# SIMULTANEOUS APPROXIMATION OF THE AMPLITUDE AND GROUP DELAY CHARACTERISTICS

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## Introduction

The measurement of a sinusoidal signal of frequency $f_0$ often requires a selective amplifier of constant amplitude and phase characteristics in a certain range of frequency $f_0$ or at least deviates from a given basic value to a slight extent only, and considerably suppressing the disturbing frequency components different from frequency $f_0$. Such problems may be encountered where the frequency of the signal to be measured are likely to change, e.g. by the uncertainty in the frequency of the signal source (as the case is with the design of devices working at industrial frequencies), and the information is carried by the amplitude of the sinusoidal signal to be measured and its phase angle relative to the reference signal [1, 2, 3]. In certain cases, to speed up the automatic balancement of a.c. measuring bridges with two components necessitates similar prescriptions. The constant phase value in a frequency range means a zero group delay, thus such selective amplifiers are very likely to have short delay time, which is advantageous, first of all, in analog circuits interconnected with digital systems.

The present paper wants to discuss the design of selective amplifiers satisfying the mentioned prescriptions.

## 1. Definition of the approximation problem

By using the method of network synthesis, the problem will be decomposed into two parts by the insertion of the complex frequency range. The first step is the approximation: the production of a network function satisfying the requirements mentioned. The second step is the realization, i.e. to find an electric network the function of which is identic with the network function obtained by approximation. The two steps cannot be separated, since the way of realization determines the type of the functions suitable to approximation. In the following application of the technique of active RC realizations will be assumed allowing to define the approximation problem, namely to produce a transfer function in the complex frequency range in the form of a rational function with real coefficients:

$$F(s) = \frac{P(s)}{H(s)} = \frac{a_0 + a_1 s + a_2 s^2 + \ldots + a_m s^m}{b_0 + b_1 s + b_2 s^2 + \ldots + b_n s^n}, \tag{1}$$

where

1. the denominator is a Hurwitz polynomial $H(s)$ (it contains roots only in the left half-plane),

2. the condition $m \leq n$ is fulfilled for the degrees of both the numerator and the denominator,

3. after the substitution $s = j\omega$ the prescriptions made for the frequency range are fulfilled.

Denote by $z_0$ the value of the transfer function at the mid-range frequency $\omega_0$:

$$z_0 = F(j\omega_0) = | F(j\omega_0) | e^{j \operatorname{arc} F(j\omega_0)}.$$

The condition that in certain mid-range regions the amplitude and phase characteristics are constant, or deviate from a basic value to a small extent only, can be exactly defined in the way that, in the case of a permissible error $\varepsilon_H$, the fulfilment of the condition

$$| F(j\omega) - z_0 | \leq \varepsilon_H \quad \omega_1 \leq \omega \leq \omega_2 \tag{1a}$$

is prescribed for the absolute value of the complex number $F(j\omega) - z_0$, where the frequencies $\omega_1$ and $\omega_2$ are the limit frequencies of the passband and, of course, $\omega_0$ is in the interior of the passband. Thus we have made a prescription for the absolute value of the complex error.

In certain cases, where the measuring system is not equally sensitive to amplitude and phase errors, the condition (1a) can be replaced by the following more general prescription :

$$k_1 \cdot (|F(j\omega)| - |z_0|)^2 + k_2 (\operatorname{arc} F(j\omega) - \operatorname{arc} z_0)^2 < \varepsilon_H,$$

where $k_1$ and $k_2$ are positive weighting factors.

The selectivity requirements can be given by the following prescriptions:

$$\begin{aligned} | F(j\omega) | &\leq \varepsilon_{s1} \quad 0 \leq \omega \leq \omega_{s1}, \\ | F(j\omega) | &\leq \varepsilon_{s2} \quad \omega_{s2} \leq \omega < \infty, \end{aligned} \tag{1b}$$

where frequencies $\omega_{s1}$ and $\omega_{s2}$ are the limit frequencies of the stopband, and $\varepsilon_{s1}$ and $\varepsilon_{s2}$ are prescribed constants. Thus the prescriptions made for the frequency range can be defined by conditions (1a) and (1b).

The special literature presents several methods for the simultaneous approximation of the amplitude and the group delay [4—8], none of them is, however, suited for the approximation of networks with zero delay time. At the same time, the realization of an arbitrary pole-zero arrangement by means of active RC networks can be considered as solved, and therefore only the problem of the approximation of prescriptions (1a,b) will be treated; realizations see in [9—12].

## 2. Theoretical realizability of the approximation

The first question is whether the approximation can be solved at all in the case of the conditions for (1). The theoretical realizability can be examined on the following considerations: Any linear network can be uniquely decomposed into a minimum phase and an all-pass section [13]. The attenuation and phase characteristics of the minimum phase network are related by the integration theorems of Bode [14]. These theorems and the monotony of the phase characteristic of the all-pass network allow to conclude on the realizability of the conditions (1). Rather than to discuss the examination in detail, only its results will be summarized.

