STABILITY DEGREE ANALYSIS OF LINEAR CONTROL SYSTEMS WITH DEAD TIME BY A DIGITAL COMPUTER

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In previous papers [6, 7] we have studied the stability region variation permitting to reach arbitrary phase margins in the case of the linear control system with dead time shown in Fig. 1, compensated in series by proportional (P), integral (I) and proportional-integral (PI)-type compensation elements, respectively. In this paper diagrams showing the variation of the stability region versus the phase margin, the dead time and the time constants of the system compensated by a proportional-differential (PD) element are presented along with a procedure written in the ALGOL language for the determination of the stability region permitting to reach arbitrary phase margins of the control system shown in Fig. 1, compensated by P, PI, PD and PID-type compensation elements, respectively.

Proportional-differential control

The transfer function of the controller (Fig. 1) is as follows:

$$Y_1(s) = K(1 + T_d s).$$

The transcendent equation derived from the transfer function of the open loop used to determine the angular frequency permitting to reach arbitrary phase margins is:

$$-\tan^{-1}\omega T_d + \omega\tau + \tan^{-1}\frac{2\zeta T\omega}{1-T^2\omega^2} = \pi - \varphi'. \tag{1}$$

It is easily seen that the transcendent equation (1) may be obtained from the relationship

$$-\frac{\pi}{2} - \omega\tau - \tan^{-1}\frac{2\zeta T\omega}{1-T^2\omega^2} + \tan^{-1}\frac{\omega T_i}{1-T_i T_d\omega^2} = -\pi + \varphi' \tag{2}$$

involving a PID-type compensation, by substituting $1/T_i = 0$. Our statement is fulfilled if:

$$\frac{\pi}{2} + \tan^{-1} \omega T_d = -\tan^{-1} \frac{1}{\omega T_d} .$$

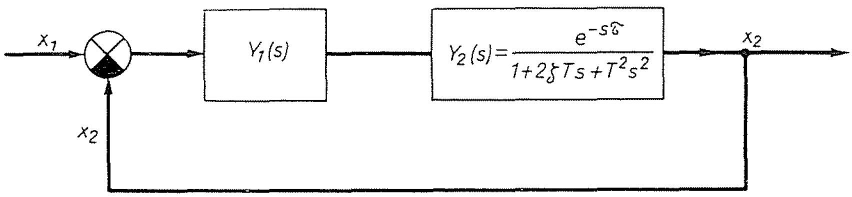

Fig. 1

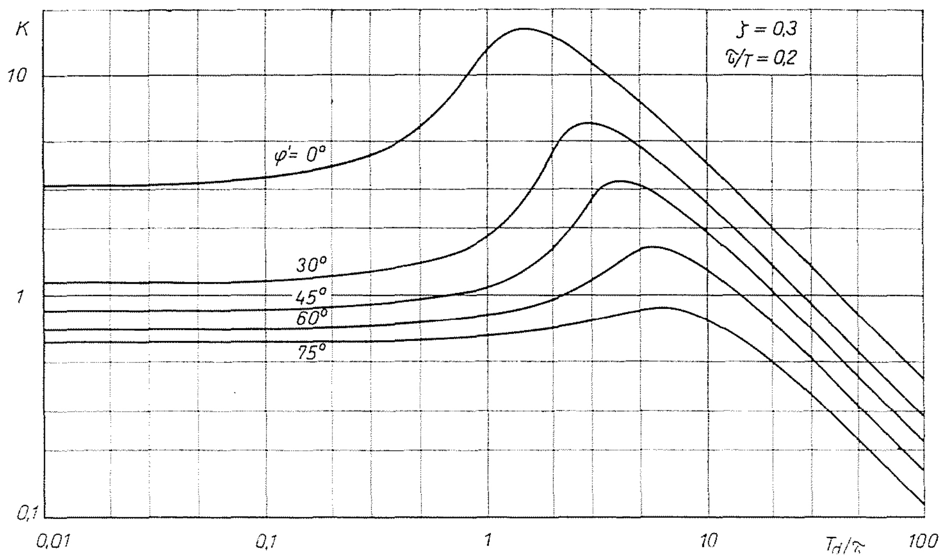

Fig. 2

Taking the tangents of both sides and after arranging we have:

$$-\frac{1}{\omega T_d} = -\frac{1}{\omega T_d} ,$$

in accordance with our above statement.

