LUMPED EQUIVALENT NETWORKS OF TRANSMISSION LINES

By

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Introduction

Lumped equivalent networks of transmission lines may be needed for simulation on analogue computer and in other tasks. Of course, an equivalent network can be used only in a finite frequency band at a prescribed accuracy. The problem of the low-frequency equivalent networks will be treated in detail here, but the presented procedure can be adopted to other frequency bands, too.

The transmission line is characterized as a two-port by its impedance parameters. The impedance parameters of a uniform transmission line of length l, with characteristic impedance Z_0 and propagation coefficient p can be expressed as

$$\begin{aligned} z_{11} &= z_{22} = Z_0 \operatorname{cth} pl \\ z_{12} &= z_{21} = Z_0 \operatorname{csch} pl, \end{aligned} \tag{1}$$

where

$$Z_0 = \sqrt{\frac{R'+sL'}{G'+sC'}} \quad \text{and} \quad p = \sqrt{(R'+sL')(G'+sC')}\,.$$

Here R', L', G' and C' represent the resistance, the inductance, the leakage conductance and the capacitance per unit length, respectively, and s is the complex frequency variable. Denote:

$$\begin{aligned} &R = R'l \quad L = L'l \quad G = G'l \quad C = C'l \quad \text{and} \\ &g = pl = \sqrt{(R+sL)(G+sC)}. \end{aligned} \tag{2}$$

With these notations:

$$\begin{aligned} z_{11} &= z_{22} = Z_0 \operatorname{cth} g \\ z_{12} &= z_{21} = Z_0 \operatorname{csch} g. \end{aligned} \tag{3}$$

The equivalent network will be determined on the basis of these relationships.

The method of determining the equivalent network

First a wholly equivalent two-port is sought for, where the branch impedances are positive real, but naturally, not rational functions. These impedance functions will be approximated by appropriate positive real rational functions, which can be easily realized by lumped networks. Positive real impedance functions are wanted in the equivalent network, because a not positive real function can be approximated along a given part of the imaginary axis by a positive real rational function only to a defined accuracy.

An equivalent network in form of a symmetrical lattice satisfies this condition. Symmetric passive two-ports, i.e. two-ports with symmetric positive real impedance matrix are known to have an equivalent network in form of symmetrical lattice in which the impedance functions are positive real.

As some disadvantages of the lattice network are known, the question may arise, whether such an equivalent network in form of grounded two-port can be found, which satisfies at the same time the above-mentioned condition. This seems to be a rather hard, maybe unsolvable problem. It does not mean of course that a transmission line cannot be substituted in a given frequency band by a lumped, grounded two-port, but the accuracy can only be increased presumably by way of using networks with ever more complicated geometries. In a lattice network the fundamental structure of the two-port does not change by increasing the accuracy, only the number of the elements in the single branches increases, as it will be shown. In this paper only the equivalent networks in form of a symmetrical lattice will be treated.

After this equivalent network has been found, the next problem is to approximate the involved positive real impedance functions by appropriate positive real rational functions. By reason of (3) the impedance functions to be approximated are expected to assume the form:

$$Z(s) = Z_0(s) f[g(s)]. \tag{4}$$

The functions $f[g(s)]$ will not be approximated directly on the basis of the variable s, but g, to simplify the calculation. For lossless transmission lines ($R = 0$ and $G = 0$) $g = s\sqrt{LC}$, i.e. the approximations made on the basis of s and g lead to the same result. It is known that the transmission lines practically used can be considered lossless except at very low (e.g. industrial) frequencies.

As for lossless transmission lines $Z_0 = \sqrt{L/C}$ and $g = s\sqrt{LC}$, $f(g)$ must be a lossless immittance function, irregarded its dimension. Hence, $f(g)$ is an odd function, so also the approximating rational function must be an odd one in g. Thus, the function $f^*(g)$ approximating $f(g)$ can be written as a sum of partial fractions in the following form:

$$f^*(g) = \sum_{i=1}^{n-1} \frac{a_i g}{g^2 + b_i} + a_n g, \tag{5}$$

the approximating impedance function being

$$Z^*(s) = Z_0(s)\, f^*[g(s)] = \sum_{i=1}^{n-1} \frac{a_i(R+sL)}{(R+sL)\,(G+sC)+b_i} + a_n(R+sL). \qquad (6)$$