1. The conditions (1a,b) are not independent, but after omitting one of them, the other can be fulfilled. If one of the prescriptions is given, a limit can be established for the other below which it can be fulfilled. A weaker, i.e. less strict, prescription can be obtained by increasing the value of the corresponding prescription $\varepsilon$.

2. Both conditions can be given stricter prescriptions simultaneously if

— approximation is by a minimum phase network, so that $F(s)$ contains no zeros in the right half-plane either;

— the number of zeros is increased;

— the amplitude in the transition range between passband and stopband can be described by a non-monotonous function and also a factoring out is permitted;

— an optimal value is chosen for arc $F(j\omega_0)$ in condition (1a) which can generally be done with networks of bandpass character. The existence of such an optimum value can be demonstrated.

## 3. The procedure of approximation

### 3.1 *Introduction of the network function* $T(s)$

In the approximation of (1) the main problem is given by the existence of simultaneous prescriptions for the amplitude and the phase characteristic in the passband and at the same time, the approximative function is subject to further conditions. The problem can be solved by introducing a new network function.

Let the network function in the complex frequency range be defined by the transfer function $F(s)$, as:

$$T(s) = \frac{R(s)}{F(s)} = \frac{F(s)-z_0}{F(s)}, \tag{2}$$

where $z_0$, usually a complex number, is the value of the transfer function at the frequency $\omega_0$ of the central band.

Examine relationship (2) as a projection of plane $F(s)$ on plane $T(s)$. The projection being conform, the circles with

$$|T(s)| = \text{constant}$$

must be circles also in the $F(s)$-plane. It is directly evident that the point $z_0$ of $F(s)$-plane transforms into the origin in the $T(s)$-plane and the origin of the $F(s)$-plane into the infinitely far point of the $T(s)$-plane, whereas the halving perpendicular of the section between the point $z_0$ and the origin of $F(s)$ will be projected by transformation (2) into the circle with unit radius of the $T(s)$-plane. Thus, by the inverse projection, the circles of $|T(s)| =$ constant with radii smaller than unity are transferred by (2) into the set of circles which can be drawn around $z_0$, and on the other hand all the circles of $|T(s)| =$ constant with radii longer than the unit radius change into the set of circles which can be drawn around the origin of the $F(s)$-plane. The sets of circles mutually corresponding to each other are shown in Fig. 1. The circles in Fig. 1*b* are sometimes referred to as Apollonius circles, since according to (2) the condition

$$|T(s)| = \frac{|R(s)|}{|F(s)|} = \text{constant}$$

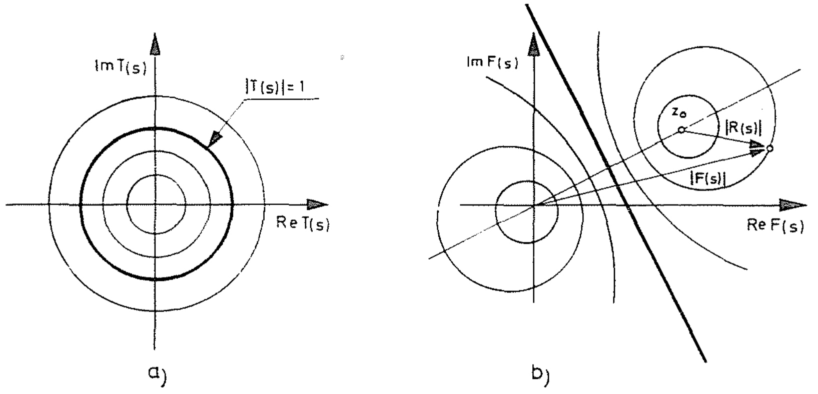


*Fig. 1.* Mutually unique projection of $T(s)$ and $F(s)$

determines the geometric places of points in the $F(s)$-plane with constant ratios of the distances from the points $z_0$ and the origin, resp. (i.e. ratios $|R(s)|$ to $|F(s)|$). This gives exactly the set of Apollonius circles.

Examine how the defined network function $T(s)$ can be used to approximate (1).

Condition (1b), in fact, means that throughout stopband the curve $F(j\omega)$ in the $F(s)$-plane remains inside the circle around the origin of radius $\varepsilon_s$ (hence $\varepsilon_s$ may be identical either with $\varepsilon_{s1}$ or with $\varepsilon_{s2}$). Let us draw that member of the

