SOME REMARKS ON MODAL TRANSFORMATION

By

F. CSÁKI

Department of Automation, Technical University, Budapest

(Received February 24, 1972)

Recently, with the event of state-space methods, great many techniques have been elaborated to determine the canonical phase-variable form. Some techniques permit to determine other canonical forms, for example, the canonical form with explicit eigenvalues. The corresponding transformation is called modal transformation. A special case of the latter leads to the so-called LUR'E form, where all the elements of the input column matrix **l** are ones. This form has some advantages and is widely used in the stability testing method of LUR'E, as a special case of that of LYAPUNOV, in determining absolute stability. Modal forms are generally advantageous because they show the natural modes of dynamic systems.

In this paper some transformations will be shown, leading from the canonical phase-variable form to the modal form of LUR'E. Not only the case of distinct eigenvalues are considered but also the case of multiple eigenvalues will be examined.

Preliminary remarks

Let us start first from the transfer function

$$G(s) = \frac{K}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} . \tag{1}$$

As it is well known, by introducing phase variables, we may obtain the canonical phase-variable form as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u \tag{2}$$

$$y = [\,K, \quad 0, \quad 0, \quad \ldots, \quad 0, \quad 0\,]\ \mathbf{x}$$

or in short-hand notation

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}_0\,\mathbf{x} + \mathbf{b}_0\,u \\ y &= \mathbf{c}_0^{\mathrm{T}}\,\mathbf{x}\end{aligned} \tag{3}$$

The phase-variable form can also be obtained by many other techniques. Let us now introduce an appropriate linear transformation

$$\mathbf{x} = \mathbf{L}\,\mathbf{z}\,, \quad \mathbf{z} = \mathbf{L}^{-1}\,\mathbf{x} \tag{4}$$

By use of the latter and assuming distinct eigenvalues, we may obtain from Eq. (3) the canonical form sought for:

$$\begin{aligned}\mathbf{z} &= \mathbf{\Lambda}\,\mathbf{z} + \mathbf{1}\,u \\ y &= \mathbf{c}^{\mathrm{T}}\,\mathbf{z}\end{aligned} \tag{5}$$

where

$$\mathbf{\Lambda} = \mathbf{L}^{-1}\,\mathbf{A}_0\,\mathbf{L} = \mathbf{diag}\;[\lambda_1, \lambda_2, \ldots, \lambda_n] \tag{6}$$

and

$$\mathbf{1} = \mathbf{L}^{-1}\,\mathbf{b}_0 = [1, 1, \ldots, 1]^{\mathrm{T}}\,, \quad \mathbf{c}^{\mathrm{T}} = \mathbf{c}_0^{\mathrm{T}}\,\mathbf{L}\,. \tag{7}$$

It is well known that the nonsingular VANDERMONDE matrix, which is a special modal matrix,

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{bmatrix} = \mathbf{M} \tag{8}$$

has the peculiarity that it transforms the phase-variable system matrix $\mathbf{A}_0$ to the eigenvalue matrix

$$\mathbf{V}^{-1}\,\mathbf{A}_0\,\mathbf{V} = \mathbf{\Lambda} \tag{9}$$

though, Eq. (7) is not fulfilled:

$$\mathbf{V}^{-1}\,\mathbf{b}_0 = \mathbf{b} \neq \mathbf{1}. \tag{10}$$

It is always possible, however, to choose a *diagonal* transformation matrix $\mathbf{T}$ such that

$$\mathbf{T}^{-1}\,\mathbf{b} = \mathbf{T}^{-1}\,\mathbf{V}^{-1}\,\mathbf{b}_0 = \mathbf{1} \tag{11}$$

Therefore the appropriate transformation matrix is

$$\mathbf{L} = \mathbf{VT} = \mathbf{MT} \tag{12}$$

We conclude our treatise concerned with distinct eigenvalues by some supplementary remarks.

First, matrix $\mathbf{V}$ is a modal matrix $\mathbf{M}$ indeed, since the column matrices $\mathbf{v}_i$ of $\mathbf{V}$ are eigenvectors,

$$\mathbf{A}_0 \mathbf{v}_i = \mathbf{v}_i \lambda_i \tag{13}$$

or

$$\mathbf{A}_0 \mathbf{V} = \mathbf{V\Lambda} \tag{14}$$

The latter equation being valid Eq. (9) holds.