If the function is required to be realizable independently of the given R, L, G and C values, and if the lossless transmission line is considered as a limiting case, it must be prescribed that a_i and b_i are nonnegative real. The function $Z(s)$ can be well approximated in the environment of one of its poles only, when $Z^*(s)$ contains the chief part of the Laurent series belonging to this pole,

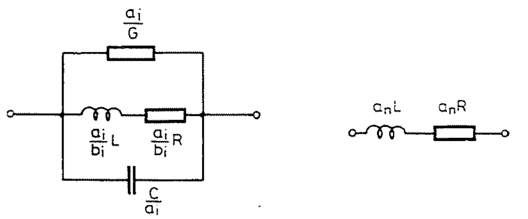

Fig. 1

As $f(g)$ is an odd positive real function, it has poles only on the imaginary axis and at these poles its residue is positive, so in the sum (6) both a_i and b_i are positive for a term corresponding to a conjugated pair of poles. Now the advantage to have positive real impedance functions in the first equivalent network becomes clear.

The realization of the terms of the sum (6) is shown in Fig. 1. So in the case of lossless transmission line the first Foster network of the approximating impedance function $Z^*(s)$ is obtained.

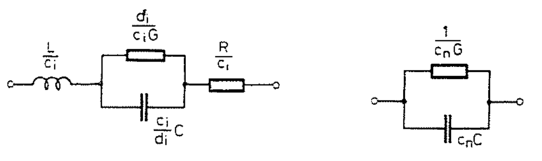

Fig. 2

Of course, also the admittance functions of the branches of the equivalent network can be approximated, to obtain a different network. A reasoning similar to the previous one leads to the conclusion that the approximating admittance function takes the following form:

$$Y^*(s) = \sum_{i=1}^{n-1} \frac{c_i(G+sC)}{(R+sL)\,(G+sC)+d_i} + c_n(G+sC). \qquad (7)$$

The realization of the terms of the previous sum is shown in Fig. 2. This network is similar to the second Foster network.

Equivalent network in form of symmetrical lattice

The following impedances figure in the equivalent network in form of symmetrical lattice (see Fig. 3):

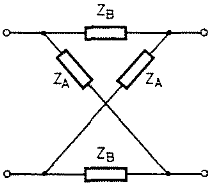

Fig. 3

$$
\begin{aligned}
Z_A &= z_{11} + z_{12} = Z_0(\mathrm{cthg} + \mathrm{cschg}) = Z_0 \,\mathrm{cth}\frac{g}{2} \\
Z_B &= z_{11} - z_{12} = Z_0\,(\mathrm{cth}g - \mathrm{csch}g) = Z_0\,\mathrm{th}\frac{g}{2}\,.
\end{aligned}
\tag{8}
$$

The admittances of the lattice are expressed as:

$$
\begin{aligned}
Y_A &= \frac{1}{Z_0}\,\mathrm{th}\,\frac{g}{2} \\
Y_B &= \frac{1}{Z_0}\,\mathrm{cth}\,\frac{g}{2}\,.
\end{aligned}
\tag{9}
$$

Obviously, the functions $\mathrm{th}(g/2)$ and $\mathrm{cth}(g/2)$ must be approximated in both cases. Their Mittag-Leffler series can be written as:

$$
\begin{aligned}
\mathrm{th}\,\frac{g}{2} &= \sum_{i=1}^{\infty} \frac{4g}{g^2 + (2i-1)^2\pi^2} \\
\mathrm{cth}\,\frac{g}{2} &= \frac{2}{g} + \sum_{i=1}^{\infty} \frac{4g}{g^2 + (2i\pi)^2}\,.
\end{aligned}
\tag{10}
$$

If the approximation has to be valid in the region $|g| < \varrho$ and in the corresponding frequency band, the approximating function has to contain the terms of the preceding series the poles of which are within a circle of radius ϱ around the origin. Accordingly, the functions $\mathrm{th}(g/2)$ and $\mathrm{cth}(g/2)$ are written in the following form:

$$
\begin{aligned}
\mathrm{th}\,\frac{g}{2} &= \sum_{i=1}^{l} \frac{4g}{g^2 + (2i-1)^2\pi^2} + t_l(g) \\
\mathrm{cth}\,\frac{g}{2} &= \frac{2}{g} + \sum_{i=1}^{k} \frac{4g}{g^2 + (2i\pi)^2} + r_k(g)\,,
\end{aligned}
\tag{11}
$$

