(1)} p'_{0_t} + \frac{\varrho}{d} (1-d_1 p_0) \frac{v_0^4}{2} \tag{31}$$

$$V_{1x}^{(1)\prime} = \frac{1}{\varrho} P_{0t}^{(1)\prime} - \frac{v_0}{\varrho} P_{0x}^{(1)\prime} - \frac{1}{\varrho} V_0^{(1)} p'_{0_x} \tag{32}$$

$$P_{2x}^{(1)\prime} = \varrho \cdot V_{2t}^{(1)\prime} + \varrho v_0^2 V_{1t}^{(1)\prime} + \varrho v_0^4 V_{0t}^{(0)\prime} - v_0 P_{1t}^{(1)\prime} - v_0^3 P_{0t}^{(1)\prime} + $$
$$+ 2v_0^3 v_1 p'_{0x} - V_1^{(1)} \cdot p'_{0_t} - v_0^2 V_0^{(1)} p'_{0_t} + \frac{\varrho}{d} (1-d_1 p_0) \frac{v_0^6}{2}; \tag{33}$$

$$V_{2x}^{(1)\prime} = \frac{1}{\varrho} P_{1t}^{(1)\prime} - \frac{v_0}{\varrho} P_{1x}^{(1)\prime} - \frac{1}{\varrho} V_1^{(1)} p'_{0x}, \tag{34}$$

and in general

$$P_{kx}^{(1)\prime} = \varrho V_{kt}^{(1)\prime} + \alpha_k^{(1)}(p_0, v_0, P_0^{(1)} \ldots, P_{k-1}^{(1)}, V_0^{(1)}, \ldots, V_{k-1}^{(1)});$$
$$V_{kx}^{(1)\prime} = \beta_k^{(1)}(p_0, v_0, P_0^{(1)}, P_1^{(1)}, \ldots, P_{k-1}^{(1)}, V_0^{(1)}, \ldots, V_{k-1}^{(1)}). \tag{35}$$

The boundary conditions are the following:

$$P_k^{(1)}(0, t) \equiv 0; \quad V_k^{(1)}(x_L, t) \equiv \alpha \cdot P_k^{(1)}(x_L, t);$$
$$k = 0, 1, 2, \ldots \tag{36}$$

Using the same procedure for (13)—(14), we get

$$p'_{n_x} = \frac{1}{1 - \frac{v_0^2}{c^2}} \left(\varrho \cdot v'_{n_t} - \frac{v_0}{c^2} (p'_{n_t} - v_n p'_{0_x} + \Psi_n) - \frac{v_n}{c^2} (p'_{0_t} - v_0 p'_{0_x}) + \varrho \varphi_n \right)$$

i.e. by expanding p_n and v_n in the circle $|\varepsilon| < |\varepsilon_n|$

$$P_{0x}^{(n)\prime} = \varrho \cdot V_{0t}^{(n)\prime} + \varrho \varphi_n(v_0, v_1, \ldots, v_{n-1}, p_0, \ldots, p_{n-1}), \tag{37}$$

$$V_{0x}^{(n)\prime} = 0, \tag{38}$$

$$P_{1x}^{(n)\prime} = \varrho \cdot V_{1t}^{(n)\prime} + \varrho \cdot v_0^2 V_{0t}^{(n)\prime} - v_0 P_{0t}^{(n)\prime} + 2v_0 V_0^{(n)} p'_{0_x} -$$
$$- v_0 \Psi_n - V_0^{(n)} p'_{0_t} + \varrho \cdot v_0^2 \varphi_n, \tag{39}$$

$$V_{1x}^{(n)\prime} = \frac{1}{\varrho} P_{0t}^{(n)\prime} - \frac{v_0}{\varrho} P_{0x}^{(n)\prime} - \frac{1}{\varrho} p'_{0_x} \cdot V_0^{(n)} + \frac{1}{\varrho} \Psi_n, \tag{40}$$

$$P_{2x}^{(n)\prime} = \varrho V_{2t}^{(n)\prime} + \varrho v_0^2 V_{1t}^{(n)\prime} + \varrho v_0^4 V_{0t}^{(n)\prime} - v_0 P_{1t}^{(n)\prime} + 2v_0 V_1^{(n)} p'_{0_x} -$$
$$- v_0^3 P_{0t}^{(n)\prime} + 2v_0^3 V_0^{(n)} p'_{0_x} - v_0^3 \Psi_n - V_1^{(n)} p'_{0_t} -$$
$$- v_0^2 V_0^{(n)} p'_{0_t} + \varrho v_0^4 \varphi_n, \tag{41}$$