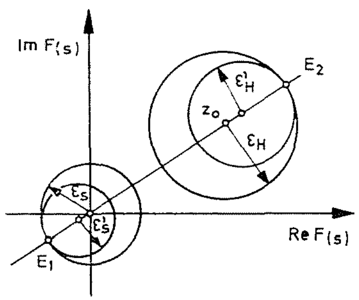


*Fig. 2.* Prescriptions in the $F(s)$-plane

set of Apollonius circles which touches the circle of radius $\varepsilon_s$ from inside (see Fig. 2) and be its radius $\varepsilon_s'$. Prescription (1b) is made stricter if, instead of to inside the circle of radius $\varepsilon_s$, the curve in the stopband is restricted to inside the circle of radius $\varepsilon_s'$. However, with $\varepsilon_s$ decreasing due to the properties of the Apollonius circles, this restriction will always remain, in practice, within the permissible limit. Let us determine what prescription follows from condition (1b) for the network function $T(s)$. On the basis of Fig. 2 and relationship (2) for the point $E_1$:

$$|R(s)| = |z_0| + \varepsilon_s ,$$

$$|F(s)| = \varepsilon_s ,$$

$$|T(s)| = \frac{|z_0| + \varepsilon_s}{\varepsilon_s} = 1 + \frac{|z_0|}{\varepsilon_s} .$$

As by the projection in question the interior the Apollonius circle of radius $\varepsilon_s'$ is transferred into the exterior of the circle of radius $\left(1 + \frac{|z_0|}{\varepsilon_s}\right)$, prescription (1b) is equivalent to the prescription

$$|T(j\omega)| \geq 1 + \frac{|z_0|}{\varepsilon_s} .$$

Since for the prescriptions in the stopband $\varepsilon_s \ll |z_0|$ is generally fulfilled and the normalization $|z_0| = 1$ is reasonable, the following very simple prescription is obtained for the network function $T(s)$:

$$|T(j\omega)| > \frac{1}{\varepsilon_s}.$$

Examine now how the prescriptions in the passband can be defined by means of the network function $T(s)$. Assuming that in the whole passband the transfer function $F(j\omega)$ remains within the circle of origin $z_0$ and radius $\varepsilon_H$, — an assumption equivalent to condition (1a), — it is evident that the conditions

$$||F(j\omega)| - |z_0|| \leq \varepsilon_{H1} \qquad \omega_2 \geq \omega \geq \omega_1$$

$$|\text{arc } F(j\omega) - \text{arc } z_0| \leq \varepsilon_{H2} \quad \omega_2 \geq \omega \geq \omega_1$$

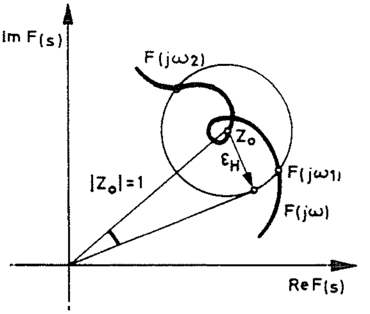


*Fig. 3.* Bandpass prescriptions in the $F(s)$-plane

are fulfilled simultaneously, and assuming that the normalization $|z_0| = 1$ has been carried out

$$\varepsilon_{H1} = \varepsilon_H,$$
$$\varepsilon_{H2} = \text{arc } \sin \varepsilon_H.$$

(see Fig. 3).

Thus, in this case the amplitude and phase prescriptions in the passband are mutually coordinated by the relationship

$$\varepsilon_{H1} = \sin \varepsilon_{H2}. \tag{4}$$

Some correlated values are compiled in Table 1.

**Table I**

| $\varepsilon_H$ | 0.01 | 0.03 | 0.05 | 0.1 | 0.15 | 0.2 | 0.3 |
|---|---|---|---|---|---|---|---|
| $\varepsilon_{H1}$ [dB] | 0.087 | 0.175 | 0.446 | 0.915 | 1.41 | 1.94 | 3.10 |
| $\varepsilon_{H2}$ [degrees] | 0.57 | 1.72 | 2.87 | 5.74 | 8.63 | 11.5 | 17.5 |

If the amplitude and phase prescriptions in the passband are to be coordinated other than by relationship (4), then, using a projection more complicated than projection (2), the circle of radius $\varepsilon_H$ can be replaced by an ellipse with suitable data, and so the correlated values $\varepsilon_{H1}$ and $\varepsilon_{H2}$ can be chosen arbitrarily. This, however, makes the whole procedure of approximation more complicated. In the following part only the case will be treated where the correlation produced by (4) is suitable.

Similarly to the prescriptions in the stopband, let the transfer function values in passband $F(j\omega)$ be limited to inside the Apollonius circle of radius $\varepsilon'_H$ instead of radius $\varepsilon_H$, as shown in Fig. 2. Then for the point $E_2$ it can be written:

$$|R(s)| = \varepsilon_H,$$

$$|F(s)| = |z_0| + \varepsilon_H,$$

$$|T(s)| = \frac{|R(s)|}{|F(s)|} = \frac{\varepsilon_H}{|z_0|+\varepsilon_H}.$$

Since projection (2) transfers the interior of the Apollonius circle of radius $\varepsilon'_H$ into the interior of the circle of radius $\varepsilon_H/(|z_0|+\varepsilon_H)$, the prescriptions in the passband will be equivalent to the prescription

$$|T(j\omega)| \leq \frac{\varepsilon_H}{|z_0|+\varepsilon_H}.$$

Since generally $\varepsilon_H \ll 1$, by using the normalization $|z_0| = 1$ the prescription in the passband, too, will be reduced to the form

$$|T(j\omega)| < \varepsilon_H.$$

Summarized: Approximation (1) gives the following prescriptions for the network function $T(s)$:

The condition

$$|T(j\omega)| < \varepsilon_H \tag{3a}$$

must be fulfilled in the passband and the condition (s)

$$|T(j\omega)| > \frac{1}{\varepsilon_s} \tag{3b}$$

in the stopband, if $|z_0| = 1$ and $\varepsilon_H = \varepsilon_{H1} = \sin \varepsilon_{H2}$.