Second, by introducing the polynomials

$$P_i(\lambda) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j}; \quad (i = 1, 2, \ldots n) \tag{15}$$

the elements n_{ij} of the inverse VANDERMONDE matrix $\mathbf{V}^{-1} = \mathbf{M}^{-1}$ can be obtained from the expanded form

$$P_i(\lambda) = \sum_{j=1}^{n} n_{ij} \lambda^{j-1} \tag{16}$$

as the coefficients of the i-th polynomial, corresponding to λ^{j-1}. On the other hand, according to Eq. (10),

$$\mathbf{b} = [v_{1n}, v_{2n}, \ldots v_{nn}]^{\mathrm{T}} \tag{17}$$

as $\mathbf{b}_0 = [0, 0, \ldots, 1]^{\mathrm{T}}$. Therefore the vector $\mathbf{b}$ is equal to the last column vector of the inverse matrix $\mathbf{V}^{-1} = \mathbf{M}^{-1}$.

Another way of computing $\mathbf{b}$ is given by

$$v_i = v_{in} = \operatorname*{Res}_{\lambda=\lambda_i} \frac{1}{D(\lambda)} = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{\lambda_i - \lambda_j} = \left.\frac{\lambda - \lambda_i}{D(\lambda)}\right|_{\lambda=\lambda_i} \tag{18}$$

where

$$D(\lambda) = |\lambda \mathbf{I} - \mathbf{A}_0| = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1 \lambda + a_0 \tag{19}$$

that is, $D(\lambda)$ is the characteristic determinant equal also to the denominator of Eq. (1) after substitution.

Third, from Eq. (11) it follows that

$$\mathbf{T}^{-1} = \mathbf{diag}\left[\frac{1}{v_{1n}}, \frac{1}{v_{2n}}, \ldots, \frac{1}{v_{nn}}\right] = \mathbf{diag}\left[\frac{1}{b_1}, \frac{1}{b_2}, \ldots, \frac{1}{b_n}\right]$$

that is,

$$\mathbf{T}^{-1} = \mathbf{diag}\left[\prod_{j=2}^{n} (\lambda_1 - \lambda_j), \prod_{\substack{j=1 \\ j \neq 2}}^{n} (\lambda_2 - \lambda_j) \ldots, \prod_{\substack{j=1 \\ j \neq n}}^{n} (\lambda_n - \lambda_j)\right] \tag{20}$$

and

$$\mathbf{T} = \mathbf{diag}\,[v_{1n}, v_{2n}, \ldots v_{nn}] = \mathbf{diag}\,[b_1, b_2, \ldots, b_n]$$

that is,

$$\mathbf{T} = \mathbf{diag}\left[\prod_{j=2}^{n} \frac{1}{(\lambda_1 - \lambda_j)}, \prod_{\substack{j=1\\ j\neq 2}}^{n} \frac{1}{(\lambda_2 - \lambda_j)}, \ldots, \prod_{\substack{j=1\\ j\neq n}}^{n} \frac{1}{(\lambda_n - \lambda_j)}\right] \tag{21}$$

Finally, the canonical modal form of LUR'E can also be obtained by partial-fraction expansion. Let the LAPLACE-transform of the input be $U(s)$ and that of the output $Y(s)$, then from Eq. (1), by expanding into partial fractions, we obtain

$$Y_i(s) = \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i} U(s) \tag{22}$$

where λ_i $(i = 1, 2, \ldots, n)$ are the distinct roots of $D(s)$, i.e. the eigenvalues of $\mathbf{A}_0$ or the poles of $G(s)$ in Eq. (1). Let us introduce phase-variables by the LAPLACE-transforms

$$X_i(s) = \frac{1}{s - \lambda_i} U(s); \quad (i = 1, 2, \ldots, n) \tag{23}$$

By inverse transformation assuming zero initial conditions

$$\dot{x}_i = \lambda_i x_i + u; \quad (i = 1, 2, \ldots, n) \tag{24}$$

which is just the detailed form of the first equation of (5). Taking Eq. (23) into consideration, the inverse transformation of Eq. (22) yields

$$y = \sum_{i=1}^{n} R_i x_i = R_1 x_1 + R_2 x_2 + \ldots + R_n x_n \tag{25}$$

which is just the detailed form of the second equation of (5) with $c_i = R_i$.

