CALCULATION OF HIGHER-ORDER SENSITIVITIES AND HIGHER-ORDER SENSITIVITY INVARIANTS

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1. Calculation of higher-order sensitivities

First-order semirelative sensitivity functions with respect to arbiratry impedances can easily be computed by the indirect method using transfer functions [1, 2, 5, 6, 9] (for notations in the transfer functions see Fig. 1):

$$Q_i = \frac{\partial K}{\partial \ln Z_i} = K_{1i} K_{i2}\,, \tag{1}$$

where

$$K = y = \frac{U_2}{U_1}; \qquad K_{1i} = \frac{U_i}{U_1}; \qquad K_{i2} = \frac{U_2}{U_{ii}}.$$

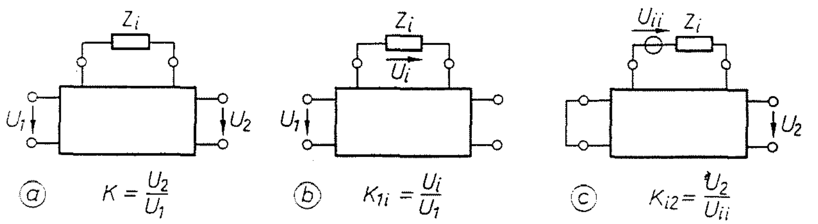

Fig. 1. Transfer functions used in calculating semirelative sensitivity functions

a) $K = \frac{U_2}{U_1}$; b) $K_{1i} = \frac{U_i}{U_1}$; c) $K_{i2} = \frac{U_2}{U_{ii}}$

The expression is applicable not only for impedances Z_i but for network parameter $x_i(R_i, L_i, C_i^{-1})$ as well, and so the formula can be rewritten as:

$$Q_i = \frac{\partial K}{\partial \ln Z_i} = \frac{\partial K}{\partial \ln x_i} \tag{2}$$

It is worthy to remark that there is only one direct path between the input and the output which touches port i and the product $K_{1i}K_{i2}$ can be regarded as the path-product (Fig. 2a). It is useful to apply this technic, for the sake of illustration, but we should like to emphasize that the path-product defined is not to be mistaken for the signal flow graph, it is only a demonstration aid.

The second-order semirelative sensitivity is to be determined with respect to the i-th and the j-th parameter:

$$Q_{ij} = \frac{\partial^2 K}{\partial \ln x_i \partial \ln x_j} = \frac{\partial Q_i}{\partial \ln x_j} = K_{1i} \frac{\partial K_{i2}}{\partial \ln x_j} + \frac{\partial K_{1i}}{\partial \ln x_j} K_{i2} \tag{3}$$

Hence, utilizing Eqs (1) and (2):

$$Q_{ij} = K_{1i} K_{ij} K_{j2} + K_{1j} K_{ji} K_{i2} \tag{4}$$

It is again remarkable that there are only two different direct paths between the input and the output which touch both the i-th and the j-th port. Two

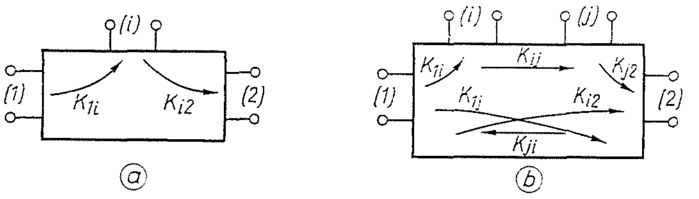

Fig. 2a. The direct path between the input and output defining the path product $K_{1i}K_{i2}$ b) The two possible direct paths between the input and output defining the products $K_{1i}K_{ij}K_{j2}$ and $K_{1j}K_{ji}K_{i2}$

path-products can be ordered to the two paths which are exactly equal to the two terms of Eq. (4). (Fig. 2b).