From relationship (2) it is directly evident that $|T(j\omega)|^2$ is a rational function with real coefficients, and thus condition (3) can be treated by the known approximation methods. The problem of how to ensure the Hurwitz character of the denominator in the transfer function $F(s)$ obtained from network function $|T(j\omega)|$ will be discussed in the chapter on the procedures of concrete approximation.

### 3.2 *Maximally flat approximation of the network function* $|T(j\omega)|$

Prescription (3a) in the passband can be approached by the maximally flat approximation based on the Taylor series. A function of $p$-th degree is maximally flat if the derivatives of the absolute value of the function at $\omega = \omega_0$ up to the $p$-th derivative equal zero.

Let the value of the transfer function in the central band be known, i.e.:

$$F(j\omega_0) = z_0$$

and make the network function $|T(j\omega)|$ in the region of this frequency maximally flat.

According to the previous considerations the condition of maximal flatness requires the simultaneous fulfilment of the equations

$$\left.\frac{\partial^i |T(j\omega)|}{\partial\omega^i}\right|_{\omega=\omega_0} = 0 \qquad i = 1, 2, 3, \ldots p. \tag{5}$$

From relationship (2) it follows that

$$T(j\omega_0) = 0. \tag{6}$$

Denote the conjugates of $z_0$ and $T(s)$ by $z_0^*$ and $T^*(s)$, resp. Relationship (2) makes the validity of

$$T^*(s) = 1 - \frac{z_0^*}{F(-s)}$$

obvious, and since the relationship $F(-j\omega_0) = z_0^*$ is fulfilled for the transfer function, the equality

$$T^*(j\omega_0) = 0 \tag{7}$$

holds as well.

Let the conditions of absolute flatness be examined. Assume $i = 1$. Then — applying the square of $|T(j\omega)|$ for sake of computation — the condition

$$\left.\frac{\partial |T(j\omega)|^2}{\partial\omega}\right|_{\omega=\omega_0} = \left.\frac{\partial[T(j\omega)\cdot T^*(j\omega)]}{\partial\omega}\right|_{\omega=\omega_0} = 0$$

has to be examined. After performing the partial derivation

$$\frac{\partial\,[T(j\omega)T^*(j\omega)]}{\partial\omega}\bigg|_{\omega=\omega_0} = T\,(j\omega_0)\,\frac{\partial T^*(j\omega)}{\partial\omega}\bigg|_{\omega=\omega_0} + T^*(j\omega_0)\,\frac{\partial T(j\omega)}{\partial\omega}\bigg|_{\omega=\omega_0} = 0\,,$$

the condition of maximum flatness for $i = 1$ is found to be fulfilled in each case if relationships (6) and (7) are taken into consideration. In the case of $i = 2$ the derivative will not be zeroed. According to relationship (5), with $i = 2$ the condition

$$\frac{\partial^2\,|T(j\omega)|^2}{\partial\omega^2}\bigg|_{\omega=\omega_0} = \frac{\partial^2\,[T(j\omega)\cdot T^*(j\omega)]}{\partial\omega^2}\bigg|_{\omega=\omega_0} = T(j\omega_0)\,\frac{\partial^2\,T^*(j\omega)}{\partial\omega^2}\bigg|_{\omega=\omega_0} +$$
$$+ 2\cdot\left(\frac{\partial T(j\omega)}{\partial\omega}\cdot\frac{\partial T^*(j\omega)}{\partial\omega}\right)\bigg|_{\omega=\omega_0} + T^*(j\omega_0)\,\frac{\partial^2 T(j\omega)}{\partial\omega^2}\bigg|_{\omega=\omega_0} = 0$$

is fulfilled in the case only if — using again equations (6) and (7) — the equality

$$\frac{\partial T\,(j\omega)}{\partial\omega}\bigg|_{\omega=\omega_0} = 0$$

holds. Applying the rule of the product functions of higher degree it becomes evident that the derivatives of the condition of maximum flatness is identical with that of the simultaneous fulfilment of the equations

$$\frac{\partial^i T(j\omega)}{\partial\omega^i}\bigg|_{\omega=\omega_0} = 0 \qquad i=1, 2, 3, \ldots p$$

and in this case the network function $|\,T(j\omega)\,|$ will be maximally flat at degree $(2p + 1)$ in the central band. For sake of simplicity, substitute $s = j\omega$ in the previous equation:

$$\frac{\partial^i\,T(s)}{\partial s^i}\bigg|_{s=j\omega_0} = 0 \quad i = 1, 2, 3, \ldots p. \tag{8}$$

