

# ON CONTROLLABLY PERIODIC PERTURBATIONS OF LIÉNARD'S EQUATION

By

M. FARKAS and R. IBRAHIM ABDEL KARIM

Department of Mechanical Engineering Mathematics, Technical University, Budapest

(Received September 3, 1971)

1. Liénard's equation plays an important role in the theory of non-linear oscillations since under certain conditions it has a unique periodic solution. In this paper, Liénard's equation under small periodic perturbations will be considered. A criterion will be given, which ensures that to any small-enough amplitude of the perturbation its period can be chosen (in a certain sense) uniquely in such a way that the perturbed equation has a unique periodic solution of the same period.

Consider Liénard's differential equation of the form

$$\frac{d^2 u}{dt^2} + \varphi(u) \frac{du}{dt} + \psi(u) = 0, \quad (1)$$

where  $\varphi \in C^1$  is an even function such that the odd function

$$\Phi(u) = \int_0^u \varphi(s) ds \quad (2)$$

has a single zero point  $u^* > 0$  with  $\Phi(u) < 0$  for  $0 < u < u^*$ , and  $\Phi(u) > 0$  and monotonously increasing for  $u > u^*$ . Further  $\psi \in C^1$  is an odd function such that  $\psi(u) > 0$  for  $u > 0$ . Moreover we assume that  $\int_0^\infty \varphi(u) du = \int_0^\infty \psi(u) du = \infty$ . Under these conditions LEVINSON and SMITH [1] proved that (1) has a unique non-constant periodic solution  $u_0(t)$ . The least positive period of the solution  $u_0(t)$  will be denoted by  $\tau_0$ .

2. Setting

$$x_1 = u, \quad x_2 = \dot{u} + \Phi(u), \quad (3)$$

where dot denotes differentiation with respect to  $t$ , (1) takes the vector form

$$\dot{x} = f(x) \quad (4)$$

with

$$\begin{aligned} x &= \text{col}(x_1, x_2) = \text{col}(u, \dot{u} + \Phi(u)), \\ f(x) &= \text{col}(x_2 - \Phi(x_1), -\psi(x_1)). \end{aligned} \quad (5)$$

System (4) has the periodic solution of period  $\tau_0$ :

$$\begin{aligned} p(t) &= \text{col}(u_0(t), \dot{u}_0(t) + \Phi(u_0(t))) = \\ &= \text{col}(u_0(t), - \int_0^t \psi(u_0(s)) ds + c) \end{aligned} \quad (6)$$

with 
$$c = \dot{u}_0(0) + \Phi(u_0(0)).$$

The first variational system of (4) corresponding to the periodic solution  $p(t)$

$$\dot{y} = f'_x(p(t))y \quad \text{with} \quad f'_x(p(t)) = \begin{bmatrix} -q(u_0(t)) & 1 \\ -\psi'(u_0(t)) & 0 \end{bmatrix}, \quad (7)$$

i.e.

$$\begin{aligned} \dot{y}_1 &= -q(u_0(t))y_1 + y_2 \\ \dot{y}_2 &= -\psi'(u_0(t))y_1 \end{aligned} \quad (8)$$

has the periodic solution  $\dot{p}(t)$  of period  $\tau_0$  [2]:

$$\dot{p}(t) = \text{col}(\dot{u}_0(t), -\psi(u_0(t))). \quad (9)$$

Without loss of generality we assume that for  $t = 0$ ,  $u_0(0) = 0$  and  $\dot{u}_0(0) = a > 0$ . Thus  $\dot{p}(0) = \text{col}(a, 0)$ .

Eliminating  $y_2$  from system (8), we get

$$\ddot{y}_1 + q(u_0(t))\dot{y}_1 + [q'(u_0(t))\dot{u}_0(t) + \psi'(u_0(t))]y_1 = 0. \quad (10)$$

This equation has already the solution  $\dot{u}_0(t)$  [see (9)]. Using the product theorem, we obtain the second linearly independent solution of (10):

$$\hat{y}_1(t) = \dot{u}_0(t) \int_0^t (\dot{u}_0(s))^{-2} I(s) ds \quad (11)$$

with

$$I(s) = \exp \left\{ - \int_0^s q(u_0(\tau)) d\tau \right\}. \quad (12)$$

Though (11) is valid only in a neighbourhood  $T$  of  $t = 0$  containing no zero point of  $\dot{u}_0(t)$ ,  $\hat{y}_1(t)$  can obviously be continued over the whole interval  $(-\infty, \infty)$ . Referring to the second equation of (8), we obtain the corresponding solution in the form

$$\hat{y}_2(t) = -\psi(u_0(t)) \int_0^t (\dot{u}_0(s))^{-2} I(s) ds + (\dot{u}_0(t))^{-1} I(t), t \in T. \quad (13)$$