$$V_{2x}^{(n)\prime} = \frac{1}{\varrho} P_{1t}^{(n)\prime} - \frac{v_0}{\varrho} P_{1x}^{(n)\prime} - \frac{1}{\varrho} V_1^{(n)} p'_{0_x}, \tag{42}$$

and in general

$$P_{kz}^{(n)\prime} = \varrho \cdot V_{kt}^{(n)\prime} + \alpha_k^{(n)}(v_0, v_1, \ldots, v_{n-1}, p_0, \ldots \\ \ldots, p_{n-1}, P_0^{(n)}, \ldots, P_{k-1}^{(n)}, V_0^{(n)}, \ldots, V_{k-1}^{(n)}), \tag{43}$$

$$V_{kz}^{(n)\prime} = \beta_k^{(n)}(v_0, \ldots, v_{n-1}, p_0, \ldots, p_{n-1}, P_0^{(n)}, \ldots \\ \ldots, P_{k-1}^{(n)}, V_0^{(n)}, \ldots, V_{k-1}^{(n)}). \tag{44}$$

The boundary conditions are the following:

$$P_k^{(n)}(0, t) \equiv 0\,; \quad V_k^{(n)}(x_L, t) \equiv \alpha \cdot P_k^{(n)}(x_L, t); \qquad k = 0, 1, 2 \ldots \tag{45}$$

Now if all our previous conditions are valid, i.e. the functions $P_k^{(n)}(x, t)$, $V_k^{(n)}(x, t)$ satisfying (19)—(45) are continuously differentiable and periodic in t for all fixed x in $0 \leq x \leq x_L$, then we can find them in form of Fourier-expansions:

$$P_k^{(n)}(x, t) = s_0^{(k,n)}(x) + \sum_{i=1}^{\infty} (s_i^{(k,n)}(x) \cos i\omega t + t_i^{(k,n)}(x) \sin i\omega t)\,; \tag{46}$$

$$V_k^{(n)}(x, t) = u_0^{(k,n)}(x) + \sum_{i=1}^{\infty} (u_i^{(k,n)}(x) \cos i\omega t + w_i^{(k,n)}(n) \cdot \sin i\omega t). \tag{47}$$

Setting these expansions into Eqs (43)—(44), we get immediately integrable differential equations in closed form for the unknown functions $s_i^{(k,n)}(x)$, $t_i^{(k,n)}(x)$, $u_i^{(k,n)}(x)$, $w_i^{(k,n)}(x)$, and the boundary conditions (27)—(36)—(45) can be met, too: namely, if $\alpha = 0$, then the initial values ($x = 0$) of s_i and t_i and the final values ($x = x_L$) of u_i and w_i are prescribed. On the other hand, if $\alpha \neq 0$ ($\alpha > 0$), then we have to deal with a system of unknown constants

$$s_i^{(k,n)}(x_L) = u_i^{(k,n)}(x_L) = \gamma_i^{(k,n)}; \qquad i, k, n = 0, 1, 2, \ldots \tag{48}$$

$$t_i^{(k,n)}(x_L) = w_i^{(k,n)}(x_L) = \delta_i^{(k,n)}; \qquad i = 1, 2, \ldots; \quad k, n = 0, 1, 2, \ldots \tag{49}$$

which must be chosen so that the initial values of s_i and t_i satisfy the conditions defined by (27)—(36)—(45). These conditions, however, separate the system of equations, defining the unknown constants so that each equation contains one and only one unknown γ_i, or δ_i in the form of a term $\gamma_i^{(k,n)}(x - x_L)$ or $\delta_i^{(k,n)}(x - x_L)$ resp., i.e. we get one and only one solution for each term in (46) and (47) resp. We shall give an example of practical interest in 4.

3. To discuss the theoretical problems of the uniqueness and existence of the solution of (1)—(2)—(4)—(5), we note first that the problem of uniqueness cannot be treated in general. The method of approximation, given in 2 proves, however, that our problem has one and only one solution in the class of functions, continuously differentiable and periodic in t for all fixed x in $0 \leq x \leq x_L$, if for (6)—(7) there exists a circle of convergence $|\lambda| < \lambda_0$ for all t, $0 \leq x \leq x_L$ and $|\varepsilon| = 1/|c^2|$ small enough and for the expansions of p_n and v_n there exists a common circle of convergence $|\varepsilon| < \varepsilon_c$. Finally, the algorithm for approximation, given in 2, demonstrates the existence of a periodic solution of (1)—(2)—(4)—(5), and also the uniqueness of this solution in the class of periodic functions, if we prove that this periodic solution is an analytic function of $\varepsilon = 1/c^2$ near to $\varepsilon = 0$ for all t, $0 \leq x \leq x_L$ and for all λ, with $|\lambda|$ small enough, and is an analytic function of λ near to $\lambda = 0$ for all t, $0 \leq x \leq x_L$ and for all ε, with $|\varepsilon|$ small enough.

