ASYMPTOTIC PROPERTIES OF THE ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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Let us define a non-negative measure $d\mu(\Theta)$ on the unit circle $z = e^{i\Theta}$ of the complex plane. Such a measure is provided by a non-decreasing function $\mu(\Theta)$ having the following condition of periodicity:

$$\mu(\Theta_2 + 2\pi) - \mu(\Theta_1 + 2\pi) = \mu(\Theta_2) - \mu(\Theta_1)$$

for any real numbers Θ_1, Θ_2.

Such a measure on the unit circle is called distribution function if $\mu(\Theta)$ takes infinitely many values on an interval of length 2π, that is, support $Br(d\mu)$ of $d\mu$ consists of infinitely many points.

Let such a distribution function be given on the unit circle of the complex plane. For each distribution function there is a uniquely determined sequence of polynomials $\{\Phi_n(d\mu, z)\}$ $(z = e^{i\Theta})$ with the following properties:

1) The grade of the polynomial $\Phi_n(d\mu, z) = \varkappa_n(d\mu)z^n + \ldots$ is exactly n,
2) its leading coefficient $\varkappa_n(d\mu)$ is positive.
3) For any pair n, m of non-negative integers the conditions of complex orthogonality holds:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{\Phi_m(d\mu,z)}\, \Phi_n(d\mu, z)\, d\mu(\Theta) = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases}. \tag{1}$$

(This is a Lebesgue—Stieltjes integral belonging to the distribution $d\mu$.)

Let $d\alpha$ be a distribution on the real line, the support $Br(d\alpha) \subset [-1, 1]$ of which consists of infinitely many points. For each distribution function of this type there is a uniquely determined sequence of polynomials with the following properties:

a) The grade of the polynomial $p_n(d\alpha, x) = \gamma_n(d\alpha)x^n + \ldots$ is exactly n,
b) its leading coefficient $\gamma_n(d\alpha)$ is positive.

(1) For proof see [1]

c) For any pair n, m of non-negative integers the following condition of orthogonality holds:

$$\int_{-\infty}^{+\infty} p_n(x)\, p_m(x)\, d\alpha(x) = \begin{cases} 1, & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}. \tag{2}$$

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Let $d\alpha$ be a distribution function on the real line with $Br(d\alpha) \subset [-1, 1]$. To any distribution function of this type we can define the function $\mu_1(\Theta)$ in the following way:

$$\mu_1(\Theta) = \begin{cases} \alpha(1) - \alpha(\cos\Theta), & \text{if } 0 \leq \Theta \leq \pi, \\ \alpha(\cos\Theta) - \alpha(1), & \text{if } -\pi \leq \Theta \leq 0 \end{cases} \tag{1.1}$$

so that it provides a distribution function on the unit circle as described above. Obviously, $d\mu_1(\Theta) = |\, d\alpha(\cos\Theta)\, |$. If $\alpha(x)$ is absolutely continuous and $\alpha'(x) = w(x)$ then $\mu_1(\Theta)$ is absolutely continuous and the weight function belonging to $d\mu_1(\Theta)$ is $f(\Theta) = w(\cos\Theta)\,|\sin\Theta\,|$.

The connection between the orthogonal polynomials on the unit circle and those on $[-1, 1]$ is well known: (1.1) is valid for $\mu_1(\Theta)$ and $\alpha(x)$.

Let $x = \cos\Theta$, $z = e^{i\theta}$ then

$$p_n(d\alpha; x) = \frac{1}{\sqrt{2\pi}}\left[1 + \frac{\Phi_{2n}(d\mu_1; 0)}{\varkappa_{2n}(d\mu_1)}\right]^{-1/2} \left[z^{-n}\,\Phi_{2n}(d\mu_1; z) + z^n\,\Phi_{2n}(d\mu_1; z^{-1})\right]. \tag{1.2}$$

Let $d\beta(x) = (1 - x^2)\,d\alpha(x)$ then

$$p_n(d\beta; x) = \sqrt{\frac{2}{\pi}}\left[1 - \frac{\Phi_{2n+2}(d\mu_1; 0)}{\varkappa_{2n+2}(d\mu_1)}\right]^{-1/2} \times \\ \times \frac{z^{-n-1}\,\Phi_{2n+2}(d\mu_1; z) - z^{n+1}\,\Phi_{2n+2}(d\mu_1; z^{-1})}{z - z^{-1}}. \tag{1.3}$$

The purpose of this work is to investigate the asymptotic properties of the orthogonal system on the unit circle corresponding to the Jacobi-polynomials.