Let the two linearly independent vector solutions of (8) be chosen in the following way:

$$y^1(t) = \frac{1}{a} \dot{p}(t) = \frac{1}{a} \operatorname{col}(\dot{u}_0(t), -\psi(u_0(t)))$$

and

$$y^2(t) = a \cdot \operatorname{col}(\hat{y}_1(t), \hat{y}_2(t)) = a \cdot \operatorname{col}\left(\dot{u}_0(t) \int_0^t (\dot{u}_0(s))^{-2} I(s) ds, -\psi(u_0(t)) \int_0^t (\dot{u}_0(s))^{-2} I(s) ds + (\dot{u}_0(t))^{-1} I(t)\right).$$

Then  $Y(t) = (y^1(t), y^2(t))$ ,  $t \in T$ , is a fundamental matrix solution of (8) such that

$$Y(0) = (y^1(0), y^2(0)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{14}$$

and

$$W(t) = \det Y(t) = \exp\left(-\int_0^t \varphi(u_0(s)) ds\right), \quad t \in T. \tag{15}$$

The right hand side of (15) is a solution of the first order differential equation

$$W' = \operatorname{Tr} f'_x(p(t)) W,$$

i.e.

$$W' = -\varphi(u_0(t)) W$$

for every  $t$ , thus by LIOUVILLE's theorem (see [2]p.28)(15), holds for  $t \in (-\infty, \infty)$ . In particular

$$W(\tau_0) = \det Y(\tau_0) = \exp\left\{-\int_0^{\tau_0} \varphi(u_0(s)) ds\right\}. \tag{16}$$

**3.** Now, consider the controllably periodically perturbed Liénard's equation

$$\ddot{u} + \varphi(u)u + \psi(u) = \mu \gamma\left(\frac{t}{\tau}, u, \dot{u}, \mu, \tau\right), \tag{17}$$

where  $\varphi$  and  $\psi$  are as in (1),  $\mu$  is a "small parameter" and the function  $\gamma \in C^1$  is periodic in  $t$  with period  $\tau$ . It is assumed that the period  $\tau$  of the perturbation can be chosen appropriately.

Using the notations and assumptions introduced above, there holds the following

**Theorem.** *If the mean value of the periodic function  $\varphi(u_0(t))$  (over a period) is different from zero, then to all sufficiently small  $|\mu|, |\vartheta|$  there belongs one and only one period  $\tau(\mu, \vartheta)$  such that the differential equation*

$$\ddot{u} + \varphi(u)\dot{u} + \psi(u) = \mu\gamma \left( \frac{t}{\tau(\mu, \vartheta)}, u, \dot{u}, \mu, \tau(\mu, \vartheta) \right) \quad (18)$$

*has one and only one periodic solution  $u_p(t; \mu, \vartheta)$  with period  $\tau(\mu, \vartheta)$  for which  $u_p(\vartheta; \mu, \vartheta) = 0$  holds; the functions  $\tau(\mu, \vartheta)$  and  $u_p(t; \mu, \vartheta)$  are in the  $C^1$  class in a neighbourhood of  $\mu = 0, \vartheta = 0$ ;  $\tau(0, 0) = \tau_0$  and  $u_p(t; 0, 0) = u_0(t)$ .*

*Proof.* By transformation (3), Eq. (17) can be brought to the equivalent normal form:

$$\dot{x} = f(x) + \mu g \left( \frac{t}{\tau}, x, \mu, \tau \right) \quad (19)$$

where  $x = \text{col}(x_1, x_2)$ ,  $f(x) = \text{col}(x_2 - \Phi(x_1), -\psi(x_1))$  and  $g(x) = \text{col}(0, \gamma(t/\tau, x_1, x_2 - \Phi(x_1), \mu, \tau))$ . Thus Theorem 1' of [3] can be applied to system (19) provided that 1 is a simple characteristic multiplier of the first variational system (8) of the unperturbed system (4). The product of the characteristic multipliers is equal to (16). Since one of the characteristic multipliers is 1, the other one is equal to  $W(\tau_0)$ . From (16) it can be seen that, if

$$\int_0^{\tau_0} \varphi(u_0(s)) ds = 0,$$

then  $W(\tau_0) = 1$  and this proves the theorem.

It is to be noted that, if

$$\int_0^{\tau_0} \varphi(u_0(s)) ds > 0, \quad (20)$$

then Theorem 3' of [3] can be applied to test the asymptotic stability of the perturbed periodic solution  $u_p(t; \mu, \vartheta)$ .

### Summary

Liénard's equation is considered under small periodic perturbations. A criterion is given which ensures that to any small enough amplitude of the perturbation its period can be chosen (in a certain sense) uniquely in such a way that the perturbed equation has a unique periodic solution of the same period.

### References

1. N. LEVINSON & O. K. SMITH: A general equation for relaxation oscillations, *Duke Math. Journal*, 9, (1942), 382—403.
2. E. CODDINGTON & N. LEVINSON: *Theory of ordinary differential equations*. Mc-Graw-Hill, New-York, Toronto, London 1955.
3. M. FARKAS: Controllably periodic perturbations of autonomous systems. *Acta Math. Acad. Sci. Hung.* XXII. (1971) in print.

Prof. Dr. Miklós FARKAS: Budapest XI., Stoczek u. 2—4. Hungary

Dr. Rahmi Ibrahim ABDEL KARIM: Mathematical Department, Faculty  
of Sciences, Cairo University, Egypt.