Expression (4) can be interpreted in the case of $Q_{ii} = \frac{\partial^2 K}{\partial (\ln x_i)^2}$ too, substituting $j = i$:

$$Q_{ii} = K_{1i} K_{ii} K_{i2} + K_{1i} K_{ii} K_{i2} = 2K_{1i} K_{ii} K_{i2} \tag{5}$$

Calculating the third-order semirelative sensitivity results in:

$$Q_{ijk} = \frac{\partial^3 K}{\partial \ln x_i \partial \ln x_j \partial \ln x_k} = K_{1i} K_{ij} K_{jk} K_{k2} + K_{1i} K_{ik} K_{kj} K_{j2} +$$
$$+ K_{1j} K_{jk} K_{ki} K_{i2} + K_{1j} K_{ji} K_{ik} K_{k2} + K_{1k} K_{ki} K_{ij} K_{j2} + K_{1k} K_{kj} K_{ji} K_{i2} \tag{6}$$

This equation contains six terms that define the six direct paths possible between input and output and the path-products (Fig. 3). All these lead to the conclusion that in the third-order case ($m = 3$) the semirelative sensitivity function contains $m! = 3! = 6$ terms containing all the direct paths between input and output supposed to touch all the $m = 3$ ports. It may also be concluded that a path-product contains $m + 1 = 4$ factors. This statement is true in general.

Theorem: The m-th order semirelative sensitivity function $Q_{12 \ldots m} = \frac{\partial^m K}{\partial \ln x_1 \partial \ln x_2 \ldots \partial \ln x_m}$ of an open circuit voltage transfer function $K = \frac{U_2}{U_1}$ with respect to R, L and C^{-1} parameters or arbitrary impedances is always expressible by the sum of $m!$ direct path-product. A path-product consists of $m + 1$ factors.

Proof: Suppose that our theorem is valid for the $m - 1$-th order sensitivity function, i.e. there are $(m - 1)!$ direct paths and a direct path defines a path-product consisting of m factors. Differentation is a linear operation so it can be derived by the terms. Deriving a term, i.e. a path product consisting of m factors, will yield m terms. There being $(m - 1)!$ path-products, after derivation the number of the terms will be $m(m - 1)! = m!$

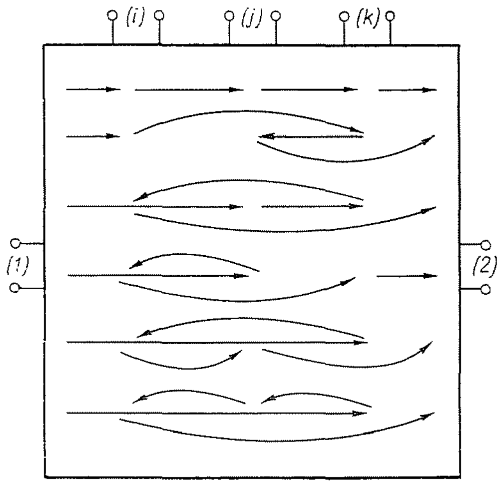

Fig. 3. The six possible direct paths appearing in calculation of the third-order semirelative sensitivity

It needs verification only that the $m!$ terms are really path-products and the number of the factors in a term is $m + 1$. Let us derive a direct path-product consisting of m factors:

$$\frac{\partial}{\partial \ln x} K_{1i} K_{ij} K_{jk} \ldots K_{m2} = \frac{\partial K_{1i}}{\partial \ln x} (K_{ij} K_{jk} \ldots K_{m2}) +$$
$$+ K_{1i} \frac{\partial K_{ij}}{\partial \ln x} (K_{jk} \ldots K_{m2}) + \ldots + K_{1i} K_{ij} K_{jk} \ldots K_{(m-1)m} \frac{\partial K_{m2}}{\partial \ln x} \quad (7)$$

The derivates in the right side of Eq. (7) are semirelative sensitivity functions, which can be calculated according to Eq. (1) and in general they are of the form $\frac{\partial K_{ij}}{\partial \ln x} = K_{ix} K_{xj}$. Substituting them into Eq. (7) the following formula will be received:

$$\frac{\partial}{\partial \ln x} K_{1i}K_{ij}K_{jk} \ldots K_{m2} = (K_{1x}K_{xi})K_{ij} \ldots K_{m2} +$$

$$+ K_{1i}(K_{ix}K_{xj})K_{jk} \ldots K_{m2} + \ldots + K_{1i}K_{ij}K_{jk} \ldots K_{mx}K_{x2} \tag{8}$$

It appears that the direct path derived $1 - i - j - k - \ldots - m - 2$ "fell to pieces", to longer paths, so as to give a product of two factors instead of one factor. But this does not alter the path characteristic of the original paths. The change is merely that there are $m + 1$ factois instead of m ones. Thereby the theorem is proved.