An illustrative example

Let the phase-variable form be

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha\beta\gamma & -\alpha\beta-\beta\gamma-\gamma\alpha & -\alpha-\beta-\gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [\,K, \quad 0, \quad 0\,]\ \mathbf{x}.$$

According to Eq. (8), the VANDERMONDE matrix, that is, the modal matrix is in our case

$$\mathbf{V} = \mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ -\alpha & -\beta & -\gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{bmatrix}$$

Taking Eq. (15) into consideration:

$$P_1(\lambda) = \frac{\beta\gamma + (\beta+\gamma)\,\lambda + \lambda^2}{(\beta-\alpha)\,(\gamma-\alpha)}$$

$$P_2(\lambda) = \frac{\gamma\alpha + (\gamma+\alpha)\,\lambda + \lambda^2}{(\gamma-\beta)\,(\alpha-\beta)}$$

$$P_3(\lambda) = \frac{\alpha\beta + (\alpha+\beta)\,\lambda + \lambda^2}{(\alpha-\gamma)\,(\beta-\gamma)}\,.$$

Therefore, by Eq. (16) the inverse matrix is

$$\mathbf{V}^{-1} = \mathbf{M}^{-1} = \begin{bmatrix} \dfrac{\beta\gamma}{(\beta-\alpha)(\gamma-\alpha)} & \dfrac{\beta+\gamma}{(\beta-\alpha)(\gamma-\alpha)} & \dfrac{1}{(\beta-\alpha)(\gamma-\alpha)} \\ \dfrac{\gamma\alpha}{(\gamma-\beta)(\alpha-\beta)} & \dfrac{\gamma+\alpha}{(\gamma-\beta)(\alpha-\beta)} & \dfrac{1}{(\gamma-\beta)(\alpha-\beta)} \\ \dfrac{\alpha\beta}{(\alpha-\gamma)(\beta-\gamma)} & \dfrac{\alpha+\beta}{(\alpha-\gamma)(\beta-\gamma)} & \dfrac{1}{(\alpha-\gamma)(\beta-\gamma)} \end{bmatrix}.$$

Now, according to Eqs (20) and (21),

$$\mathbf{T}^{-1} = \mathbf{diag}\left[(\beta-\alpha)\;(\gamma-\alpha),\;(\gamma-\beta)\;(\alpha-\beta),\;(\alpha-\gamma)\;(\beta-\gamma)\right]$$

and

$$\mathbf{T} = \mathbf{diag}\left[\frac{1}{(\beta-\alpha)(\gamma-\alpha)},\;\frac{1}{(\gamma-\beta)(\alpha-\beta)},\;\frac{1}{(\alpha-\gamma)(\beta-\gamma)}\right]$$

respectively.

Multiple eigenvalues

Let us assume now that $G(s)$ in Eq. (1) has multiple poles, that is to say, $D(\lambda)$ in Eq. (19) has multiple roots and $\mathbf{A}_0$ in Eq. (3) has multiple eigenvalues.

We may determine an appropriate co-ordinate transformation of the character (4) by which a canonical modal form similar to Eq. (5), but not identical with it, can be obtained:

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{J}\,\mathbf{z} + \mathbf{l}\,u \\ \mathbf{y} &= \mathbf{c}^T\,\mathbf{z} \end{aligned} \tag{26}$$

where, by introducing the pseudodiagonal form,

$$\mathbf{J} = \mathbf{L}^{-1}\,\mathbf{A}_0\,\mathbf{L} = \mathbf{diag}\,[\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_m] \tag{27}$$

and

$$\mathbf{J}_i = \begin{bmatrix} \lambda_i & 1 & & & & & \\ & \lambda_i & 1 & & & & \\ & & \lambda_i & 1 & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & \cdot & \cdot \\ & & & & & \lambda_i & 1 \\ & & & & & & \lambda_i \end{bmatrix} \quad (i = 1, 2, \ldots, m)$$

The latter are called JORDAN blocks and $\mathbf{J}$ is called JORDAN matrix.