Now, considering Eqs (1)—(2) with fixed, continuously differentiable initial conditions $p(0, t)$ and $v(0, t)$, and with fixed (complex) parameters λ and $\varepsilon = 1/c^2$, resp., then the solution of this problem is uniquely determined and continuously differentiable in $0 \leq x \leq x_L$ (s.f.e. [2, 3]). We can immediately prove that the system

$$\frac{\partial p'_\lambda}{\partial x} = \varrho \left(\frac{\partial v'_\lambda}{\partial t} - v'_\lambda \frac{\partial v}{\partial x} - v \frac{\partial v'_\lambda}{\partial x} + \frac{1}{d} (1 - d_1 p) \frac{v^2}{2} + \right.$$
$$\left. + \frac{\lambda}{d} (1 - d_1 p)\, v \cdot v'_\lambda - \lambda \frac{d_1}{2d} v^2 \cdot p'_\lambda \right), \tag{50}$$

$$\varrho \cdot \frac{\partial v'_\lambda}{\partial x} = \varepsilon \left(\frac{\partial p'_\lambda}{\partial t} - v'_\lambda \frac{\partial p}{\partial x} - v \frac{\partial p'_\lambda}{\partial x} \right) \tag{51}$$

has a uniquely determined pair of continuous solutions, satisfying the initial conditions

$$p'_\lambda(0, t) \equiv v'_\lambda(0, t) \equiv 0$$

for all t and $0 \leq x \leq x_L$, and that these solutions are the partial derivatives of $p(x, t; \lambda)$ and of $v(x, t; \lambda)$ resp., for the given value of λ; in the same manner it can be proved that the system

$$\frac{\partial p'_\varepsilon}{\partial x} = \varrho \left(\frac{\partial v'_\varepsilon}{\partial t} - v'_\varepsilon \frac{\partial v}{\partial x} - v \frac{\partial v'_\varepsilon}{\partial x} + \frac{\lambda}{d} (1 - d_1 p)\, v \cdot v'_\varepsilon - \frac{\lambda}{d} \frac{v^2}{2} p'_{1\varepsilon}\, d_1 \right), \tag{52}$$

$$\varrho \frac{\partial v'_\varepsilon}{\partial x} = \frac{\partial p}{\partial t} - v \frac{\partial p}{\partial x} + \varepsilon \left(\frac{\partial p'_\varepsilon}{\partial t} - v'_\varepsilon \frac{\partial p}{\partial x} - v \frac{\partial p'_\varepsilon}{\partial x} \right) \tag{53}$$

has a uniquely determined pair of continuous solutions, satisfying the initial conditions

$$p'_\varepsilon(0, t) \equiv v'_\varepsilon(0, t) \equiv 0 \tag{54}$$

for all t and $0 \leq x \leq x_L$, and that these solutions are the partial derivatives of $p(x, t; \varepsilon)$ and of $v(x, t; \varepsilon)$, resp., for the given values of ε. It means that the initial value problem, connected with (1)—(2), has a regular, and thus, analytical pair of solutions in the parameters λ and ε for all values of λ and ε. Though, a mixed, initial-final boundary problem is considered and to prove the existence of an analytic solution of this problem means some difficulties, as against the previous problem, because a change of λ or ε causes a change in $p(x_L, t)$ and in $v(x_L, t)$ at fixed initial functions $p(0, t)$ and $v(0, t)$ — the infinitesimal change of $p(x_L, t)$ and of $v(x_L, t)$ resp. can be characterized by $p'_\lambda(x_L, t)d\lambda$; $v'_\lambda(x_L, t)d\lambda$ and by $p'_\varepsilon(x_L, t)d\varepsilon$; $v'_\varepsilon(x_L, t)\,d\varepsilon$, resp. these derivatives being solutions of (50)—(51) and of (52)—(53), resp., — and after this change the condition $v(x_L, t) \equiv \alpha p(x_L, t)$ is not yet satisfied. We must therefore prove, that for given λ, ε, and solutions $p(x, t; \lambda, \varepsilon)$, $v(x, t; \lambda, \varepsilon)$ of (1)—(2)—(4)—(5), belonging to this pair of parameters, a change $\Delta v(0, t; \lambda, \varepsilon; \Delta\lambda)$ $\Delta v(0, t; \lambda, \varepsilon; \Delta\varepsilon)$ can be given to a given variation $\Delta\lambda$ and $\Delta\varepsilon$, resp. so that the solutions