Szegő mentions in [2] that the orthogonal polynomials on the unit circle corresponding to the Jacobi polynomials can be expressed, using suitable constants, as linear combinations of certain Jacobi polynomials but he does not indicate these constants.

So let $d\alpha(x) = (1 - x)^a (1 + x)^b\, dx$, where $a > -1$, $b > -1$.

(2), (1.2), (1.3) For proof see [1]

Multiplying by suitable factors the orthogonal polynomials $p_n(d\alpha, x)$ on $[-1, 1]$ belonging to the distribution $d\alpha(x)$ give the Jacobi polynomials $P_n^{(a,b)}(x)$. Further, let $d\beta(x) = (1-x^2)d\alpha(x) = (1-x)^{a+1}(1+x)^{b+1}\,dx$. As usually, let us denote the orthogonal polynomials on $[-1, 1]$ belonging to the distribution $d\beta(x)$ by $p_n(d\beta; x)$.

In the following we shall work with orthogonal polynomials on the unit circle belonging to $d\mu_1$ that has been defined in (1.1). If no confusion is risked we shall use the simpler notations $\Phi_n(d\mu_1; z) \equiv \Phi_n(z)$, $\varkappa_n(d\mu_1) \equiv \varkappa_n$.

Since

$$x = \cos\Theta = \frac{e^{i\Theta}+e^{-i\Theta}}{2} = \frac{z+z^{-1}}{2},$$

we have

$$x^n = \frac{(z+z^{-1})^n}{2^n}.$$

Let us apply (1.3) to $n-1$, from this and from (1.2) we get by comparing the coefficients:

$$\frac{\gamma_n(d\alpha)}{2^n} = \frac{1}{\sqrt{2\pi}}\left[1+\frac{\Phi_{2n}(0)}{\varkappa_{2n}}\right]^{-1/2}[\varkappa_{2n}+\Phi_{2n}(0)],$$

$$\frac{\gamma_{n-1}(d\beta)}{2^{n-1}} = \sqrt{\frac{2}{\pi}}\left[1-\frac{\Phi_{2n}(0)}{\varkappa_{2n}}\right]^{-1/2}[\varkappa_{2n}-\Phi_{2n}(0)].$$

Solving the system of equations we have

$$\varkappa_{2n}^2 = \frac{\pi}{4^n}[\gamma_n^2(d\alpha)+\gamma_{n-1}^2(d\beta)],$$

and since $\varkappa_{2n}$ is positive,

$$\varkappa_{2n} = \frac{\sqrt{\pi}}{2^n}\sqrt{\gamma_n^2(d\alpha)+\gamma_{n-1}^2(d\beta)}; \tag{1.4}$$

$$\Phi_{2n}(0) = \frac{\sqrt{\pi}}{2^n}\,\frac{\gamma_n^2(d\alpha)-\gamma_{n-1}^2(d\beta)}{\sqrt{\gamma_n^2(d\alpha)+\gamma_{n-1}^2(d\beta)}}. \tag{1.5}$$

Since the equality

$$\sum_{\nu=0}^{n}|\Phi_\nu(0)|^2 = \varkappa_n^2 \tag{3}$$

holds, we also have

$$\varkappa_{2n-1}^2 = \varkappa_{2n+2}^2 - |\Phi_{2n+2}(0)|^2 \tag{1.6}$$

and

$$|\Phi_{2n+1}(0)|^2 = \varkappa_{2n+1}^2 - \varkappa_{2n}^2. \tag{1.7}$$

(3) For proof see [1]

Since the coefficients of $\Phi_n(d\mu_1; z)$ are real we get (4)

$$|\Phi_{2n+2}(0)|^2 = \Phi_{2n+2}^2(0) .$$

Using equalities (1.4) to (1.7) we obtain that

$$\varkappa_{2n+1}^2 = \frac{\pi}{4^n} \frac{\gamma_{n+1}^2(d\alpha)\,\gamma_n^2(d\beta)}{\gamma_{n+1}^2(d\alpha)+\gamma_n^2(d\beta)} ,$$

and since $\varkappa_{2n+1}$ is positive,

$$\varkappa_{2n+1} = \frac{\sqrt{\pi}}{2^n} \frac{\gamma_{n+1}(d\alpha)\,\gamma_n(d\beta)}{\sqrt{\gamma_{n+1}^2(d\alpha)+\gamma_n^2(d\beta)}} . \tag{1.8}$$

$$|\Phi_{2n+1}(0)|^2 = \frac{\pi}{4^n}\left[\frac{\gamma_{n+1}^2(d\alpha)\,\gamma_n^2(d\beta)}{\gamma_{n+1}^2(d\alpha)+\gamma_n^2(d\beta)} - \gamma_n^2(d\alpha) - \gamma_{n-1}^2(d\beta)\right]. \tag{1.9}$$

It should be mentioned that $\Phi_{2n+1}(0)$ can be determined up to its sign from (1.9) since $\Phi_{2n+1}(0)$ is a real number.