Using the definition equation (2) the condition for its derivatives to become zero can be determined for the transfer function $F(s)$. For the case $i = 1$:

$$\frac{\partial\,T(s)}{\partial s}\bigg|_{s=j\omega_0} = \frac{\partial}{\partial s}\left(1 - \frac{z_0}{F(s)}\right)\bigg|_{s=j\omega_0} = -\frac{\partial\,\dfrac{H(s)}{P(s)}}{\partial s}\Bigg|_{s=j\omega_0} = 0\,,$$

where also relationship (1) has been used. After derivation and arrangement, the following relationship is obtained:

$$\left.\frac{\dfrac{\partial P(s)}{\partial s}}{\dfrac{\partial H(s)}{\partial s}}\right|_{s=j\omega_0} = \left.\frac{P(s)}{H(s)}\right|_{s=j\omega_0}. \tag{9}$$

Since the transfer function assumes the value $z_0$ at $s = j\omega_0$, the detailed form of Eq. (9) — after derivation — will give

$$\left.\frac{a_1+2\,a_2\,s+3\,a_3\,s^2+\ldots+ma_m\,s^{m-1}}{b_1+2\,b_2\,s+3\,b_3\,s^2+\ldots+na_n\,s^{n-1}}\right|_{s=j\omega_0} = z_0$$

as a condition for the coefficients of the transfer function. In a similar manner, using (1) and (2), relationship (8) gives the condition system

$$\left.\frac{\dfrac{\partial^i P(s)}{\partial s^i}}{\dfrac{\partial^i H(s)}{\partial s^i}}\right|_{s=j\omega_0} = z_0 \qquad i = 1, 2, 3, \ldots, p. \tag{10}$$

After performing the derivation and completing Eqs (10) with the condition $F(j\omega_0) = z_0$, the network function $|\,T(j\omega)\,|$ will be found maximally flat at the central band frequency $\omega_0$ on the $(2p + 1)$th degree if following conditions are fulfilled:

$$z_0 = \left.\frac{\sum\limits_{k=i}^{m}\binom{k}{i}\cdot i!\cdot a_k\cdot s^{k-i}}{\sum\limits_{k=i}^{n}\binom{k}{i}\cdot i!\cdot b_k\cdot s^{k-i}}\right|_{s=\omega_0} \qquad i = 0, 1, 2, \ldots, p. \tag{11}$$

Thus equation system (11) gives the maximally flat approximation of the prescriptions in the passband. It may be simplest to fulfil the prescriptions in the stopband by zeroing a corresponding number of coefficients

$$a_0,\ a_1,\ \ldots,\ a_l$$

in transfer function (1) and by suitably choosing the degrees $n$ and $m$. Let us refer again to the statement made on the theoretical realizability. The denominator $H(s)$ of the transfer function must be a Hurwitz polynomial. This condition can be fulfilled in the simplest way by suitably pre-defining the polynomial $H(s)$, and the equation system (11) for coefficients $a_k$ will correspond to a linear equation system of order $(2p+2)$. In fact, the equality of the real and imaginary parts in the right- and left-hand sides of equation system (11) means a linear equation system of order $(p+1)$ each. Then for the degrees the condition

$$2p+1+l=m \tag{12}$$

will necessarily be fulfilled, where

$l$ the number of zeros at zero frequency,
$m$ the degree in the numerator of the transfer function,
$(n-m)$ the number of zeros at infinity.

From the form

$$z_0 = |z_0| \cdot e^{j\varphi_0} = |z_0| \cdot (\cos\varphi_0 + j\sin\varphi_0)$$

where $\varphi_0$ is the phase shift value assumed at frequency $\omega_0$ it appears that equation system (11) can simply be solved also by treating $\varphi_0$ as a parameter — as, also in this case, a linear equation system is obtained for the coefficients $a_k$ — offering a possibility to find the optimum phase value $\varphi_0$. Or some of the coefficients $b_k$ in the denominator $H(s)$ of the transfer function may be considered as unknowns in equation system (11), and then the condition of maximum flatness can be fulfilled at a higher degree than the $p$ obtained from condition (12). In this case, however, care must be taken in solving (11) that $H(s)$ be a Hurwitz polynomial. Treating $\varphi_0$ as a parameter can be advantageous also in this case.