For the sake of simplicity let us first suppose that only the first eigenvalue has a multiplicity, let us say, ν, whereas the other eigenvalues are distinct. We introduce modified VANDERMONDE matrices,

$$V_2 = \begin{bmatrix} 1 & 0 & 1 & \ldots & 1 \\ \lambda_1 & 1 & \lambda_2 & \ldots & \lambda_{n-1} \\ \lambda_1^2 & 2\lambda_1 & \lambda_2^2 & \ldots & \lambda_{n-1}^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & (n-1)\lambda_1^{n-2} & \lambda_2^{n-1} & \ldots & \lambda_{n-1}^{n-1} \end{bmatrix} \tag{28}$$

or

$$V_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & \ldots & 1 \\ \lambda_1 & 1 & 0 & \lambda_2 & \ldots & \lambda_{n-2} \\ \lambda_1^2 & 2\lambda_1 & 1 & \lambda_2^2 & \ldots & \lambda_{n-2}^2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & (n-1)\lambda_1^{n-2} & \frac{1}{2}(n-1)(n-2)\lambda^{n-3} & \lambda_2^{n-1} & \ldots & \lambda_{n-2}^{n-1} \end{bmatrix} \tag{29}$$

and so on. By use of the appropriate matrix $\mathbf{V}_\nu$ we obtain

$$\mathbf{V}_\nu^{-1}\,\mathbf{A}_0\,\mathbf{V}_\nu = \mathbf{J} \tag{30}$$

where

$$\mathbf{J} = \mathbf{diag}\,[\mathbf{J}_1, \lambda_2, \lambda_3 \;\ldots\; \lambda_{n-\nu+1}]\,. \tag{31}$$

The modified matrices may also be expressed as

$$V_2 = [\mathbf{v}_1, \mathbf{v}_1', \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}] \tag{32}$$

and

$$V_3 = \left[\mathbf{v}_1, \mathbf{v}_1', \frac{1}{2}\mathbf{v}_1'', \mathbf{v}_2, \ldots, \mathbf{v}_{n-2}\right] \tag{33}$$

and so on. If we had m multiple eigenvalues, then the modified VANDERMONDE matrix had the form

$$\mathbf{V}_{(m)} = [\mathbf{v}_1, \mathbf{v}_1' \ldots, \mathbf{v}_2, \mathbf{v}_2', \ldots, \ldots, \mathbf{v}_m, \mathbf{v}_m' \ldots] \tag{34}$$

and $\mathbf{J}$ could be expressed as

$$\mathbf{J} = \mathbf{diag}\,[\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_m] \tag{35}$$

We emphasize that Eq. (7) is not satisfied and Eq. (10) holds. In case of multiple eigenvalues the transformation matrix $\mathbf{T}$ in Eq. (11) is not a diagonal matrix any more but an *upper triagonal matrix*, and so is also his inverse.

It is obvious that the commutativity condition

$$\mathbf{T}^{-1}\,\mathbf{JT} = \mathbf{J} \quad \text{or} \quad \mathbf{JT} = \mathbf{TJ} \tag{36}$$

holds. Using the latter equation, adjusting the diagonal elements to unity and employing a trial-and-error method for the other triangle elements over the diagonal, $\mathbf{T}$ and $\mathbf{T}^{-1}$ can finally be determined in accordance with Eq. (11). Then Eq. (11) comes true and the whole transformation matrix is given in Eq. (12).

As concerns multiple eigenvalues, let us make some further remarks. Previously, in Eqs (15) and (16) a computational method was shown, simply delivering the inverse of the VANDERMONDE matrix. For multiple eigenvalues this method is to be modified. Let us suppose that we have only one eigenvalue λ_1 with multiplicity n. For this case we have

$$P_n(\lambda) = (\lambda - \lambda_1)^{n-1} = \sum_{j=1}^{n} n_{nj}\,\lambda^{j-1} \tag{37}$$

and the corresponding coefficients yield the elements of the last row of the inverse matrix. The elements of the $(n-1)$th row are obtained from

$$P_{n-1}(\lambda) = \frac{1}{n-1}\,\frac{d}{d\lambda}\,P_n(\lambda) = (\lambda - \lambda_1)^{n-2} = \sum_{j=1}^{n-1} n_{n-1,j}\,\lambda^{j-1}.$$

This procedure can be continued to yield

$$P_i(\lambda) = (\lambda - \lambda_1)^{i-1} = \sum_{j=1}^{i} n_{ij}\,\lambda^{j-1} \qquad (i = 1, 2, \ldots, n) \tag{38}$$

The computation of the inverse modified VANDERMONDE matrix is more complicated when besides the multiple eigenvalue also distinct eigenvalues or separate multiple eigenvalues are present.