The values $\gamma_n(d\alpha)$ are known (see SZEGŐ [2])

$$\gamma_n(d\alpha) = \frac{1}{2^n}\left\{\frac{2n+a+b+1}{2^{a+b+1}} \frac{\Gamma(n+1)\,\Gamma(n+a+b+1)}{\Gamma(n+a+1)\,\Gamma(n+b+1)}\right\}^{1/2} \frac{\Gamma(a+b+2n+1)}{\Gamma(n+1)\,\Gamma(a+b+n+1)} \tag{1.10}$$

$$n = 1, 2, \ldots$$

Taking into consideration the equalities (1.4), (1.5), (1.8), (1.9) and (1.10) we obtain

$$\varkappa_{2n}^2 = \frac{\pi}{(4^n)^2} \frac{1}{2^{a+b+1}} \frac{\Gamma^2(a+b+2n+2)}{\Gamma(n+1)\,\Gamma(a+b+n+2)\,\Gamma(n+a+1)\,\Gamma(n+b+1)} , \tag{1.11}$$

$$\Phi_{2n}^2(0) = \frac{\pi}{(4^n)^2} \frac{(a+b+1)^2}{2^{a+b+1}} \frac{\Gamma^2(a+b+2n+1)}{\Gamma(n+1)\,\Gamma(a+b+n+2)\,\Gamma(n+a+1)\,\Gamma(n+b+1)} , \tag{1.12}$$

$$\varkappa_{2n+1}^2 = \frac{\pi}{(4^n)^2} \frac{1}{2^{a+b+3}} \frac{\Gamma^2(a+b+2n+3)}{\Gamma(a+b+n+2)\,\Gamma(n+1)\,\Gamma(n+a+2)\,\Gamma(n+b+2)} , \tag{1.13}$$

$$\Phi_{2n+1}^2(0) = \frac{\pi}{(4^n)^2} \frac{1}{2^{a+b+1}} \frac{\Gamma^2(a+b+2n+2)}{\Gamma(a+b+n+2)\,\Gamma(n+1)\,\Gamma(n+a+1)\,\Gamma(n+b+1)} \times$$

$$\times \frac{(a-b)^2}{4(n+a+1)\,(n+b+1)} . \tag{1.14}$$

(4) For proof see [1]

Using

$$\Gamma(z) = e^{-z} z^{z-1/2} \sqrt{2\pi} \left(1 + O\left(\frac{1}{n}\right)\right) \quad (|\arg z| < \pi)$$

we obtain from (1.11)—(1.14):

$$\varkappa_{2n}^2 = 2^{a+b+1}\left[1 + O\left(\frac{1}{n}\right)\right], \tag{1.15}$$

$$\Phi_{2n}^2(0) = \frac{2^{a+b-1}(a+b+1)^2}{n^2}\left[1 + O\left(\frac{1}{n}\right)\right], \tag{1.16}$$

$$\Phi_{2n+1}^2(0) = \frac{2^{a+b-1}(a-b)^2}{n^2}\left[1 + O\left(\frac{1}{n}\right)\right], \tag{1.17}$$

$$\varkappa_{2n+1}^2 = 2^{a+b+1}\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{1.18}$$

We should like to estimate $\Phi_n(z)$ using Jacobi polynomials. For this purpose let us apply equalities (1.2) and (1.3) whence

$$\begin{aligned}\Phi_{2n}(d\mu_1; z) = \frac{z^n}{2}\Bigg[\sqrt{2\pi}\, p_n(d\alpha; x)\left(1 + \frac{\Phi_{2n}(0)}{\varkappa_{2n}}\right)^{1/2} + \\ + \sqrt{\frac{\pi}{2}}\, p_{n-1}(d\beta; x)\left(1 - \frac{\Phi_{2n}(0)}{\varkappa_{2n}}\right)^{1/2}(z - z^{-1})\Bigg],\end{aligned} \tag{1.19}$$

$$\begin{aligned}\Phi_{2n}(d\mu_1; z^{-1}) = \frac{z^{-n}}{2}\Bigg[\sqrt{2\pi}\, p_n(d\alpha; x)\left(1 + \frac{\Phi_{2n}(0)}{\varkappa_{2n}}\right)^{1/2} - \\ - \sqrt{\frac{\pi}{2}}\, p_{n-1}(d\beta, x)\left(1 - \frac{\Phi_{2n}(0)}{\varkappa_{2n}}\right)^{1/2}(z - z^{-1})\Bigg].\end{aligned} \tag{1.20}$$