The above statement is useful in developing the program for the determination of the stability region variation for P, PI, PD and PID-type controls.

By substituting $1/T_i = 0$, the limit position of the stability region may be calculated from the following equality:

$$K = \frac{\sqrt{(1 - T^2 \omega^2)^2 + (2 \zeta T \omega)^2}}{\sqrt{(1/T_i - T_d \omega^2)^2 + \omega^2}} \tag{3}$$

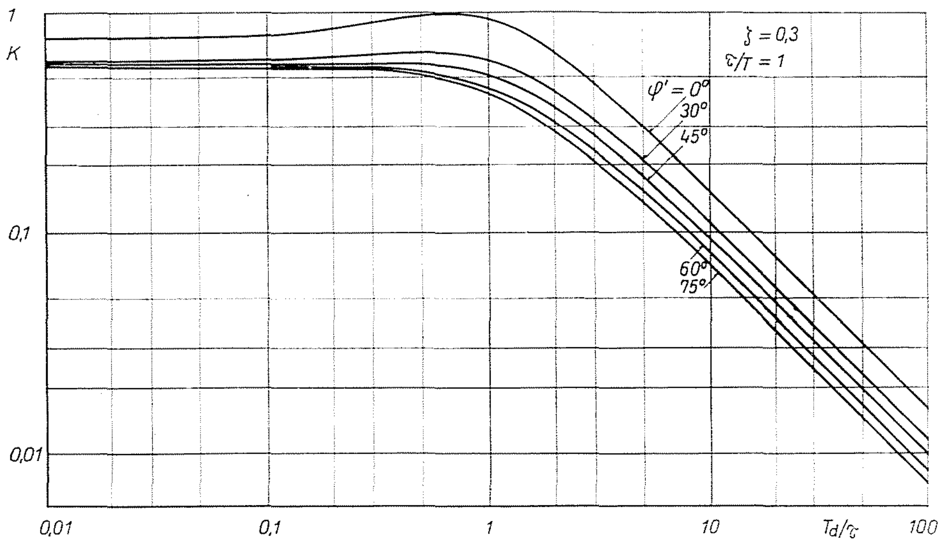
1
K
0,1
0,01
0,01
0,1
1
10
Td/τ
100
ζ = 0,3
τ/T = 1
φ' = 0°
30°
45°
60°
75°

Fig. 3

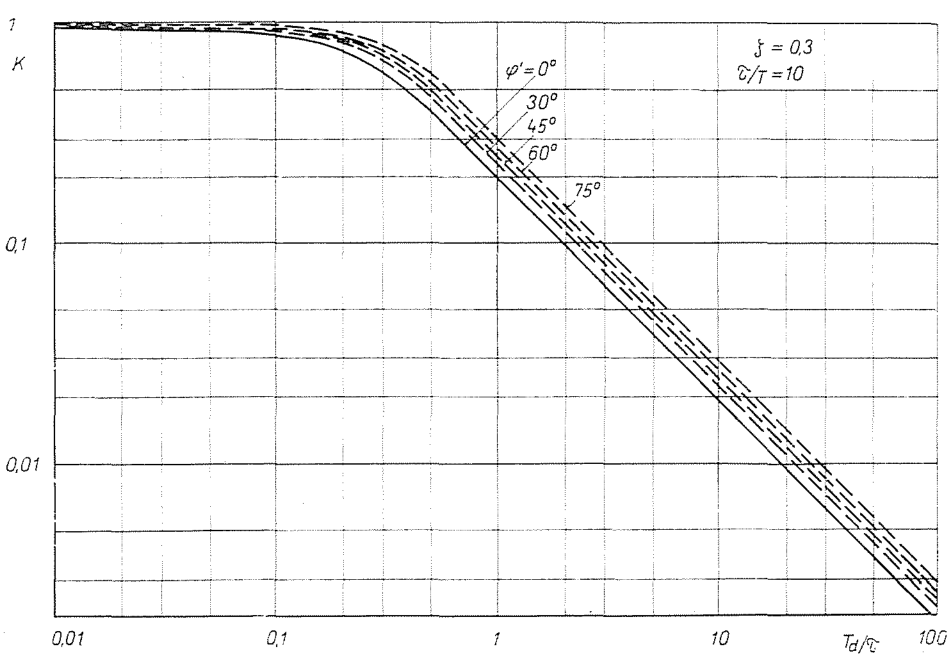
1
K
0,1
0,01
0,01
0,1
1
10
Td/τ
100
ζ = 0,3
τ/T = 10
φ' = 0°
30°
45°
60°
75°

Fig. 4

involving a PID-type compensation, with the angular frequency value obtained by the numerical evaluation of the relationship (1) with arbitrary accuracy.