$$p^*(x, t; \lambda, \varepsilon),\; v^*(x, t; \lambda, \varepsilon) \text{ resp. } p^{**}(x, t; \lambda + \Delta\lambda, \varepsilon)\,,$$
$$v^{**}(x, t; \lambda + \Delta\lambda, \varepsilon) \text{ resp. } p^{***}(x, t; \lambda, \varepsilon + \Delta\varepsilon)\,,$$
$$v^{***}(x, t; \lambda, \varepsilon + \Delta\varepsilon)$$

of the initial value problems

$$p^*(0, t; \lambda, \varepsilon) \equiv p^{**}(0, t; \lambda + \Delta\lambda, \varepsilon) \equiv p^{***}(0, t; \lambda, \varepsilon + \Delta\varepsilon) \equiv p(0, t)\,,$$
$$v^{**}(0, t; \lambda + \Delta\lambda, \varepsilon) \equiv v^*(0, t; \lambda, \varepsilon) + \Delta v(0, t; \lambda, \varepsilon; \Delta\lambda)$$
$$v^{***}(0, t; \lambda, \varepsilon + \Delta\varepsilon) \equiv v^*(0, t; \lambda, \varepsilon) + \Delta v(0, t; \lambda, \varepsilon; \Delta\varepsilon)$$

connected with the system (1)—(2) for parameter values λ, ε; $\lambda + \Delta\lambda, \varepsilon$; or $\lambda, \varepsilon + \Delta\varepsilon$ satisfy the final condition

$$v^*(x_L, t) \equiv \alpha p^*(x_L, t);\; v^{**}(x_L, t) \equiv \alpha \cdot p^{**}(x_L, t);\; v^{***}(x_L, t) \equiv \alpha \cdot p^{***}(x_L, t) \quad (55)$$

and that the functions

$$\lim_{\Delta\lambda=0} \frac{p^{**} - p^*}{\Delta\lambda}\,;\; \lim_{\Delta\lambda=0} \frac{v^{**} - v^*}{\Delta\lambda}\,;\; \lim_{\Delta\varepsilon=0} \frac{p^{***} - p^*}{\Delta\varepsilon}\,;\; \lim_{\Delta\varepsilon=0} \frac{v^{***} - v^*}{\Delta\varepsilon}$$

exist for all t and $0 \leq x \leq x_L$, and are continuous in λ and ε. Now it is clear, that all the differences $p^{**} - p^*$; $v^{**} - v^*$; $p^{***} - p^*$, $v^{***} - v^*$ can be divided in two parts; $\Delta_1 p^{**}$, $\Delta_1 v^{**}$; $\Delta_1 p^{***}$ and $\Delta_1 v^{***}$ is the part of $p^{**} - p^*$; $v^{**} - v^*$; $p^{***} - p^*$; $v^{***} - v^*$, caused by the change of the initial value of v^* with λ and ε, resp., unchanged in (1)—(2), whereas the other part of these differences, $\Delta_2 p^{**}$, $\Delta_2 v^{**}$ and $\Delta_2 p^{***}$, $\Delta_2 v^{***}$ is caused by the change of λ and ε, resp., in (1)—(2) with unchanged initial values $p^{**}(0, t)$, $v^{**}(0, t)$ and $p^{***}(0, t)$, $v^{***}(0, t)$, resp. We must now prove on one hand that $\Delta v(0, t; \lambda, \varepsilon; \Delta\lambda)$ and $\Delta v(0, t; \lambda, \varepsilon; \Delta\varepsilon)$ can be chosen so that the final condition is still satisfied, and on the other hand, that the functions

$$\lim_{\Delta\lambda=0} \frac{\Delta_1 p^{**}}{\Delta\lambda}; \quad \lim_{\Delta\lambda=0} \frac{\Delta_2 p^{**}}{\Delta\lambda}; \quad \ldots$$

exist for all t and $0 \leq x \leq x_L$, and are continuous in λ and ε. In proving these properties we shall make advantage from the fact, that p and v are continuously differentiable and periodic in t for all fixed x (more exactly, that we consider the existence and uniqueness of the solutions only in this class). That means practically to consider in constructing solutions for (1)—(2)—(4)—(5) only those solutions of the initial value problems — with variable initial values — for which $p(0, t)$ and $v(0, t)$ are continuously differentiable and periodic in t. Thereby $v(0, t)$ must be continuously differentiable and periodic, too, and so it can be chosen in the form of a convergent Fourier-expansion:

$$\Delta v = \mu_0 + \sum_{i=0}^{\infty} (\mu_i \cos i\omega t + \nu_i \sin i\omega t) \,. \tag{56}$$