The $\Phi_n(d\mu_1; z)$ of odd indices can be expressed by those of even ones. To do so, we use the following identity:

$$\varkappa_n \Phi_n(z) - \Phi_n(0)\, \Phi_n^*(z) = \varkappa_{n-1}\, z\, \Phi_{n-1}(z)\,, \tag{1.21}$$

where $\Phi_n^*(z) = z^n \Phi_n(z^{-1})$.

Thus

$$\varkappa_{2n} \Phi_{2n}(z) - \Phi_{2n}(0)\, z^{2n}\, \Phi_{2n}(z^{-1}) = \varkappa_{2n-1}\, z \Phi_{2n-1}(z)$$

and

$$\Phi_{2n-1}(z) = \frac{\varkappa_{2n}}{\varkappa_{2n-1}} z^{-1} \Phi_{2n}(z) - \frac{\Phi_{2n}(0)}{\varkappa_{2n-1}} z^{2n-1} \Phi_{2n}(z^{-1})\,. \tag{1.22}$$

(1.21) For proof see [1]

The asymptotic behaviour of the Jacobi polynomials $P_n^{(a,b)}(x)$ is known (see SZEGŐ [2]).

$$\left(\sin\frac{\Theta}{2}\right)^a\left(\cos\frac{\Theta}{2}\right)^b P_n^{(a,b)}(\cos\Theta)=\left(n+\frac{a+b+1}{2}\right)^{-a}\times$$
$$\times\frac{\Gamma(n+a+1)}{n!}\left(\frac{\Theta}{\sin\Theta}\right)^{1/2}J_0\left[\left(n+\frac{a+b+1}{2}\right)\Theta\right]+ \tag{1.23}$$
$$+\begin{cases}\Theta^{1/2}\,O(n^{-3/2}) & \text{if } cn^{-1}\leq\Theta\leq\pi-\varepsilon\\ \Theta^{a+2}\,O(n^a) & \text{if } 0<\Theta\leq cn^{-1}\end{cases},$$

where $J_0(z)$ denotes the Bessel function of the first kind with index 0, c and ε are fixed positive numbers. On the interval $[\pi-\varepsilon,\pi)$ a symmetric formula can be obtained by interchanging the roles of a and b.

For

$$p_n(d\alpha,x)=\left\{\frac{2n+a+b+1}{2^{a+b+1}}\;\frac{\Gamma(n+1)\,\Gamma(n+a+b+1)}{\Gamma(n+a+1)\,\Gamma(n+b+1)}\right\}^{1/2}P_n^{(a,b)}(x), \tag{1.24}$$

we get

$$p_n(d\alpha,x)=\frac{n^{1/2}}{2^{\frac{a+b}{2}}}\left[1+O\left(\frac{1}{n}\right)\right]P_n^{(a,b)}(x). \tag{1.25}$$

Using (1.23) we obtain that

$$p_n(d\alpha,\cos\Theta)=\frac{n^{1/2}}{2^{\frac{a+b+1}{2}}}\,\Theta^{1/2}\left(\sin\frac{\Theta}{2}\right)^{-a-1/2}\left(\cos\frac{\Theta}{2}\right)^{-b-1/2}\times$$
$$\times J_0\left(\left[n+\frac{a+b+1}{2}\right]\Theta\right)\left[1+O\left(\frac{1}{n}\right)\right]+\left(\sin\frac{\Theta}{2}\right)^{-a}\times \tag{1.26}$$
$$\times\left(\cos\frac{\Theta}{2}\right)^{-b}\begin{cases}\Theta^{1/2}\,O(n^{-1}) & \text{if } cn^{-1}\leq\Theta\leq\pi-\varepsilon\\ \Theta^{a+2}O(n^{a+1/2}) & \text{if } 0<\Theta\leq cn^{-1}\end{cases},$$