In this case the inverse matrix can be obtained by matrix inversion or by applying the identity

$$\mathbf{V}_m^{-1}\mathbf{A}_0 = \mathbf{J}\mathbf{V}_m^{-1}, \qquad \mathbf{M}^{-1}\mathbf{A}_0 = \mathbf{J}\mathbf{M}^{-1} \tag{39}$$

and using a trial-and-error procedure.

Denoting the row vectors of the inverse matrix by $\mathbf{n}_j^{\mathrm{T}}$, for a certain JORDAN block $\mathbf{J}_i$ of order $\nu_i = \nu$, according to (39) we may write

$$\begin{aligned} \mathbf{n}_\nu^{\mathrm{T}}\ \mathbf{A}_0 &= \lambda_i\,\mathbf{n}_\nu^{\mathrm{T}} \\ \mathbf{n}_{\nu-1}^{\mathrm{T}}\ \mathbf{A}_0 &= \lambda_i\,\mathbf{n}_{\nu-1}^{\mathrm{T}} + \mathbf{n}_\nu^{\mathrm{T}} \\ &\ \vdots \\ \mathbf{n}_1^{\mathrm{T}}\ \mathbf{A}_0 &= \lambda_i\,\mathbf{n}_1^{\mathrm{T}} + \mathbf{n}_2^{\mathrm{T}}\,. \end{aligned} \tag{40}$$

Both the matrix inversion and the trial-and-error procedure are facilitated by applying (37) and (15), (16) for the computation of certain row vectors.

Of course, the canonical form of LUR'E can be obtained by partial-fraction expansion also in the case of multiple eigenvalues.

Modal transformations with multiple eigenvalues

For the case in which multiple eigenvalues arise and matrix $\mathbf{A}$ is nonsymmetric as, for example, $\mathbf{A}_0$, the determination of the number of independent column vectors of the modal matrix is not so simple. The reason for the ambiguity is that there is no unique correspondence between the order of a multiple root of the characteristic equation $D(\lambda) = 0$ and the degeneracy of the corresponding characteristic matrix $[\lambda_i\mathbf{I} - \mathbf{A}]$.

If, let us say, λ_i is a multiple root of order p_i, the degeneracy of the characteristic matrix cannot be greater than p_i, and the dimension of the associated vector space spanned by the corresponding modal vectors $\mathbf{m}_i$ is not greater than p_i. The problem is more complicated if the order of multiple root λ_i is p_i and the degeneracy q_i of $[\lambda_i\,\mathbf{I} - \mathbf{A}]$ is less than p_i. In this case only $q_i < p_i$ linearly independent solutions can be found for the characteristic equation

$$[\lambda_i\mathbf{I} - \mathbf{A}]\,\mathbf{m}_i = \mathbf{0} \qquad (i = 1, 2, \ldots, m) \tag{41}$$

The dimension of the associated vector space for the $\mathbf{m}_i$ is less than p_i, and no p_i linearly independent characteristic vectors corresponding to λ_i can be obtained. Only in case of a *symmetric* $n \times n$ matrix $\mathbf{A}$ is the degeneracy of $[\lambda_i \mathbf{I} - \mathbf{A}]$ definitely equal to p_i for a p_i-fold root, so that p_i linearly independent eigenvectors can be found.

For the case where the degeneracy of $[\lambda_i \mathbf{I} - \mathbf{A}]$ is equal to one, that is to say, for a simple degeneracy, the modal column vector can be chosen to be proportional to any non-zero column of $\mathbf{Adj}\ [\lambda_i \mathbf{I} - \mathbf{A}]$. This is the only column vector that can be obtained for the set of p_i equal roots. The other additional vectors which are necessary in constructing the transformation matrix have to be determined by some other method (see below).

