A MODERN METHOD HANDLING ONLY REAL QUANTITIES FOR STATE TRANSITION MATRIX DETERMINATION

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In the case of real eigenvalues, the state transition matrix is simple to be determined from the canonical form by real-number arithmetics. When complex or multiple complex eigenvalues are involved, then the quasi-canonical form and transformation described in the present paper or the methods suggested in [1] and [2] are advised. The advantage of the transformation presented in the following is that it relies on the facilities of modern mathematics, linear algebra and matrix analysis expressing results in a compact economical form.

The ultimate formulae in [1, 2] and in the present paper agree and require only real-number arithmetics.

State transition matrix determination through the canonical form

Let us consider the differential equation of a one variable section of constant parameter:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t) \tag{1}$$

where $\mathbf{x}$ is the state vector (an $n \times 1$ dimension matrix), $\mathbf{b}$ is a column vector (an $n \times 1$ dimension matrix) and $u(t)$ is a scalar. If the eigenvalues of the matrix $\mathbf{A}$ of $n \times n$ are not different, then the so-called JORDAN-type canonical form [3, 4]

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{m_1} & & & & & \mathbf{0} \\ & \mathbf{J}_{m_2} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ \mathbf{0} & & & & & \mathbf{J}_{m_\nu} \end{bmatrix} \tag{2}$$

can be produced by a similarity-transformation, where

$$\mathbf{J}_{m_i} = \begin{bmatrix} s_i & 1 & 0 & . & . & 0 & 0 \\ 0 & s_i & 1 & . & . & 0 & 0 \\ . & . & . & & & . & . \\ . & . & . & & & . & . \\ . & . & . & . & . & s_i & 1 \\ 0 & 0 & 0 & . & . & 0 & s_i \end{bmatrix}$$

is the so-called JORDAN partial matrix of dimension $m_i \times m_i$ and

$$\sum_{i=1}^{\nu} m_i = n \tag{4}$$

The JORDAN partial matrix $\mathbf{J}_{m_i}$ can be evolved also as the sum of a diagonal and a nilpotent matrix by the relationship

$$\mathbf{J}_{m_i} = s_i \mathbf{I} + \mathbf{H}_{m_i} \tag{5}$$

with $\mathbf{I}$ being the unit matrix of dimension $m_i \times m_i$. Considering that

$$\mathbf{H}_{m_i}^{m_i} = \mathbf{0} \tag{6}$$

and with the similarity transformation matrix $\mathbf{L}$ of $n \times n$ the relationship for the state transition matrix can be written up directly as

$$\mathbf{\Phi}(\Delta t) = e^{\Delta t \mathbf{J}} = e^{\Delta t \mathbf{L}^{-1} \mathbf{A} \mathbf{L}} = \begin{bmatrix} e^{\Delta t \mathbf{J}_m} & & & & \\ & e^{\Delta t \mathbf{J}_m} & & & \\ & & . & & \\ & & & . & \\ & & & & . \\ & & & & & e^{\Delta t \mathbf{J}_{m_\nu}} \end{bmatrix} \tag{7}$$

where

$$e^{\Delta t \mathbf{J} m_i} = e^{s_i \Delta t} \begin{bmatrix} 1 & \Delta t & \frac{\Delta t^2}{2!} & \dots & \frac{\Delta t^{m_i - 1}}{(m_i - 2)!} \\ 0 & 1 & \Delta t & \dots & \frac{\Delta t^{m_i - 2}}{(m_i - 2)!} \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} . \tag{8}$$

In the case of real and multiple eigenvalues the state transition matrix $\mathbf{\Phi}(\Delta t)$ can be calculated according to relationships (7) and (8) by real-number arithmetics [4].

The calculation of the state transition matrix in the case of complex eigenvalues using the quasi-canonical form

Let us introduce the notations

$$\mathbf{S}_i = \begin{bmatrix} s_i & 0 \\ 0 & \hat{s}_i \end{bmatrix} \tag{9}$$

and

$$\mathbf{E}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{10}$$

By regarding a conjugate complex pair of eigenvalues as a hyper-eigenvalue a quasi-canonical form can be defined where the hyper-eigenvalue $\mathbf{S}_i$ defined by the formula (9) will appear r times if a conjugate complex pair of eigenvalues $(s_i, \hat{s}_i)$ occurs r times in the main diagonal. So, according to formula (3) the partial matrix of dimension $2m_i \times 2m_i$ can be written as

$$\mathbf{J}^*_{m_i} = \begin{bmatrix} \mathbf{S}_i & \mathbf{E}_2 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_i & \mathbf{E}_2 & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \dots & \mathbf{S}_i & \mathbf{E}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \cdot & \mathbf{S}_i \end{bmatrix} =$$