Diagrams 2 through 16 show the functionality $K = K(T_d/\tau)$ determined with the help of a digital computer for the phase margin values $\varphi' = 0°, 30°, 45°, 60°, 75°$, with the time constants of the plant as parameters.

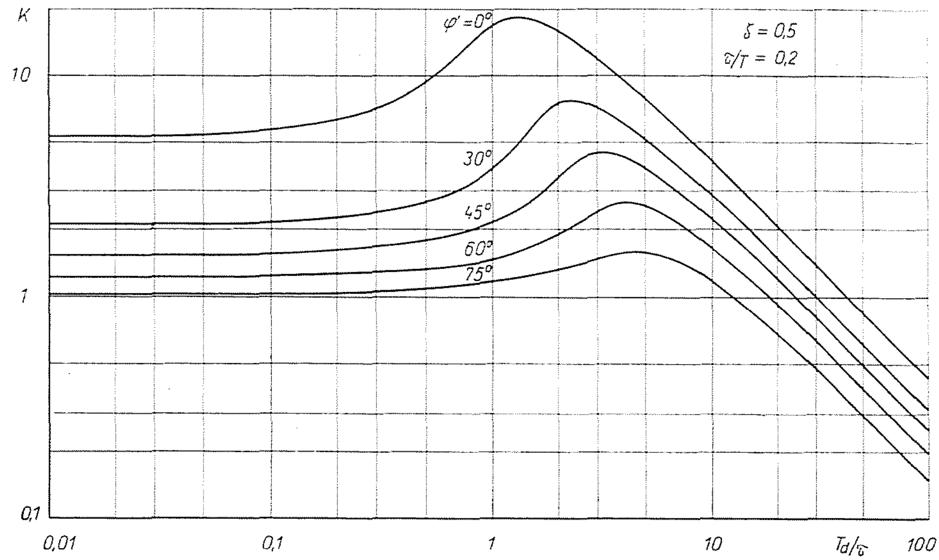

Fig. 5