For the case where the degeneracy of $[\lambda_i \mathbf{I} - \mathbf{A}]$ is equal to $q_i > 1$, $\mathbf{Adj}\ [\lambda_i \mathbf{I} - \mathbf{A}]$ and all its derivatives, up to and including

$$\left\{\frac{d^{q_i-2}}{d\lambda^{q_i-2}} \mathbf{Adj}\, [\lambda \mathbf{I} - \mathbf{A}]\right\}\Bigg|_{\lambda=\lambda_i}, \tag{42}$$

are zero matrices. The q_i linearly distinct solutions for the modal column vectors can be obtained from the column vectors of differentiated adjoint matrices which are non-zero ones. For example, in case of full degeneracy, $q_i = p_i$, the p_i linearly independent modal column vectors can be obtained from the non-zero columns of

$$\left\{\frac{d^{p_i-1}}{d\lambda^{p_i-1}} \mathbf{Adj}\ [\lambda \mathbf{I} - \mathbf{A}]\right\}\Bigg|_{\lambda=\lambda_i}. \tag{43}$$

All the above remarks are concerned with general matrices $\mathbf{A}$. In case of special phase-variable matrices $\mathbf{A}_0$, as in Eq. (2), the rank of $[\lambda_i \mathbf{I} - \mathbf{A}_0]$ is always $r = n - 1$, that is, the degeneracy q_0 is always one, and therefore there are as many Jordan blocks as there are separate multiple eigenvalues, and all the superdiagonal elements in each Jordan block are unity.

Now, let us consider the determination of additional column vectors. Let the column vectors of a transformation matrix, that is, of the modified modal matrix $\mathbf{M}$, be denoted by $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n$. In the canonical system matrix $\mathbf{J}$ there is a Jordan block of order ν_i associated with λ_i if and only if the $\nu = \nu_i$ linearly independent $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_\nu$ column vectors satisfy the equations

$$\begin{aligned} \mathbf{A}_0 \mathbf{m}_1 &= \mathbf{m}_1 \lambda_i \\ \mathbf{A}_0 \mathbf{m}_2 &= \mathbf{m}_2 \lambda_i + \mathbf{m}_1 \\ &\vdots \\ \mathbf{A}_0 \mathbf{m}_\nu &= \mathbf{m}_\nu \lambda_i + \mathbf{m}_{\nu-1} \end{aligned} \tag{44}$$

the detailed form of

$$\mathbf{A}_0 \mathbf{M} = \mathbf{M}\mathbf{J}_i \tag{45}$$

based on Eq. (27).

We emphasize that the reason of choosing the modified VANDERMONDE matrices in form given by Eqs (28), (29), (32), (33), (34) is that these matrices do satisfy Eq. (44).

Two illustrative examples

First example

Let us start from the phase variable form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\alpha^3 & -3\alpha^2 & -3\alpha\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y=[K, \quad 0, \quad 0]\ \mathbf{x}$$

According to Eq. (29), the modified modal matrix $\mathbf{M}$ and its inverse $\mathbf{M}^{-1}$ can be expressed as

$$\mathbf{M}=\begin{bmatrix}1 & 0 & 0\\ -\alpha & 1 & 0\\ \alpha^2 & -2\alpha & 1\end{bmatrix}=\mathbf{V}_3;\quad \mathbf{M}^{-1}=\begin{bmatrix}1 & 0 & 0\\ \alpha & 1 & 0\\ \alpha^2 & 2\alpha & 1\end{bmatrix}=\mathbf{V}_3^{-1}$$

The column vectors of $\mathbf{M}=\mathbf{V}_3$ satisfy Eq. (44), and the row vectors of $\mathbf{M}^{-1}=\mathbf{V}_3{}^{-1}$ satisfy Eq. (40). Eq. (30) is also fulfilled. It is also seen that the last row of the inverse matrix can be taken from Eq. (37), the second row from Eq. (38) with $i=2$, and so on. The appropriate triangular transformation matrix $\mathbf{T}$ and its inverse matrix $\mathbf{T}^{-1}$ are obtained as

$$\mathbf{T}=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix};\quad \mathbf{T}^{-1}=\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