$$= \begin{bmatrix} s_i & 0 & 1 & & & & & \mathbf{0} \\ & \hat{s}_i & 0 & 1 & & & & \\ & & s_i & 0 & \cdot & & & \\ & & & \hat{s}_i & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & \cdot & \cdot & 1 \\ & & & & & & s_i & 0 \\ \mathbf{0} & & & & & & & \hat{s}_i \end{bmatrix} \tag{11}$$

or in the form:

$$\mathbf{J}^*_{m_i} = \mathbf{diag}\,[\mathbf{S}_i, \mathbf{S}_i, \mathbf{S}_i, \dots, \mathbf{S}_i]\,\mathbf{I} + \mathbf{H}^*_{m_i} \tag{12}$$

where

$$\mathbf{H}^*_{m_i} = \begin{bmatrix} \mathbf{0} & \mathbf{E}_2 & & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{E}_2 & & & \\ & & \mathbf{0} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & \cdot & \mathbf{E}_2 \\ & & & & & \mathbf{0} \end{bmatrix}. \tag{13}$$

Now, $\mathbf{H}^*_{m_i}$ is the nilpotent matrix of dimension $2m_i \times 2m_i$, i.e. the equality

$$(\mathbf{H}^*_{m_i})^{m_i} = \mathbf{0} \tag{14}$$

is met.

In the following, our consideration may be restricted to the case of a conjugate complex pair of eigenvalues appearing r times. As it was mentioned in the introduction, the calculation is to involve nothing but real-number arithmetics. For this purpose an appropriate transformation must be applied. From the formulae

$$\mathbf{K}_2 = \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

$$\mathbf{K}_2^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ j/2 & -j/2 \end{bmatrix} \tag{15}$$

and $s_i = \sigma_i + j\omega_i$
the relationship

$$\mathbf{P}_i = \mathbf{K}_2^{-1}\ \mathbf{S}_i\,\mathbf{K}_2 = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \tag{16}$$

is easily justified. It appears that our aim is best approached by the linear transformation

$$\mathbf{K} = \mathbf{diag}\,[\mathbf{K}_2, \ldots, \mathbf{K}_2, \ldots, \mathbf{K}_2] \tag{17}$$

The introduction of this transformation means that the partial matrix $\mathbf{J}^*_{m_i}$

$$\mathbf{J}^{**}_{m_i} = \mathbf{K}^{-1}\,\mathbf{diag}\,[\mathbf{S}_i, \mathbf{S}_i, \ldots, \mathbf{S}_i]\,\mathbf{K} + \mathbf{H}^*_{m_i} \tag{18}$$

is to be substituted by the partial matrix $\mathbf{J}^{**}_{m_i}$, where

$$\mathbf{K}^{-1}\,\mathbf{H}^*_{m_i}\,\mathbf{K} = \mathbf{H}^*_{m_i} \tag{19}$$

For supplying final results we must demonstrate how quantities $e^{\Delta t \mathbf{P}_i}$ are calculated. It is obvious that

$$\mathbf{P}_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} = \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} + \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} \tag{20}$$

Therefore

$$e^{\mathbf{P}_i} = e^{\sigma_i \mathbf{I}} \, e^{\begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}} = e^{\sigma_i} e^{\mathbf{B}_i}. \tag{21}$$

$\mathbf{B}_i$ is a skew symmetrical matrix diagonalized by the unitary matrix

$$\mathbf{U} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -j/\sqrt{2} & j/\sqrt{2} \end{bmatrix}. \tag{22}$$

$\mathbf{U}^{-1}\mathbf{B}_i \mathbf{U}$ being the main diagonal, the correctness of the relationship

$$e^{\mathbf{B}_i} = \mathbf{U} \, e^{\mathbf{U}^{-1}\mathbf{B}_i\mathbf{U}} \, \mathbf{U}^{-1} \tag{23}$$

is easily admitted leading to

$$\mathbf{D}_i = e^{\Delta t \mathbf{B}_i} = \mathbf{U} \begin{bmatrix} e^{-j\omega_i \Delta t} & 0 \\ 0 & e^{j\omega_i \Delta t} \end{bmatrix} \mathbf{U}^{-1} = \begin{bmatrix} \cos\Theta_i & \sin\Theta_i \\ -\sin\Theta_i & \cos\Theta_i \end{bmatrix} \tag{24}$$

with $\Theta_i = \omega_i \Delta t$

In the final issue — in the case of a complex, multiple eigenvalue — relationship (8) is to be replaced by the formula

$$e^{\Delta t \mathbf{J}^{**}_{m_i}} = e^{\sigma_i \Delta t} \begin{bmatrix} \mathbf{D}_i & \Delta t\, \mathbf{D}_i & \dots & \frac{\Delta t^{m_i - 1}}{(m_i - 1)!} \mathbf{D}_i \\ & \mathbf{D}_i & \dots & \frac{\Delta t^{m_i - 2}}{(m_i - 2)!} \mathbf{D}_i \\ & & & \vdots \\ \mathbf{0} & & & \mathbf{D}_i \end{bmatrix} \tag{25}$$