where $(2l+1)\pi > \varrho$ and $(2k+2)\pi > \varrho$. As it has been shown, the terms in the preceding sums, except $t_l(g)$ and $r_k(g)$, lead to the circuits in Figs 1 and 2. The functions $t_l(g)$ and $r_k(g)$ are zeroed at $g = 0$ and are analytic throughout the circle of radius ϱ and so in a sense they can be considered small compared to the other terms. The functions $t_l(g)$ and $r_k(g)$ are to be approximated in form (5). At $g = 0$, some derivatives of the approximating function should equal the corresponding derivatives of the original function. As both the original and the approximating functions are odd, the first m derivatives of odd order are necessarily equal. The approximating function and its Taylor series around the point $g = 0$ take the following form:

$$h(g) = \sum_{i=1}^{m/2} \frac{A_i g}{g^2+B_i} = \sum_{p=1}^{\infty} (-1)^{p-1} g^{2p-1} \sum_{i=1}^{m/2} \frac{A_i}{B_i^p}, \quad \text{if } m \text{ is even}$$

$$h(g) = A_0 g + \sum_{i=1}^{(m-1)/2} \frac{A_i g}{g^2+B_i} = \tag{12}$$

$$= g\left(A_0 + \sum_{i=1}^{(m-1)/2} \frac{A_i}{B_i}\right) + \sum_{p=2}^{\infty} (-1)^{p-1} g^{2p-1} \sum_{i=1}^{(m-1)/2} \frac{A_i}{B_i^p},$$

if m is odd.

Similarly, the functions $t_l(g)$ and $r_k(g)$ are expanded to Taylor series around the point $g = 0$:

$$t_l(g) = \sum_{p=1}^{\infty} D_p g^{2p-1}$$
$$r_k(g) = \sum_{p=1}^{\infty} E_p g^{2p-1}. \tag{13}$$

The detailed calculations give the following expressions for the coefficients of these Taylor series:

$$D_p = \frac{4(-1)^{p+1}}{\pi^{2p}} \sum_{i=l+1}^{\infty} \frac{1}{(2i-1)^{2p}}$$
$$E_p = \frac{4(-1)^{p+1}}{\pi^{2p}} \sum_{i=k+1}^{\infty} \frac{1}{(2i)^{2p}}. \tag{14}$$

The first m coefficients of the power series (12) and (13) are to be made equal and so a nonlinear system of equations is got for the m unknowns A_i and B_i. It is rather difficult to solve this system of equations for a high m value. As it has been shown for the sake of realizability it is necessary that $A_i \geq 0$ and $B_i \geq 0$, furthermore for the validity of the approximation throughout the circle of radius ϱ it is necessary that $B_i > \varrho^2$. In the general case these conditions cannot be proved to be satisfied. Simple calculations show that for low m values (for $m = 1, 2, 3$ and presumably for $m = 4$, too) these conditions are always met, independent of the value of k and l.

Obviously, the choice of k, l and m plays an important role in many respects. It influences the number of the circuit elements used in the equivalent network, the error of the equivalent network and the width of the frequency band in which the equivalence is acceptable.

The number N of the reactive circuit elements used in the lattice network is given by the following relationship:

$$N = 4(k + l + m) + 2. \tag{15}$$

The same number of resistors is necessary, if the transmission line is not lossless.

The region of the g-plane where the presented approximation of the functions $\mathrm{th}(g/2)$ and $\mathrm{cth}(g/2)$ is valid has already been indicated. As a consequence the frequency band, in which the approximation is acceptable, fulfils the following condition:

$$[(R^2 + \omega^2 L^2)(G^2 + \omega^2 C^2)]^{1/4} < \min[(2k + 2)\pi,\ (2l + 1)\pi]. \tag{16}$$

The choice $l = k$ or $l = k + 1$ is seen to be generally suitable.

The error of the approximation under the impact of k, l, m and other factors will be now investigated in detail.

The error of the approximation

The error of the approximation will be investigated for a transmission line terminating in an impedance Z_2. In this case the input impedance of the network in Fig. 3 is given by

$$Z_{11} = \frac{2Z_A Z_B + (Z_A + Z_B) Z_2}{Z_A + Z_B + 2Z_2}\ . \tag{17}$$

The error of the input impedance is

$$\begin{aligned} \Delta Z_{11} &\approx \frac{\partial Z_{11}}{\partial Z_A} \Delta Z_A + \frac{\partial Z_{11}}{\partial Z_B} \partial Z_B = \\ &= \frac{2}{(Z_A + Z_B + 2Z_2)^2} [(Z_B + Z_2)^2 \Delta Z_A + (Z_A + Z_2)^2 \Delta Z_B]. \end{aligned} \tag{18}$$