Now, (56) can satisfy the final condition (55) for $|\Delta\lambda|$ and $|\Delta\varepsilon|$ small enough, if $p(x, t)$ and $v(x, t)$ are differentiable functions of the Fourier coefficients μ_0, μ_i, ν_i $(i = 1, 2, \ldots)$, and if the partial derivatives

$$\frac{\partial p}{\partial \mu_i}, \quad \frac{\partial v}{\partial \mu_i}, \quad \frac{\partial p}{\partial \nu_i}, \quad \frac{\partial v}{\partial \nu_i}$$

are continuously differentiable and periodic in t for all fixed x, and, applying the notations

$$\frac{\partial p}{\partial \mu_i} = a_0^{(i)}(x) + \sum_{j=1}^{\infty} \left(a_j^{(i)}(x) \cos j\omega t + b_j^{(i)}(x) \sin j\omega t\right);$$

$$\frac{\partial v}{\partial \mu_i} = c_0^{(i)}(x) + \sum_{j=1}^{\infty} \left(c_j^{(i)}(x) \cos j\omega t + d_j^{(i)}(x) \sin j\omega t\right);$$

$$\frac{\partial p}{\partial \nu_i} = e_0^{(i)}(x) + \sum_{j=1}^{\infty} \left(e_j^{(i)}(x) \cos j\omega t + f_j^{(i)}(x) \sin j\omega t\right);$$

$$\frac{\partial v}{\partial \nu_i} = g_0^{(i)}(x) + \sum_{j=1}^{\infty} \left(g_j^{(i)}(x) \cos j\omega t + h_j^{(i)}(x) \sin j\omega t\right);$$

the sequence of matrices (57)

$$\mathbf{B}_n = \mathbf{A}_n^{-1}\,;\ \mathbf{A}_n = (l_{ij})_n;\ \ i,j = 1,2,\ldots,2n+1,$$

$$l_{ij}=\begin{cases}\alpha\cdot a_{\frac{i-1}{2}}^{\left(\frac{j-1}{2}\right)}(x_L)-c_{\frac{i-1}{2}}^{\left(\frac{j-1}{2}\right)}(x_L), & \text{if } i,j=2k+1;\ k=0,1,\ldots,n;\\ \alpha\cdot b_{\frac{i}{2}}^{\left(\frac{j-1}{2}\right)}(x_L)-d_{\frac{i}{2}}^{\left(\frac{j-1}{2}\right)}(x_L), & \text{if } i=2k;\ k=1,\ldots,n \text{ and } j=2k+1;\ k=0,1,\ldots,n;\\ \alpha\cdot e_{\frac{i-1}{2}}^{\left(\frac{j}{2}\right)}(x_L)-g_{\frac{i-1}{2}}^{\left(\frac{j}{2}\right)}(x_L), & \text{if } i=2k+1;\ k=0,1,\ldots,n \text{ and } j=2k;\ k=1,2,\ldots,n;\\ \alpha\cdot f_{\frac{i}{2}}^{\left(\frac{j}{2}\right)}(x_L)-h_{\frac{i}{2}}^{\left(\frac{j}{2}\right)}(x_L), & \text{if } i,j=2k;\ k=1,2,\ldots,n,\end{cases} \tag{58}$$

converges to a matrix $\mathbf{B} = (b_{ij});\ i,j = 1,2,\ldots$, and this $\mathbf{B}$ has the feature that to each continuously differentiable function

$$\sigma(t) = k_0 + \sum_{i=1} (k_i \cos i\omega t + l_i \sin i\omega t)$$

another differentiable function corresponds, defined as follows:

$$\begin{aligned} B[\sigma(t)] = {} & \sum_{j=1}^{\infty} (l_j\, b_{1,2j} + k_{j-1}\; b_{1,2j-1}) + \\ & + \sum_{i=1}^{\infty} \left\{ \left(\sum_{j=1}^{\infty} l_j\, b_{2i+1,2j} + k_{j-1}\, b_{2i+1,2j-1}) \right) \cos i\omega t + \right. \\ & \left. + \left(\sum_{j=1}^{\infty} (l_j\, b_{2i,2j} + k_{j-1}\, b_{2i,2j-1}) \right) \sin i\omega t \right\}. \end{aligned} \tag{59}$$