Example

Let us consider a system with a one-fold conjugate complex pair of poles:

$$s_1 = \sigma_1 + j\omega_1 \text{ and } \hat{s}_1 = \sigma_1 - j\omega_1.$$

The transfer function $Y(s)$ of the system decomposed into partial fractions will have the form:

$$Y(s) = \frac{1}{s-\sigma_1-j\omega_1} + \frac{1}{s-\sigma_1+j\omega_1} . \tag{26}$$

Denoting the LAPLACE transforms of the input and output signals of the system by $U(s)$ and $V(s)$, respectively, and choosing the state variable transfer functions according to the relationships

$$\begin{aligned} \frac{X_1(s)}{U(s)} &= \frac{1}{s-s_1} \\ \frac{X_2(s)}{U(s)} &= \frac{1}{s-\hat{s}_1} \end{aligned} \tag{27}$$

the LAPLACE transform of the output signal can be written in the form:

$$V(s) = c_1X_1(s) + \hat{c}_1X_2(s) \tag{28}$$

If in the calculations are to be avoided complex arithmetics, then the state variable transfer functions must be chosen in a different way. In the present example the decomposition of the transfer function $Y(s)$:

$$Y(s) = \frac{(c_1+\hat{c}_1)\,(s-\sigma_1)}{(s-\sigma_1)^2+\omega_1^2} - \frac{j(\hat{c}_1-c_1)\,\omega_1}{(s-\sigma_1)^2+\omega_1^2} \tag{29}$$

i.e. the choice of the state variable transfer functions according to

$$\begin{aligned} \frac{Z_1(s)}{U(s)} &= \frac{s-\sigma_1}{(s-\sigma_1)^2+\omega_1^2} \\ \frac{Z_2(s)}{U(s)} &= \frac{-\omega_1}{(s-\sigma_1)^2+\omega_1^2} \end{aligned} \tag{30}$$

permits the LAPLACE-transform of the output signal to be calculated by the formula

$$V(s) = (c_1 + \hat{c}_1)Z_1(s) + j(\hat{c}_1 - c_1)Z_2(s) \tag{31}$$

The choice of the state variable transfer functions according to (30) is seen to permit to calculate exclusively with real numbers.

For sake of comparison let us examine now how the canonical form is influenced by the linear transformation $\mathbf{x} = \mathbf{K}_2\mathbf{z}$ in the tested case. With the considered conjugate complex pair of eigenvalues the canonical equations develops according to the pair of formulae

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} s_1 & 0 \\ 0 & \hat{s}_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \qquad y = [c_1 \, \hat{c}_1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{32}$$

Introducing the linear transformation $\mathbf{x} = \mathbf{K}_2\mathbf{z}$ we obtain:

$$\begin{aligned} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} \sigma_1 & \omega_1 \\ -\omega_1 & \sigma_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= [c_1 \, \hat{c}_1] \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \\ &= [(c_1 + \hat{c}_1) \;\; j(c_1 - \hat{c}_1)] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \end{aligned} \tag{33}$$

For the LAPLACE-transforms of the state variables z_1, z_2 in these equations we can write up on the basis of (33) the following relationships:

$$\begin{aligned} sZ_1(s) &= \sigma_1 Z_1(s) + \omega_1 Z_2(s) + U(s) \\ sZ_2(s) &= -\omega_1 Z_1(s) + \sigma_1 Z_2(s) \end{aligned} \tag{34}$$

From these the expressions for $Z_1(s)$ and $Z_2(s)$ are derived as:

$$\begin{aligned} Z_1(s) &= \frac{s - \sigma_1}{(s - \sigma_1)^2 + \omega_1^2} U(s), \\ Z_2(s) &= \frac{-\omega_1}{(s - \sigma_1)^2 + \omega_1^2} U(s) \end{aligned} \tag{35}$$

respectively. Relationships (30) and (35) immediately appear to be identical, i.e. it can be stated that the linear transformation used in the example is equivalent with the choice of the state variable transfer functions according to (30) (i.e. requiring only real-number arithmetics). A purposeful choice of the state variable transfer (weighting-) functions applies to the general case as well (see [1, 2]), just as the equivalence of the discussed transformation **K**.

Summary

An economical, modern method for the determination of the transition matrix is based on real-number arithmetics. The use of the suggested quasi-canonical form and linear transformation permits computations storing a minimum number of parameters, i.e. high accuracy derivation of the state transition matrix at a minimum number of operations. Finally, the equivalence between the proposed transformation of the quasi-canonical form and a parallel decomposition is illustrated by the example of a conjugate complex pair of eigenvalues.

References

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