Denote the error of the approximation of the functions $\mathrm{cth}(g/2)$ and $\mathrm{th}(g/2)$ by Δ_1 and Δ_2, respectively; if the branch impedances of the lattice are approximated, then:

$$\Delta Z_A = Z_0 \Delta_1 \quad \text{and} \quad \Delta Z_B = Z_0 \Delta_2 \,. \tag{19}$$

If the admittances are approximated, then:

$$\begin{aligned} \Delta Z_A &= \frac{1}{Y_A + \Delta Y_A} - \frac{1}{Y_A} \approx - \frac{\Delta Y_A}{Y_A^2} = - \frac{Z_A^2}{Z_0} \Delta_2 \quad \text{and} \\ \Delta Z_B &\approx - \frac{Z_B^2}{Z_0} \Delta_1 \,. \end{aligned} \tag{20}$$

All these substituted into (18), the following expressions arise for the error in the two cases:

$$\Delta Z'_{11} \approx \frac{Z_0^3}{2 (Z_0 \operatorname{cth} g + Z_2)^2} \left[\left(\operatorname{th} \frac{g}{2} + \frac{Z_2}{Z_0} \right)^2 \Delta_1 + \left(\operatorname{cth} \frac{g}{2} + \frac{Z_2}{Z_0} \right)^2 \Delta_2 \right] \tag{21}$$

and

$$\Delta Z''_{11} \approx - \frac{Z_0^3}{2 (Z_0 \operatorname{cth} g + Z_2)^2} \left[\left(1 + \frac{Z_2}{Z_0} \operatorname{th} \frac{g}{2} \right)^2 \Delta_1 + \left(1 + \frac{Z_2}{Z_0} \operatorname{cth} \frac{g}{2} \right)^2 \Delta_2 \right], \text{resp.} \tag{22}$$

The expressions in square brackets are now compared with each other for equal Δ_1 and Δ_2. As $\operatorname{cth}(g/2)$ and $\operatorname{th}(g/2)$ may have values much greater than unity, so if $|Z_2| > |Z_0|$, then generally $|\Delta Z'_{11}| < |\Delta Z''_{11}|$. On the other hand, if $|Z_2| < |Z_0|$, then generally $|\Delta Z'_{11}| > |\Delta Z''_{11}|$. If the terminating impedance is great compared to the characteristic impedance, it is likely to be advisable to approximate the branch impedances, in the opposite case the branch admittances.

The transfer impedance is investigated now. If the short circuit current of the current generator at the input terminals is I_g and its parallel impedance is Z_1, then the transfer impedance is expressed as:

$$Z_{21} = \frac{U_2}{I_g} = \frac{(Z_A - Z_B) Z_1 Z_2}{2 Z_A Z_B + (Z_A + Z_B)(Z_1 + Z_2) + 2 Z_1 Z_2} \,. \tag{23}$$

In the two cases the error is given by the following expressions, using the previous notations:

$$\begin{aligned} \Delta Z'_{21} \approx &- \frac{Z_1 Z_2 Z_0^3}{2 [Z_0^2 + Z_1 Z_2 + Z_0 (Z_1 + Z_2) \operatorname{cth} g]^2} \times \\ &\times \left\{ \left[\frac{Z_1 Z_2}{Z_0^2} + \operatorname{th}^2 \frac{g}{2} + \frac{Z_1 + Z_2}{Z_0} \operatorname{th} \frac{g}{2} \right] \Delta_1 + \right. \\ &+ \left. \left[\frac{Z_1 Z_2}{Z_0^2} + \operatorname{cth}^2 \frac{g}{2} + \frac{Z_1 + Z_2}{Z_0} \operatorname{cth} \frac{g}{2} \right] \Delta_2 \right\} \end{aligned} \tag{24}$$

$$\Delta Z''_{21} \approx -\frac{Z_1 Z_2 Z_0^3}{2[Z_0^2+Z_1Z_2+Z_0(Z_1+Z_2)\,\mathrm{cthg}]^2} \times$$

$$\times\left\{\left[\frac{Z_1 Z_2}{Z_0^2}\,\mathrm{th}^2\frac{g}{2}+1+\frac{Z_1+Z_2}{Z_0}\,\mathrm{th}\,\frac{g}{2}\right]\Delta_1+\right. \tag{25}$$

$$\left.+\left[\frac{Z_1 Z_2}{Z_0^2}\,\mathrm{cth}^2\frac{g}{2}+1+\frac{Z_1 Z_2}{Z_0}\,\mathrm{cth}\,\frac{g}{2}\right]\Delta_2\right\}.$$

Comparison of (24) and (25) shows that if $|Z_1Z_2| > |Z_0|^2$, then generally $|\Delta Z'_{21}| < |\Delta Z''_{21}|$, and if $|Z_1Z_2| < |Z_0|^2$, then generally $|\Delta Z'_{21}| > |\Delta Z''_{21}|$.