Let us make mention of the special case where zero frequency is the central band and thus prescriptions of low-pass character are to be approximated. Then necessarily $\omega_0=0$, $z_0=|z_0|$ and $\varphi_0=0$. Applying normalization $|z_0|=1$, equation system (11) simplifies to the condition system

$$a_k=b_k \quad k=0,1,2,\ldots p. \tag{13}$$

This result is evident. Tending to zero frequency, the value of the transfer function $F(j\omega)$ is determined by the coefficients of the members with the lowest exponents. Thus, if the deviation of the amplitude and the phase from the value of the central band in this frequency range is to be minimized, then the dominant coefficients must really coincide.

For example, according to (13) the network function $|T(j\omega)|$ of the low-pass transfer function

$$F(s)=\frac{1+2s+2s^2}{1+2s+2s^2+s^3}$$

is maximally flat at the fifth degree in the environment of zero frequency and has a distance selectivity of 20 dB/decade.

In Figs 4a through f the pole-zero arrangement, the network function $|T(j\omega)|$, the Nyquist diagram of the transfer function $F(j\omega)$, the amplitude curve, the phase characteristic and the step response function, respectively, are represented.

### 3.3 *Example for the maximally flat approximation of the network function $|T(j\omega)|$ of a bandpass network*

Assume a slope of 40 dB/decade as high-frequency attenuation characteristic and 20 dB/decade as attenuation characteristic in the environment of zero frequency. Let the prescription of maximum flatness for $|T(j\omega)|$ on the third degree at the frequency of normalized central band with $\omega_0 = 1$ be fulfilled.

Then with the degrees

$$n - m = 2,\ l = 1,\ 2p + 1 = 3,\ \text{hence } p = 1$$

and, from (12)

$$m = 4,\ n = 6,$$

arbitrarily assuming the polynomial $H(s)$, the coefficients of the numerator in the transfer function $F(s)$ can be determined from (11). As the assumption $l = 1$ makes $a_0 = 0$, (10) and (11) give for the coefficients:

$$|z_0|\, e^{j\varphi_0} = \frac{a_1(j1)+a_2(j1)^2+a_3(j1)^3+a_4(j1)^4}{H(j1)}, \qquad i = 0 \tag{14}$$

$$|z_0|\, e^{j\varphi_0} = \frac{a_1+2a_2(j1)+3a_3(j1)^2+4a_4(j1)^3}{\left.\frac{\partial H(s)}{\partial s}\right|_{s=j1}}, \qquad i = p = 1. \tag{15}$$

The choice of the polynomial $H(s)$ affects the amplitude characteristic between passband and stopband and the limit frequencies of the stopband. Assume

$$H(s) = \left(1 + \frac{s}{3} + s^2\right)^3$$

i.e. a polynomial with an absolute value of three times the unity and with poles having a $Q$-factor $Q = 3$. Then the condition $n = 6$ is fulfilled, and

$$H(j1) = -\frac{j}{27},$$

$$\left.\frac{\partial H(s)}{\partial s}\right|_{s=j1} = -\frac{1}{9} - j\frac{2}{3}.$$