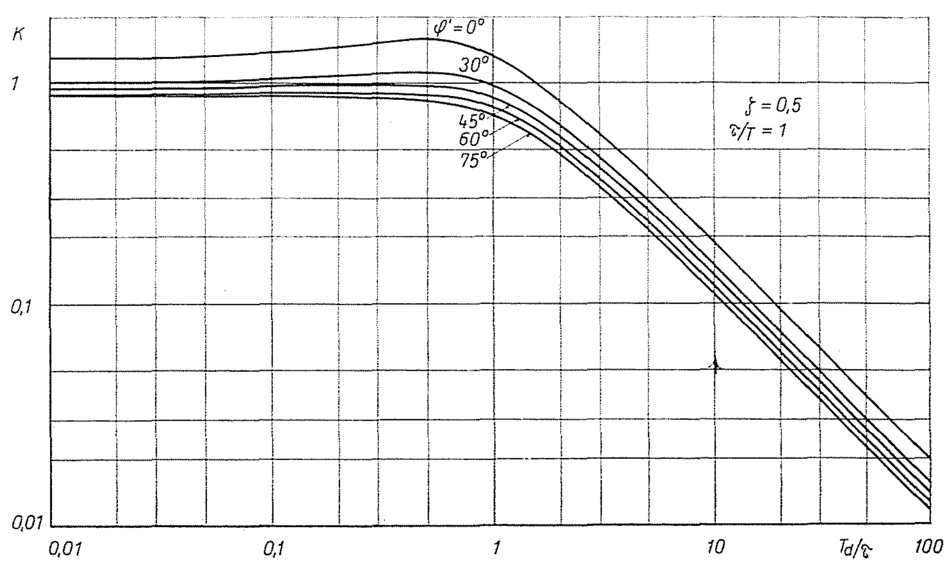

Fig. 6

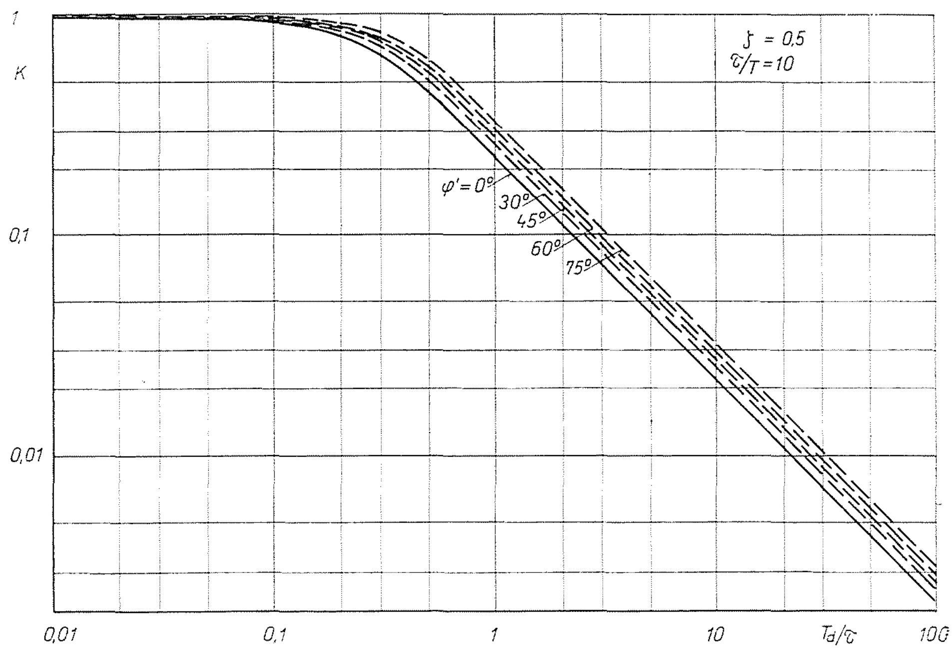
1
K
0,1
0,01
ζ = 0,5
τ/T = 10
φ' = 0°
30°
45°
60°
75°
0,01
0,1
1
10
T_d/τ
100

Fig. 7

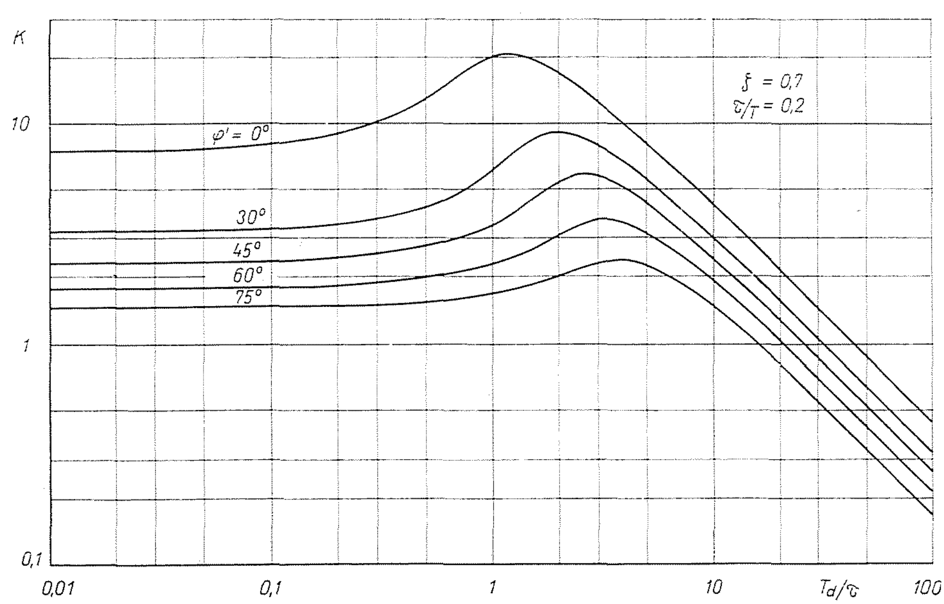
K
10
1
0,1
ζ = 0,7
τ/T = 0,2
φ' = 0°
30°
45°
60°
75°
0,01
0,1
1
10
T_d/τ
100