Eqs (36) and (11) are satisfied indeed. According to Eq. (12), the whole transformation matrix $\mathbf{L}$ and its inverse $\mathbf{L}^{-1}$ can be expressed as

$$\mathbf{L}=\begin{bmatrix}1 & -1 & 0\\ -\alpha & 1+\alpha & -1\\ \alpha^2 & -2\alpha-\alpha^2 & 1+2\alpha\end{bmatrix};\quad \mathbf{L}^{-1}=\begin{bmatrix}1+\alpha+\alpha^2 & 1+2\alpha & 1\\ \alpha+\alpha^2 & 1+2\alpha & 1\\ \alpha^2 & 2\alpha & 1\end{bmatrix}$$

and this leads to the final form

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix}=\begin{bmatrix}-\alpha & 1 & 0\\ 0 & -\alpha & 1\\ 0 & 0 & -\alpha\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}+\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u$$

$$y=[K, \quad -K, \quad 0].\ z$$

For the sake of comparison we solve the problem by expanding in partial fractions, too. The transfer function can be expressed now as

$$G(s) = \frac{K}{(s+\alpha)^3}$$

that is directly in partial fraction form. Let us introduce canonical variables by the LAPLACE transformed relations:

$$Z_1(s) = \frac{1}{s+\alpha}U(s) + \frac{1}{(s+\alpha)^2}U(s) + \frac{1}{(s+\alpha)^3}U(s)$$

$$Z_2(s) = \frac{1}{s+\alpha}U(s) + \frac{1}{(s+\alpha)^2}U(s)$$

$$Z_3(s) = \frac{1}{s+\alpha}U(s).$$

Inverse transformation yields exactly the previous canonical form.

Second example

Let the phase-variable form given as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\alpha^2\beta & -2\alpha\beta-\alpha^2 & -2\alpha-\beta\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

$$y = [\;K, \quad 0, \quad 0\;]\,\mathbf{x}$$

According to (28), the modified modal matrix and its inverse matrix can be expressed as

$$\mathbf{M} = \begin{bmatrix}1 & 0 & 1\\ -\alpha & 1 & -\beta\\ \alpha^2 & -2\alpha & \beta^2\end{bmatrix} = \mathbf{V}_2;\quad \mathbf{M}^{-1} = \frac{1}{(\beta-\alpha)^2}\begin{bmatrix}\beta^2-2\alpha\beta & -2\alpha & -1\\ \alpha\beta^2-\alpha^2\beta & \beta^2-\alpha^2 & \beta-\alpha\\ \alpha^2 & 2\alpha & 1\end{bmatrix}$$

Applying Eq. (36), the triangular transformation matrix $\mathbf{T}$ can be obtained as

$$\mathbf{T} = \begin{bmatrix}\frac{1}{\beta-\alpha} & \alpha-\beta+1 & 0\\ 0 & \frac{1}{\beta-\alpha} & 0\\ 0 & 0 & \frac{1-(\beta-\alpha)^2+\beta-\alpha}{\alpha-\beta}\end{bmatrix}$$

whereas its inverse matrix can be expressed as

$$\mathbf{T}^{-1}=\begin{bmatrix} \beta-\alpha & (\beta-\alpha-1)(\beta-\alpha)^2 & 0 \\ 0 & \beta-\alpha & 0 \\ 0 & 0 & \dfrac{\alpha-\beta}{1-(\beta-\alpha)^2+\beta-\alpha} \end{bmatrix}.$$

These matrices are seen to have an intermediate form as compared with the corresponding matrices in the previous two examples.

The inverse modal matrix $\mathbf{M}^{-1}$ has been determined by matrix inversion technique, which is relatively simple for $n \leq 3$. We remark, however, that the last row of $\mathbf{M}^{-1}$ could also be obtained from the coefficients of

$$P_3(\lambda) = \frac{(\lambda+\alpha)^2}{(\alpha-\beta)^2} = \frac{\alpha^2+2\alpha\lambda+\lambda^2}{(\beta-\alpha)^2}$$

the latter being just the expression which can be derived from Eq. (15) by substituting $\lambda_1 = -\alpha$, $\lambda_2 = -\alpha$ and $\lambda_3 = -\beta$.