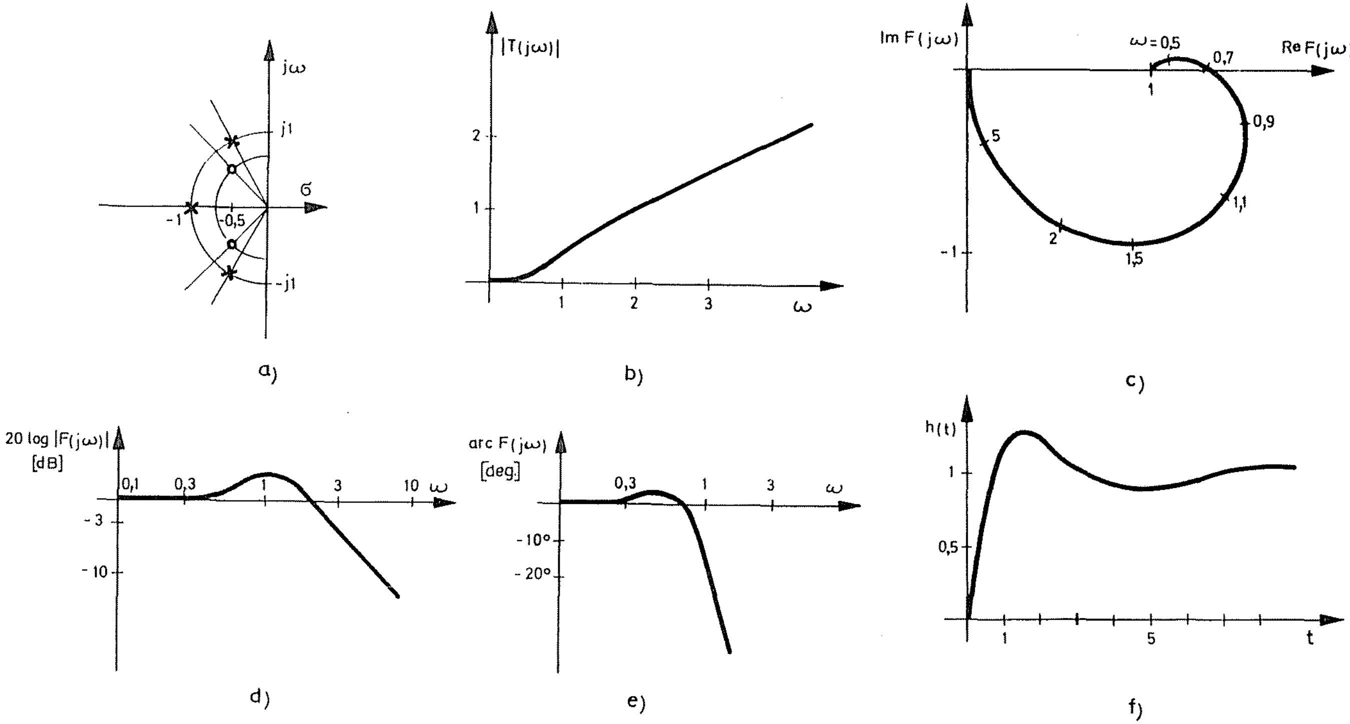


*Fig. 4.* Network functions of the low-pass network

Substituting them into Eqs (14) and (15) and assuming the normalization $|z_0| = 1$, the solution of the equations will be

$$\begin{aligned} a_1 &= -\frac{\sin\varphi_0}{3} \\ a_2 &= \frac{\cos\varphi_0}{3} - \frac{\sin\varphi_0}{54} \\ a_3 &= \frac{\cos\varphi_0}{27} - \frac{\sin\varphi_0}{3} \\ a_4 &= \frac{\sin\varphi_0}{54} + \frac{\cos\varphi_0}{3}\,. \end{aligned} \tag{16}$$

If e.g. $\varphi_0 = 40°$, the substitution into (16) will yield $a_1 = 0.214263$, $a_2 = 0.267252$, $a_3 = 0.242635$, $a_4 = 0.243445$, and thus the transfer function can

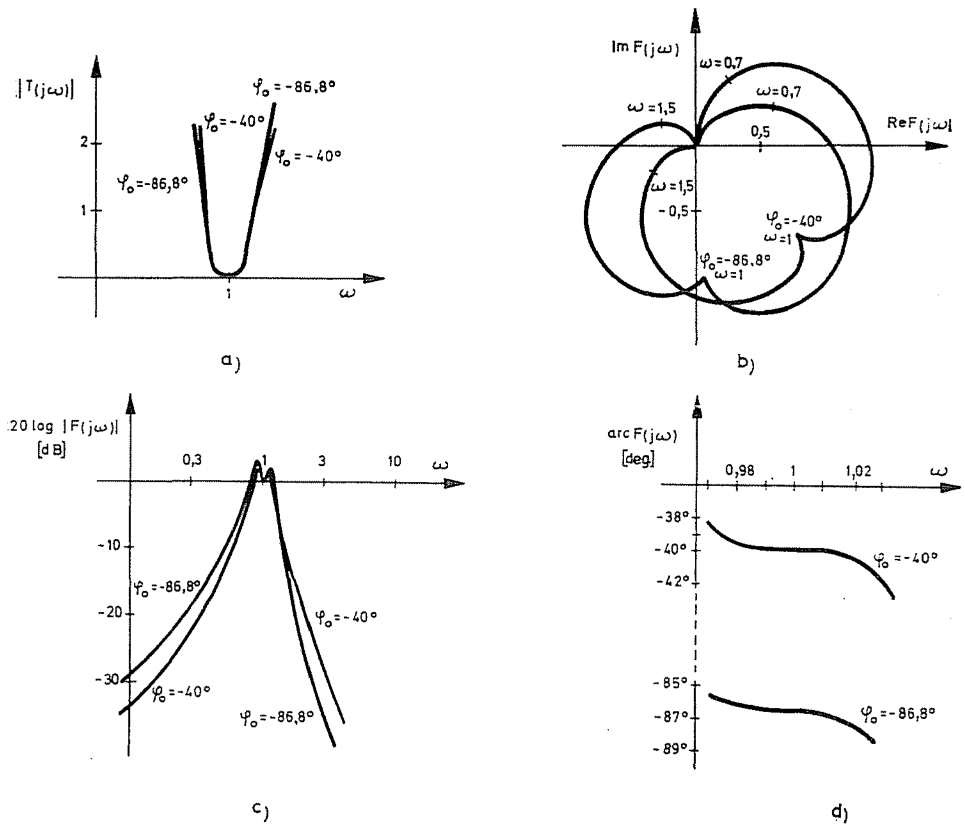


*Fig. 5.* Network functions of the bandpass network

be written as

$$F(s)=\frac{-s\cdot(0.243445\,s^3+0.242635\,s^2+0.267252\,s+0.214263)}{\left(s^2+\frac{s}{3}+1\right)^3}\,.$$