If the transmission line acts between great terminating impedances compared to the characteristic one, it is likely to be advisable to approximate the impedance functions of the equivalent network, otherwise to approximate the admittance functions.

The values of k, l and m influence the error of the input and transfer impedances through Δ_1 and Δ_2. This is why now the problem of choosing k, l and m again arises. The frequency band where the approximation is valid has been shown to define the minimal value of k and l through (16). If the complexity of the equivalent network, i.e. the number of the used circuit elements is limited, then (15) defines the maximal value of the sum $(k + l + m)$. The question now arises how to choose k, l and m between these limits. If k and l have the minimal value defined by (16), the error can be diminished by increasing m. In this case, however, the error remains great in the upper part of the frequency band corresponding to the chosen k and l. This is why it is generally more advisable to diminish the error by increasing k and l and to choose $m = 1$. This has also the advantage to avoid all the mentioned difficulties which arise for high values of m. It is practical to choose m greater than one only, if approximation in a narrow frequency band at a great accuracy is wanted. In critical cases the error must be investigated as a function of the frequency and the result must be compared with the accuracy prescription.

Here the error of the impedance parameters will be determined in detail, but only for the case $m = 1$ and $k = l$. (The results would be similar for the admittance parameters.) The error of the impedance parameters, divided by the characteristic impedance is written in form of Taylor series:

$$\frac{\Delta z_{11}}{Z_0} = \sum_{p=2}^{\infty} u_p\, g^{2p-1}$$
$$\frac{\Delta z_{21}}{Z_0} = \sum_{p=2}^{\infty} v_p\, g^{2p-1}, \tag{26}$$

where on the basis of (12) and (13)

$$
\begin{aligned}
u_p &= -\frac{1}{2}(D_p+E_p) = \frac{2(-1)^p}{\pi^{2p}} \sum_{i=k+1}^{\infty}\left[\frac{1}{(2i)^{2p}} + \frac{1}{(2i-1)^{2p}}\right] \\
v_p &= \frac{1}{2}(D_p-E_p) = \frac{2(-1)^p}{\pi^{2p}} \sum_{i=k+1}^{\infty}\left[\frac{1}{(2i)^{2p}} - \frac{1}{(2i-1)^{2p}}\right].
\end{aligned} \tag{27}
$$

The absolute value of the Taylor series is seen to increase the most rapidly on the imaginary axis, because for imaginary values of g the members of the series are of the same sign. The error of the parameter z_{11} and z_{21} is seen in Figs 4*a*

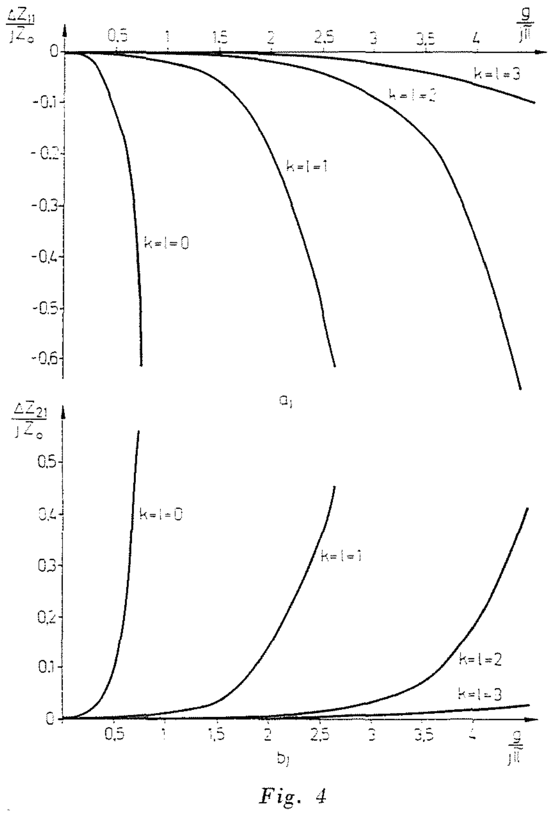

Fig. 4

and 4*b*, resp., for some k and l and for imaginary values of g. If g is not pure imaginary, the error is smaller than the error corresponding to a pure imaginary g of the same absolute value. If the transmission line is lossless, g is imaginary, and so the curves give directly the error of the parameters as a function of the frequency. In this case the scale of the horizontal axis shows how many times the length of the transmission line is larger than the half wavelength.