Namely, if all the above conditions are satisfied, then to each unbalanced, but nearly balanced final condition

$$\alpha\cdot\tilde{p}(x_L,t) - \tilde{v}(x_L,t) = \Delta\vartheta\cdot\left(\tilde{\mu}_0 + \sum_{j=1}^{\infty}(\tilde{\mu}_j\cos j\omega t + \tilde{\nu}_j \sin j\omega t)\right) \not\equiv 0\,,$$

continuously differentiable in t ($\Delta\vartheta$ denoting either $\Delta\lambda$ or $\Delta\varepsilon$) the initial deviation Δv in the form (56) is uniquely determined by

$$\begin{pmatrix}\mu_0\\ \nu_1\\ \mu_1\\ \nu_2\\ \mu_2\\ \vdots\end{pmatrix} = \Delta\lambda\cdot\mathbf{B}\cdot\begin{pmatrix}\tilde{\mu}_0\\ \tilde{\nu}_1\\ \tilde{\mu}_1\\ \tilde{\nu}_2\\ \tilde{\mu}_2\\ \vdots\end{pmatrix} + \sigma(\Delta\lambda)\cdot\mathbf{B}\cdot\begin{pmatrix}\tilde{\tilde{\mu}}_0\\ \tilde{\tilde{\nu}}_1\\ \tilde{\tilde{\mu}}_1\\ \tilde{\tilde{\nu}}_2\\ \tilde{\tilde{\mu}}_2\\ \vdots\end{pmatrix}, \tag{60}$$

where the features of **B** safeguard that Δv is continuously differentiable in t for all $\Delta\lambda$, $|\Delta\lambda|$ small enough. (60) implies too, that

$$\lim_{\Delta\vartheta=0}\frac{\Delta_1 p^{**}}{\Delta\vartheta};\ \lim_{\Delta\vartheta=0}\frac{\Delta_1 v^{**}}{\Delta\vartheta};\ \lim_{\Delta\vartheta=0}\frac{\Delta_1 p^{***}}{\Delta\vartheta};\ \lim_{\Delta\vartheta=0}\frac{\Delta_1 v^{***}}{\Delta\vartheta}$$

exist. (Note that the existence of $\lim\limits_{\Delta\lambda=0}\frac{\Delta_2 p^{**}}{\Delta\lambda}$; $\lim\limits_{\Delta\lambda=0}\frac{\Delta_2 v^{**}}{\Delta\lambda}$; etc. follows from the analyticity — in λ and in ε — of the solutions of the corresponding initial value problems). Now it remains only to prove the facts used above with respect to the derivatives $\frac{\partial p}{\partial \mu_i}$; $\frac{\partial p}{\partial \nu_i}$; $\frac{\partial v}{\partial \mu_i}$; $\frac{\partial v}{\partial \nu_i}$ and to the matrix **B**.

For this reason, let us consider the following initial value problems

$$\begin{aligned}
&\frac{\partial p'_{\mu_i}}{\partial x}=\varrho\left(\frac{\partial v'_{\mu_i}}{\partial t}-v'_{\mu_i}\frac{\partial v}{\partial x}-v\frac{\partial v'_{\mu_i}}{\partial x}+\frac{\lambda}{d}(1-d_1p)\,vv'_{\mu_i}-\lambda\frac{d_1}{d}\frac{v^2}{2}p'_{\mu_i}\right);\\
&\varrho\frac{\partial v'_{\mu_i}}{\partial x}=\varepsilon\left(\frac{\partial p'_{\mu_i}}{\partial t}-v'_{\mu_i}\frac{\partial p}{\partial x}-v\frac{\partial p'_{\mu_i}}{\partial x}\right);\\
&p'_{\mu_i}(0,t)\equiv 0;\ v'_{\mu_i}(0,t)\equiv\cos i\omega t,
\end{aligned}\tag{61}$$

and

$$\begin{aligned}
&\frac{\partial p'_{\nu_i}}{\partial x}=\varrho\left(\frac{\partial v'_{\nu_i}}{\partial t}-v'_{\nu_i}\frac{\partial v}{\partial x}-v\frac{\partial v'_{\nu_i}}{\partial x}+\frac{\lambda}{d}(1-d_1p)\,vv'_{\nu_i}-\lambda\frac{d_1}{d}\frac{v^2}{2}p'_{\nu_i}\right);\\
&p'_{\nu_i}(0,t)\equiv 0;\ v'_{\nu_i}(0,t)\equiv\sin i\omega t,
\end{aligned}\tag{62}$$