$$p_n(d\beta,\cos\Theta)=\frac{n^{1/2}}{2^{\frac{a+b+3}{2}}}\,\Theta^{1/2}\left(\sin\frac{\Theta}{2}\right)^{-a-3/2}\left(\cos\frac{\Theta}{2}\right)^{-b-3/2}\times$$
$$\times J_0\left(\left[n+\frac{a+b+3}{2}\right]\Theta\right)\left[1+O\left(\frac{1}{n}\right)\right]+\left(\sin\frac{\Theta}{2}\right)^{-a-1}\times \tag{1.27}$$
$$\times\left(\cos\frac{\Theta}{2}\right)^{-b-1}\begin{cases}\Theta^{1/2}\,O(n^{-1}) & \text{if } cn^{-1}\leq\Theta\leq\pi-\varepsilon\\ \Theta^{a+3}\,O(n^{a+3/2}) & \text{if } 0<\Theta\leq cn^{-1}\end{cases}.$$

In fact, we have proved the following theorem by using results (1.15)—(1.18) and putting the estimates (1.26), (1.27) into equalities (1.19), (1.20), (1.22):

Theorem

Using the notations introduced above, let $z = e^{i\theta}$, $x = \cos\Theta$

$$\Phi_{2n}(d\mu_1; e^{i\Theta}) = \frac{e^{in\Theta}}{2}\sqrt{2\pi}\left[\frac{n^{1/2}}{2^{\frac{a+b+1}{2}}}\Theta^{1/2}\left(\sin\frac{\Theta}{2}\right)^{-a-1/2}\left(\cos\frac{\Theta}{2}\right)^{-b-1/2}\times\right.$$

$$\times J_0\left(\left[n+\frac{a+b+1}{2}\right]\Theta\right)(1+i)+\Theta^{1/2}\left(\sin\frac{\Theta}{2}\right)^{-a-1/2}\times$$

$$\times\left(\cos\frac{\Theta}{2}\right)^{-b-1/2}J_0\left(\left[n+\frac{a+b+1}{2}\right]\Theta\right)\left(O(n^{-1/2})+\right.$$

$$\left.+iO(n^{-1/2})\right)+\left(\sin\frac{\Theta}{2}\right)^{-a}\left(\cos\frac{\Theta}{2}\right)^{-b}\times$$

$$\left.\times\begin{cases}\Theta^{1/2}[O(n^{-1})+iO(n^{-1})] & \text{if } cn^{-1}\leq\Theta\leq\pi-\varepsilon\\ \Theta^{a+2}O(n^{a+1/2})(1+i\Theta O(n)) & \text{if } 0<\Theta\leq cn^{-1}\end{cases}\right],$$

$$\Phi_{2n-1}(d\mu_1; e^{i\Theta}) = \frac{e^{i(n-1)\Theta}}{2}\sqrt{2\pi}\left[\frac{n^{1/2}}{2^{\frac{a+b+1}{2}}}\Theta^{1/2}\left(\sin\frac{\Theta}{2}\right)^{-a-1/2}\times\right.$$

$$\times\left(\cos\frac{\Theta}{2}\right)^{-b-1/2}J_0\left(\left[n+\frac{a+b+1}{2}\right]\Theta\right)(1+i)+\Theta^{1/2}\times$$

$$\times\left(\sin\frac{\Theta}{2}\right)^{-a-1/2}\left(\cos\frac{\Theta}{2}\right)^{-b-1/2}J_0\left(\left[n+\frac{a+b+1}{2}\right]\Theta\right)\left(O(n^{-1/2})+\right.$$

$$\left.+iO(n^{-1/2})\right)+\left(\sin\frac{\Theta}{2}\right)^{-a}\left(\cos\frac{\Theta}{2}\right)^{-b}\times$$

$$\left.\times\begin{cases}\Theta^{1/2}[O(n^{-1})+iO(n^{-1})] & \text{if } cn^{-1}\leq\Theta\leq\pi-\varepsilon\\ \Theta^{a+2}O(n^{a+1/2})[O(1)+i\Theta O(n)] & \text{if } 0<\Theta\leq cn^{-1}\end{cases}\right].$$

Let us also mention that interchanging the roles of a and b we obtain a symmetric formula on the interval $[\pi - \varepsilon, \pi)$.

Summary

The purpose of this work is to examine the asymptotic properties of the orthogonal system on the unit circle corresponding to the Jacobi-polynomials; we give an asymptotic expansion in closed form for the above mentioned system of orthonormed polynomials.

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