Equivalent networks not valid for low frequencies

The presented method can be modified if the equivalent network need not be valid for low frequencies, but it must be valid in a frequency band $[\omega_1; \omega_2]$. Again the series (10) underlie the choice of the rational functions approximating the functions th$(g/2)$ and cth$(g/2)$. Those members of the series must occur in the approximating function the poles of which are on the s-plane near the intervals $[j\omega_1; j\omega_2]$ and $[-j\omega_1; -j\omega_2]$ of the imaginary axis. The poles of a member form a complex conjugated pair, and the poles of the i-th member can be determined from the equation

$$(R + sL)(G + sC) + n^2\pi^2 = 0, \tag{28}$$

where $n = 2i - 1$ for the function th$(g/2)$, and $n = 2i$ for the function cth$(g/2)$. The roots of Eq. (28) are:

$$s_n = -\frac{1}{2}\left(\frac{G}{C} + \frac{R}{L}\right) \pm \sqrt{\frac{1}{4}\left(\frac{G}{C} - \frac{R}{L}\right)^2 - \frac{n^2\pi^2}{LC}}, \tag{29}$$

i.e. the poles are on a line parallel to the imaginary axis. If the transmission line is lossless, the poles given in (29) are on the imaginary axis, and the members of the series (10) must obviously occur in the approximating function, for which

$$\omega_1 \leq \frac{n\pi}{\sqrt{LC}} \leq \omega_2. \tag{30}$$

If the transmission line is not lossless, the decision is less easy. For example it seems suitable to choose from the series (10) the members, with poles inside the circles which have as diameter the intervals $[j\omega_1; j\omega_2]$ and $[-j\omega_1; -j\omega_2]$ of the imaginary axis. Accordingly, both functions th$(g/2)$ and cth$(g/2)$ can be written as sums of three terms:

$$\begin{aligned} \text{th}\frac{g}{2} &= \sum_{i=l_1}^{l_2} \frac{4g}{g^2+(2i-1)^2\pi^2} + \sum_{i=1}^{l_1-1} \frac{4g}{g^2+(2i-1)^2\pi^2} + \sum_{i=l_2+1}^{\infty} \frac{4g}{g^2+(2i-1)^2\pi^2} \\ \text{cth}\frac{g}{2} &= \sum_{i=k_1}^{k_2} \frac{4g}{g^2+(2i\pi)^2} + \left[\frac{2}{g} + \sum_{i=1}^{k_1-1} \frac{4g}{g^2+(2i\pi)^2}\right] + \sum_{=k_2+1}^{\infty} \frac{4g}{g^2+(2i\pi)^2}\,. \end{aligned} \tag{31}$$

The first sum will occur unchanged in the approximating function, but the two other terms must yet be approximated. The method shown for solving similar problems for the low frequency approximation can hardly be generalized. That is why only the approximation corresponding to the case $m = 1$ wil be used. The members of the second and the third sum will be approximated in the forms $4/g$ and $4g/(n\pi)^2$, respectively. So the final approximating functions are:

$$
\begin{aligned}
\operatorname{th}\frac{g}{2} &\approx \sum_{i=l_1}^{l_2}\frac{4g}{g^2+(2i-1)^2\pi^2}+\frac{4l_1-1}{g}+\frac{4g}{\pi^2}\sum_{i=l_2+1}^{\infty}\frac{1}{(2i-1)^2}\\
\operatorname{cth}\frac{g}{2} &\approx \sum_{i=k_1}^{k_2}\frac{4g}{g^2+(2i\pi)^2}+\frac{4k_1-2}{g}+\frac{g^2}{\pi^2}\sum_{i=k_2+1}^{\infty}\frac{1}{i^2}\,.
\end{aligned}
\tag{32}
$$

Similarly to the low frequency case, here $k_1 \leq l_1 \leq k_1 + 1$ and $k_2 \leq l_2 \leq k_2+1$. The equivalent network is now easy to determine.

The error of the approximation is determined by the value of k_1, k_2, l_1 and l_2. Here the error cannot be diminished by increasing m, but by decreasing k_1, l_1 and increasing k_2, l_2.