respectively. It is now obvious, that solutions of (61) and (62) are just the partial derivatives p'_{μ_i}, v'_{μ_i}, and p'_{ν_i}, v'_{ν_i}, resp. and so their existence and their continuous differentiability have been proved (the coefficients in (61) and (62) are the solutions of the corresponding initial value problems related to $p(x,t)$ and $v(x,t)$, resp. and hence they are continuously differentiable). Now the conditions, related to the matrix **B** can only be treated directly for $\varepsilon=\lambda=0$. In this case (1)—(2) is reduced to

$$\frac{\partial p}{\partial x}=\varrho\left(\frac{\partial v}{\partial t}-v\frac{\partial v}{\partial x}\right);\ \frac{\partial v}{\partial x}=0;\ p(0,t)\equiv I(t);\ v(x_L,t)\equiv\alpha p(x_L,t)$$

i.e. (for $\alpha\neq 0$)

$$\begin{aligned}
&v(x,t)\equiv v(t);\ p(x,t)\equiv\varrho\cdot x\cdot v'(t)+I(t),\\
&v(t)\equiv\alpha\cdot\varrho\cdot x_L v'(t)+\alpha\cdot I(t);\ v(t)=ce^{\frac{t}{\alpha\varrho x_L}}+\int_0^t e^{\frac{t-\tau}{\alpha\varrho x_L}}I(\tau)\,d\tau
\end{aligned}$$

where c must be 0, because we consider only periodic solutions of our problem. So, in the case of $\lambda = \varepsilon = 0$ we have

$$v(x, t) = \int_0^t e^{\frac{t-\tau}{\alpha\varrho x_L}} I(\tau) d\tau; \ p(x, t) = I(t) + \frac{x}{\alpha x_L} \int_0^t e^{\frac{t-\tau}{\alpha\varrho x_L}} I(\tau) d\tau,$$

for $\alpha \neq 0$ and

$$v(x, t) \equiv 0 ; \qquad p(x, t) \equiv I(t)$$

for $\alpha = 0$.

(61) is then reduced to

$$\frac{\partial p'_{\mu i}}{\partial x} = \varrho \left(\frac{\partial v'_{\mu i}}{\partial t} - v'_{\mu i} \cdot 0 - v \frac{\partial v'_{\mu i}}{\partial x} \right) ;$$

$$\frac{\partial v'_{\mu i}}{\partial x} = 0 ; \ p'_{\mu i}(0, t) \equiv 0; \ v'_{\mu i}(0, t) \equiv \cos i\omega t ,$$

i.e.

$$v'_{\mu_i}(x, t) \equiv \cos i\omega\, t; \ p'_{\mu_i}(x, t) = -i\omega\varrho x \sin i\omega t. \tag{63}$$

(62) is reduced to

$$\frac{\partial p'_{\nu i}}{\partial x} = \varrho \left(\frac{\partial v'_{\nu i}}{\partial t} - v'_{\nu i} \cdot 0 - v \frac{\partial v'_{\nu i}}{\partial x} \right) ;$$

$$\frac{\partial v'_{\nu i}}{\partial x} = 0; \ p'_{\nu i}(0, t) \equiv 0; \ v'_{\nu i}(0, t) \equiv \sin i\omega t$$

i.e.

$$v'_{\nu i}(x, t) \equiv \sin i\omega t; \ p'_{\nu i}(x, t) \equiv i\omega\varrho x \cos i\omega t . \tag{64}$$

From (63) and (64) it appears that matrices $\mathbf{A}_n$ are blockdiagonal with blocks

$$\begin{pmatrix} -1 & \alpha i\omega\varrho x_L \\ -\alpha i\omega\varrho x_L & -1 \end{pmatrix}$$

i.e. the sequence of the matrices $\mathbf{B}_n = \mathbf{A}_n^{-1}$ converges to a blockdiagonal one with blocks

$$\begin{pmatrix} -\dfrac{1}{1+\alpha^2 i^2 \omega^2 \varrho^2 x_L^2} & -\dfrac{\alpha i\omega\varrho x_L}{1+\alpha^2 i^2 \omega^2 \varrho^2 x_L^2} \\ \dfrac{\alpha i\omega\varrho x_L}{1+\alpha^2 i^2 \omega^2 \varrho^2 x_L^2} & -\dfrac{1}{1+\alpha^2 i^2 \omega^2 \varrho^2 x_L^2} \end{pmatrix} . \tag{65}$$