Furthermore, the elements of the second row could be obtained from

$$P_2(\lambda) = (\lambda+\alpha)\frac{\lambda+\beta}{\beta-\alpha} = \frac{\alpha\beta(\beta-\alpha)+(\beta^2-\alpha^2)\lambda+(\beta-\alpha)\lambda^2}{(\beta-\alpha)^2}.$$

We remark that $P_2(\lambda)$ does not follow any more directly from Eq. (15) but it is a mixed expression based partly on the first relationship in Eq. (38) and partly on Eq. (15).

Finally, the first row of $\mathbf{M}^{-1}$ may be obtained from the coefficients of

$$P_1(\lambda) = \frac{(\beta-2\alpha-\lambda)(\lambda+\beta)}{(\beta-\alpha)^2} = \frac{(\beta^2-2\alpha\beta)-2\alpha\lambda-\lambda^2}{(\beta-\alpha)^2}$$

but the construction rule of the latter polynomial is not quite obvious.

Problems with numerator dynamics

The outlined procedure of obtaining Lur'e forms can also be applied to problems where numerator dynamics are also available. If the transfer function is

$$G(s) = \frac{c_{n-1}s^{n-1}+\ldots+c_1 s+c_0}{s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

then Eq. (2) is further valid, the only difference being that now

$$y = [c_0, c_1, c_2, \ldots, c_{n-2}, c_{n-1}]\,\mathbf{x} = \mathbf{c}_\iota^{\mathrm{T}}\mathbf{x}.$$

As $\mathbf{A}_0$ and $\mathbf{b}_0$ are the same as before, the procedure remains unaltered, and only $\mathbf{c}^{\mathrm{T}} = \mathbf{c}_0^{\mathrm{T}}\mathbf{L}$ will result in a somewhat different form.

Conclusions

The canonical forms with explicite eigenvalues are very useful because they constitute the base for modal analysis of dynamic systems. One of the most important forms is the LUR'E form.

In this paper a method has been proposed for obtaining the LUR'E form when starting from the phase-variable form.

In case of distinct eigenvalues, the original VANDERMONDE matrix, as given in Eq. (8), can be applied. Since Eq. (10) holds, the introduction of a further diagonal transformation matrix $\mathbf{T}$ becomes necessary. $\mathbf{T}$ has to satisfy Eq. (11). The complete transformation matrix is then given in Eq. (12). The elements of the inverse VANDERMONDE matrix can be obtained from Eqs (15) and (16). In case of distinct eigenvalues, $\mathbf{T}$ and $\mathbf{T}^{-1}$ can easily be expressed by Eqs (21) and (20), respectively. Of course, the desired canonical form can also be obtained by expanding the transfer function into partial fractions, introducing appropriate canonical variables, and inverse LAPLACE-transformation.

In case of multiple eigenvalues the problem becomes more complicated. Instead of $\mathbf{\Lambda}$ we have now a JORDAN matrix $\mathbf{J}$, as given in Eq. (27). The VANDERMONDE matrix is to be modified as shown by Eqs (28), (29), (30), (31) or more generally by Eq. (34). Eqs (44) and (45) may also be employed. In this case the transformation matrix $\mathbf{T}$ satisfying condition (11) is not a diagonal matrix any more but becomes an upper triangular matrix together with its inverse matrix. The computation of the inverse modified VANDERMONDE matrix is simple only in the case of a single multiple eigenvalue, when Eq. (38) can be applied. If there are more than one multiple eigenvalue or besides the multiple eigenvalue there are also distinct eigenvalues the computation of the inverse modified VANDERMONDE matrix becomes more or less complicated. In this case Eqs (39) and (40) can be used. The desired canonical form can be obtained through LAPLACE transformation technique, for multiple eigenvalues, however, also this method becomes somewhat complicated.

Summary

The peculiarities of the modal transformation leading to the LUR'E form are discussed. Together with the systems of distinct eigenvalues the systems of multiple eigenvalues are also treated. Some examples serve as illustrations.

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Prof. Dr. Frigyes Csáki, 1052 Budapest, XI., Garami E. tér 3, Hungary