Examples

1. The approximation is the poorest, if $k = l = m = 0$. Then

$$\operatorname{th}\frac{g}{2} \approx 0$$

$$\operatorname{cth}\frac{g}{2} \approx \frac{2}{g}\,.$$

Depending on whether the impedance functions or the admittance functions are approximated the networks in Figs $5a$ or $5b$ result.

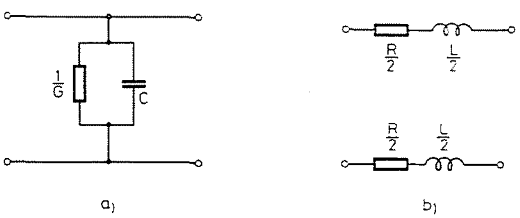

Fig. 5

2. The network is by one degree more complicated, if $k + l + m = 1$, i.e. the number of the reactive elements is $N = 6$. In this case there are two alternatives possible: $k = l = 0$ and $m = 1$, or $k = m = 0$ and $l = 1$. The latter one is to be avoided, because for $m = 0$ the functions $t_l(g)$ and $r_k(g)$ are entirely neglected, involving great error for small values of k and l. In the

first case:

$$\operatorname{th}\frac{g}{2} \approx \frac{g}{2}$$

$$\operatorname{cth}\frac{g}{2} \approx \frac{2}{g} + \frac{g}{6}\,.$$

If the impedance functions are approximated on the basis of the above formulae, the following expressions are got:

$$Z_A = Z_0\left(\frac{2}{g} + \frac{g}{6}\right) = \frac{2}{G + sC} + \frac{R + sL}{6}$$

$$Z_B' = Z_0\frac{g}{2} = \frac{R+sL}{2}\,.$$

The network is seen in Fig. 6*a*, applying a notation customary for symmetrical lattices.

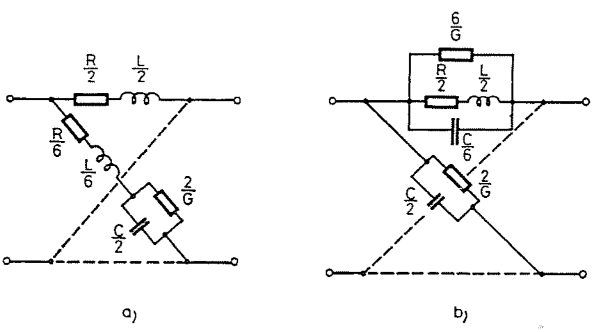

Fig. 6

If the admittances of the lattice are approximated, then

$$Y_A = \frac{1}{Z_0}\frac{g}{2} = \frac{1}{2}(G+sC)$$

$$Y_B = \frac{1}{Z_0}\left(\frac{2}{g} + \frac{g}{6}\right) = \frac{2}{R+sL} + \frac{1}{6}(G+sC).$$

The network is seen in Fig. 6*b*.

3. Finally, let us consider a more complicated equivalent network. For instance, let the number of the used reactive element be 30. Then, according to (15), $k + l + m = 7$. Suppose that an equivalent network is sought for, valid in a narrow frequency band, but at a possibly great accuracy. So a relatively

great value is chosen for m; let $m = 4$. Consequently, $k = 1$ and $l = 2$. If the transmission line is lossless, the frequency band where the approximation is valid is, according to (16):

$$\omega < \frac{4\pi}{\sqrt{LC}},$$

i.e., the wave-length must be at least half the length of the transmission line.

According to the formula (11)

$$\mathrm{th}\,\frac{g}{2} = \frac{4g}{g^2+\pi^2} + \frac{4g}{g^2+9\pi^2} + t_2(g)$$

$$\mathrm{cth}\,\frac{g}{2} = \frac{2}{g} + \frac{4g}{g^2+4\pi^2} + r_1(g)\,.$$

Now the Taylor series of $r_1(g)$ and $t_2(g)$ must be determined. Rather than (14), the Taylor series of the other functions occurring in the above relationships will be applied, to result in:

$$\begin{aligned} r_1(g) &= 6.5345\cdot 10^{-2}g - 2.1128\cdot 10^{-4}g^3 + \\ &+ 1.1275\cdot 10^{-6}g^5 - 6.7140\cdot 10^{-9}g^7 + \ldots \\ t_2(g) &= 4.9684\cdot 10^{-2}g - 9.5772\cdot 10^{-5}g^3 + \\ &+ 3.129\cdot 10^{-7}g^5 - 1.101\cdot 10^{-9}g^7 + \ldots \end{aligned}$$