After finding the roots of the third degree equation in the numerator, the pole to zero arrangement of the transfer function will be:

zeros:

$$z_1=0$$
$$z_2=-0.882646$$
$$z_{3,4}=-0.057013\pm j\,0.996944$$

poles:

$$p_{1,2}=p_{3,4}=p_{5,6}=-0.166667\pm j\,0.986013.$$

Figs 5a through d show the network function $|T(j\omega)|$, the Nyquist diagram of the transfer function, the related amplitude and phase characteristics, respectively. From the condition $a_4=0$ we obtain the value of the basic phase shift, i.e.

$$\varphi_0=\text{arc}\tan(-18)=-86.82^\circ.$$

In this case the slope of high-frequency distance selectivity will be 60 dB/decade. Fig. 5 shows also the characteristics belonging to this basic phase shift.

### 3.4 *Chebychev approximation of the transfer function $|T(j\omega)|$*

For low-pass networks the network function $|T(j\omega)|^2$ contains only the even powers of $\omega$. Then prescription (3a) can be fulfilled by equal-ripple approximation in the whole passband (see Fig. 6), if the network function $|T(j\omega)|$ is produced from the relationship

$$|T(j\omega)|^2=\frac{1}{2}\cdot\varepsilon_H^2\cdot[1+Q(\omega^2)]\,, \tag{17}$$

where

$$Q(\omega^2)=\cos\left[2n\ \text{arc}\cos\omega-2\cdot\sum_{i=1}^{t}\text{arc}\tan\frac{c_i\,\omega}{\sqrt{1-\omega^2}}+c_0\right].$$

For the meaning of the constants in the formula and the properties of the rational function $Q(\omega^2)$ see [15] and [16].

Thus relationship (17) is suitable for producing the network function $|T/(j\omega)|$ of equal-ripple approximation in the passband. From this the transfer function $F(s)$ is obtained with the help of the definition relationship (2). Omitting the calculations we present only the final results.

It can be demonstrated that, using this method, the denominator $H(s)$ of the transfer function $F(s)$ can be a Hurwitz polynomial only if the equality

$$n = m, \text{ or } n = m + 1$$

holds for the degrees of both the numerator and the denominator of $F(s)$.

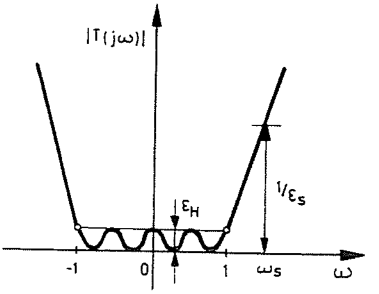


*Fig. 6.* Equal-ripple approximation for the network function $|T(j\omega)|$

In the following, some low-pass transfer functions with oscillation in the passband $\varepsilon_H$ are presented. They may be of importance for the design of selective amplifiers.

If $n = m = 1$

$$F(s) = \frac{1+s\sqrt{c^2-1}}{1+s[c\cdot\varepsilon_H+\sqrt{c^2-1}]},$$

$n = 2,\ m = 1$

$$F(s) = \frac{1+s\sqrt{c^2-1}}{1+\varepsilon_H+s\sqrt{c^2-1}+s^2\cdot\varepsilon_H\cdot(1+c)},$$

$n = 2,\ m = 2$

$$F(s) = \frac{(1+s\sqrt{c_1^2-1})\cdot(1+s\sqrt{c_2^2-1})}{1+\varepsilon_H+s\cdot(\sqrt{c_1^2-1}+\sqrt{c_2^2-1})+s^2[\varepsilon_H\cdot(1+c_1c_2)+\sqrt{c_1^2-1}\sqrt{c_2^2-1}]},$$

$n = 3,\ m = 2$

$$F(s) = \frac{P(s)}{H(s)} \qquad P(s) = (1+s\sqrt{c_1^2-1})\cdot(1+s\sqrt{c_2^2-1})$$

$$H(s) = 1+s[\varepsilon_H(1+c_1+c_2)+\sqrt{c_1^2-1}+\sqrt{c_2^2-1}]+s^2\sqrt{c_1^2-1}\cdot\sqrt{c_2^2-1}+$$
$$+s^3[\varepsilon_H\cdot(1+c_1+c_2+c_1c_2)]$$

The parameters $c$ and $c_1$, $c_2$ in the transfer function — either real or conjugate complex numbers — have to be chosen in the way that the amplitude characteristic may have a suitable slope in the frequency range between passband and stopband. The problem can be solved by computer optimization.

## Summary

Approximation problems of selective amplifiers are discussed with simultaneous by prescribed amplitude and phase characteristics. The prescriptions related to the amplitude characteristic are the usual ones, whereas for the phase characteristic in the passband the approximation of zero group delay is aimed at. It can be stated that these prescriptions are not independent. For the case, however, where the prescriptions can be fulfilled in principle, an approximation method of practical use is presented by defining a new transfer function $T(s)$. Attention is drawn to the importance of the value of optimum basic phase shift.

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