2. Higher-order sensitivity invariants

It is known that the sum of relative sensitivities with respect to different circuit parameters is invariant [2, 3, 7]:

$$\sum_{i=1}^{n} S_i^y = M \tag{9}$$

Investigating networks consisting of resistors (R), inductors (L), capacitors ($D = C^{-1}$), current-controlled voltage sources (transfer resistances r), gyrators (R_G) as well as transformers (a), impedance converters (k), voltage controlled voltage sources (μ), current controlled current sources (β) and operational amplifiers (A) it can be shown [3] that if the network characteristic y is a

1. voltage or current transfer function (K), the sensitivity sum is $M = 0$,
2. transfer (or driving point) admittance (Y), the sensitivity sum is $M = -1$, and
3. transfer (or driving point) impedance (Z), the sensitivity sum is $M = +1$.

The summation must be extended to all of the parameters R, L, D, r and R_G. In the summation n means the sum of the numbers of resistors, inductors, capacitors, current controlled voltage sources and gyrators. Using semirelative sensitivities only number M has to be multiplied by the network characteristic y to give the semirelative sensitivity sum:

$$\sum_{i=1}^{n} Q_i = My \tag{10}$$

Eq. (10) can be easily generalized for higher-order sensitivity sums.

The necessary second-order sensitivities can be drawn in a quadratic matrix:

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ Q_{21} & Q_{22} & \vdots & \vdots \\ \vdots & \cdots & Q_{ij} \cdots & \vdots \\ \vdots & & \vdots & \vdots \\ Q_{1n} & & \cdots & Q_{nn} \end{bmatrix}$$

The sum of all the matrix elements $\mathbf{Q}$ gives the second order semirelative sensitivity sum.

Deriving the first-order semirelative sensitivity sum with respect to $\ln x_j$:

$$\frac{\partial}{\partial \ln x_j} \sum_{i=1}^{n} Q_i = \sum_{i=1}^{n} Q_{ij} = \frac{\partial}{\partial \ln x_j} (My) = MQ_j \tag{11}$$

Thus, summarizing the elements of matrix $\mathbf{Q}$ in the j-th column will result in MQ_j. Summarizing all the columns of matrix $\mathbf{Q}$:

$$\sum_{j=1}^{n} \sum_{i=1}^{n} Q_{ij} = \sum_{j=1}^{n} MQ_j = M^2 y \tag{12}$$

The third-order semirelative sensitivity sum can be calculated in a similar manner:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} Q_{ijk} = M^3 y \tag{13}$$

Theorem: The sum of the m-th order semirelative sensitivities of a linear network consisting of parameters R, L, D, r, R_G, a, k, μ, β and A is

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_m=1}^{n} Q_{i_1, i_2, \ldots i_m} = M^m y \tag{14}$$

where n is the sum of the number of parameters R, L, D, r and R_G.

Proof: Let the above statement be valid for the $(m-1)$-th order, i.e.

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_{(m-1)}=1}^{n} Q_{i_1, i_2 \ldots i_{(m-1)}} = M^{m-1} y \tag{15}$$

Differentating Eq. (15) with respect to $\ln x_r$:

$$\frac{\partial}{\partial \ln x_r} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_{(m-1)}=1}^{n} Q_{i_1, i_2 \ldots i_{(m-1)}} = M^{m-1} \frac{\partial y}{\partial \ln x_r} = M^{m-1} Q_r \tag{16}$$

Summarizing Eq. (16) from $r = 1$ to $r = n$ and substituting $r = i_m$, the obtained formula will be exactly the Eq. (14).

Summary

In this paper it has been shown that the higher-order semirelative sensitivity functions of an open circuit voltage transfer function can always be calculated by the method using voltage transfer function, i.e. without derivation. It has been shown, too, that the first-order sensitivity invariants can be generalized to higher-order sensitivity invariants. These theorems may be used for high-speed calculation of higher-order sensitivities with an immediate check of the received results.

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