The approximating function of $r_1(g)$ is taken in the following form:

$$h(g) = \frac{A_1 g}{g^2+B_1} + \frac{A_2 g}{g^2+B_2}\,.$$

Having equalized the coefficients of the Taylor series of the function $h(g)$ to the coefficients of the series $r_1(g)$ we get the following system of equations:

$$\frac{A_1}{B_1} + \frac{A_2}{B_2} = 6.5345\cdot 10^{-2}$$

$$\frac{A_1}{B_1^2} + \frac{A_2}{B_2^2} = 2.1128\cdot 10^{-4}$$

$$\frac{A_1}{B_1^3} + \frac{A_2}{B_2^3} = 1.1275\cdot 10^{-6}$$

$$\frac{A_1}{B_1^4} + \frac{A_2}{B_2^4} = 6.7140\cdot 10^{-9}.$$

After some transformations, the above system of equations is reduced to a single equation of second order, to lead to the following roots:

$$A_1 = 4.931 \quad B_1 = 164.7$$
$$A_2 = 42.34 \quad B_2 = 1196.$$

Accordingly, the function cth$(g/2)$ is approximated by the following rational function:

$$\operatorname{cth}\frac{g}{2} \approx \frac{2}{g} + \frac{4g}{g^2+39.48} + \frac{4.931g}{g^2+164.7} + \frac{42.34g}{g^2+1196}\,.$$

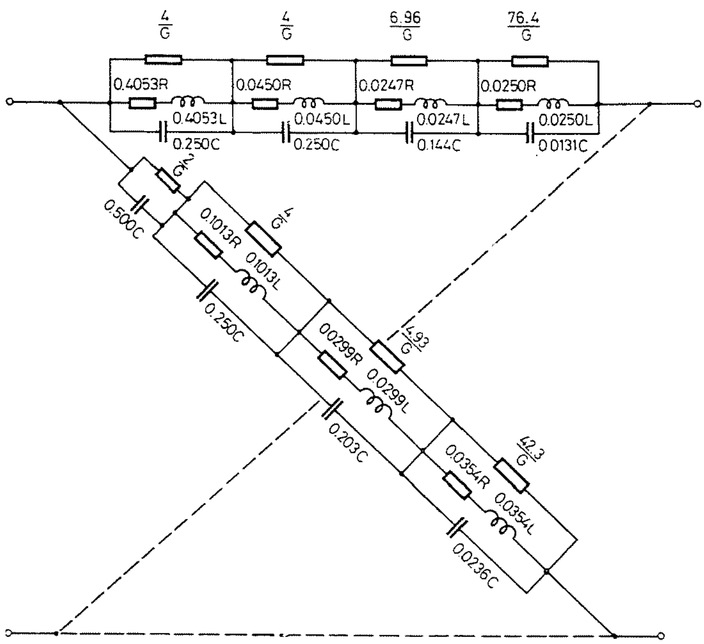

Fig. 7

The approximation of the function th$(g/2)$ is, similarly:

$$\operatorname{th}\frac{g}{2} \approx \frac{4g}{g^2+9.870} + \frac{4g}{g^2+88.13} + \frac{6.96g}{g^2+282} + \frac{76.4g}{g^2+3060}\,.$$

If the above functions are used for determining the impedances, the network in Fig. 7 arises.

It is instructive to compare the error of the network in Fig. 7 to the error of the network belonging to the values $k = l = 3$ and $m = 1$, because both of them contain 30 reactive network elements. The error of the parameters z_{11}

and z_{22} of the latter network is plotted in Fig. 4. According to the calculation, the value of $|\Delta z_{11}/Z_0|$ and $|\Delta z_{21}/Z_0|$ is about $5 \cdot 10^{-4}$ for the network in Fig. 7, if $g = j2.5\pi$; this is by one order smaller than the error of the other network; for $g = j3.5\pi$ the errors are of the same order, and for $g = j4\pi$, the error of the network in Fig. 7 is infinite, while the error of the other network is rather low. The equivalent network shown in this example can be used in a relatively narrow frequency band, but there at a great accuracy, as it has been expected.

Summary

A systematical method has been given to find a lumped equivalent network of a transmission line in a given frequency band. The equivalent network has been chosen in form of symmetrical lattice. The low-frequency equivalent networks have been treated in detail, and for this case the error of the approximation as a function of the frequency has been investigated.

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