Now the form of $\mathbf{B}$ guarantees that the expression in (59) is the Fourier-expansion of a differentiable function, because the i-th Fourier coefficients of

$\mathbf{B}[\sigma(t)]$ are linear combinations of the i-th Fourier coefficients of $\sigma(t)$ with factors of $O(1/i)$ and $O(1/i^2)$, resp. Now, if λ or ε is near enough to 0, then the solutions of (61) and of (62) lie near to (63) and to (64), resp., i.e. matrices $\mathbf{A}_n$ are nearly blockdiagonal; denoting the change of $\mathbf{A}_n$ by $\Delta\mathbf{A}_n$, so all the elements of $\Delta\mathbf{A}_n$ are of the order $O(|\lambda| + |\varepsilon|)$. Now if $|\lambda|$ and $|\varepsilon|$ are small enough, then

$$(\mathbf{A}_n + \Delta\mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1} - \mathbf{A}_n^{-1}\cdot\Delta\mathbf{A}_n\cdot\mathbf{A}_n^{-1} + (\mathbf{A}_n^{-1}\Delta\mathbf{A}_n)^2\,\mathbf{A}_n^{-1} - + \ldots \sim \sim \mathbf{A}_n^{-1} - \mathbf{A}_n^{-1}\cdot\Delta\mathbf{A}_n\cdot\mathbf{A}_n^{-1}. \tag{66}$$

Using now (65) and (66), we get the relationship

$$b_{ij} = \theta\left(\frac{1}{ij}\right) \quad \text{for} \quad i \neq j-1,\ j+1 \tag{67}$$

whereas

$$b_{ij} = \theta\left(\frac{1}{i}\right) \quad \text{for} \quad i = j-1,\ j+1, \tag{68}$$

and also this guarantees the differentiability of $\mathbf{B}[\sigma(t)]$. Q.E.D.

THEOREM. The initial-boundary-value problem (1), (2), (4), and (5) with continuously differentiable and periodic initial function $I(t) = p(0, t)$ has one and only one continuously differentiable solution, periodic for fixed x in the domain $0 \leq x \leq x_L$ if $|\lambda|$ and $|\varepsilon| = 1/|c^2|$ are small enough. This periodic solution can be then approximated by the algorithm given in **2.** to the desired accuracy.

4. To characterize the method of approximation on one hand, and the effect of the frequency on the other hand, let us consider the problem for the case

$$I(t) = 100 + 25\cos 10t + 10\cos 40t\,;$$

$$\varrho = 1\,;\quad \lambda = 0.2\,;\quad c = 500\,;\quad x_L = 50\,;\quad d = 0.2\,;$$

$$d_1 = 0.005\,;\quad \alpha = 0\,;$$

truly representing a thrombosis occurring in a normal arteria.

Now, from (19) and (20)

$$V_0^{(0)}(x,t) \equiv 0;\quad P_{0)}^{(0}(x,t) \equiv I(t).$$

And from (21) and (22)

$$V_1^{(0)}(x,t) \equiv (50-x)\,(250\sin 10t + 400\sin 40t);$$

$$P_1^{(0)}(x,t) \equiv \left[\frac{50^2}{2} - \frac{(50-x)^2}{2}\right](2500 \cos 10t + 16.000 \cos 40t) =$$

$$= \frac{x}{2}(100-x)(2500 \cos 10t + 16.000 \cos 40t)$$

and so

$$v_0 \sim \frac{50-x}{500}\left(\frac{1}{2}\sin 10t + \frac{4}{5}\sin 40t\right),$$

$$p_0 \approx 100 + 25 \cos 10t + 10 \cos 40t +$$

$$+ \frac{x}{2} \cdot \frac{100-x}{500}(5 \cos 10t + 32 \cos 40t).$$

Summary

An algorithm is given for the approximation of a solution, periodic in t for all fixed $0 \leq x \leq x_L$ of the initial-final boundary problem

$$p'_x = \varrho\left(v'_t - v \cdot v'_x + f(p; d)\frac{v^2}{2}\right)$$

$$\varrho c^2 v'_x = p'_t - v\, p'_x;$$

$$p(0,t) = p_0 + \sum_{k=1}^{\infty}(p_k \cos k\,\omega\, t + r_k \sin k\,\omega\, t);$$

$$v(x_L, t) = \alpha \cdot p(x_L, t).$$

The theorem is proved, too, that the above problem has one and only one solution in the class of the periodic functions, if $p(0, t)$ is continuously differentiable.

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