Fig. 8

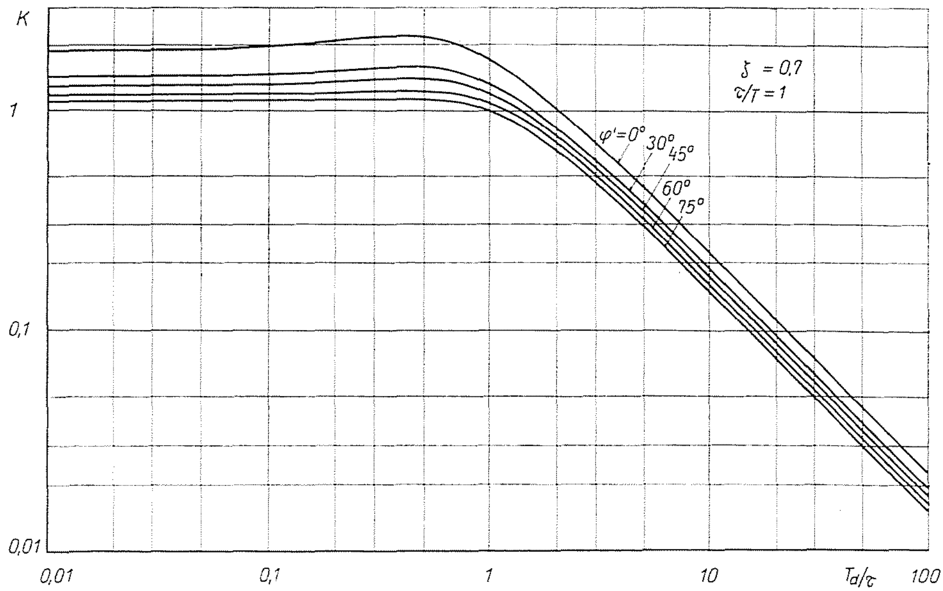
K
1
0,1
0,01
0,01
0,1
1
10
Td/τ
100
ζ = 0,7
τ/T = 1
φ' = 0°
30°
45°
60°
75°

Fig. 9

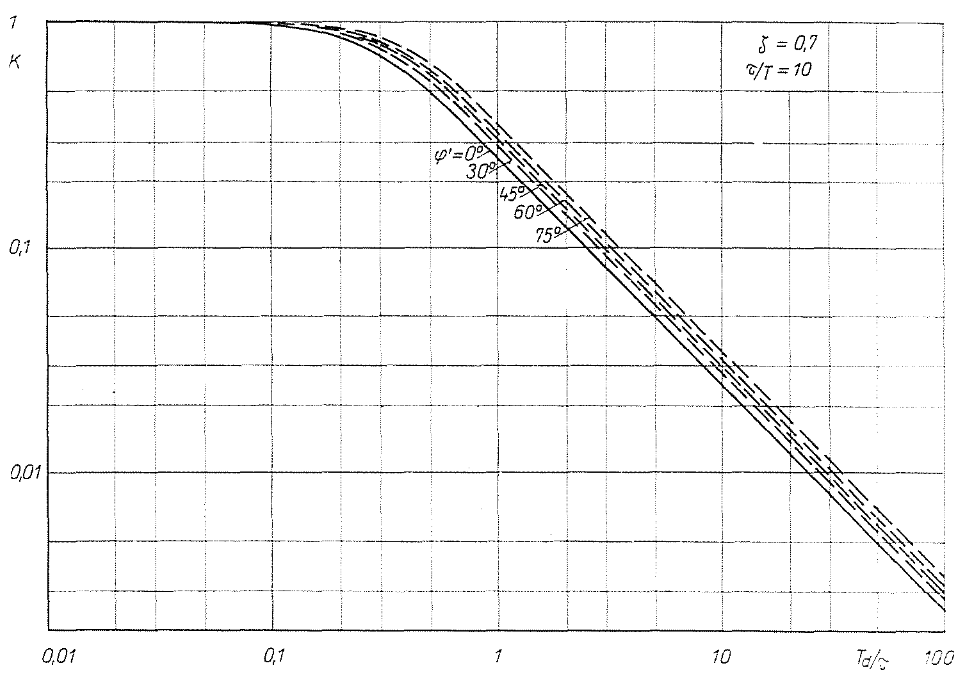
1
K
0,1
0,01
0,01
0,1
1
10
Td/τ
100
ζ = 0,7
τ/T = 10
φ' = 0°
30°
45°
60°
75°

Fig. 10

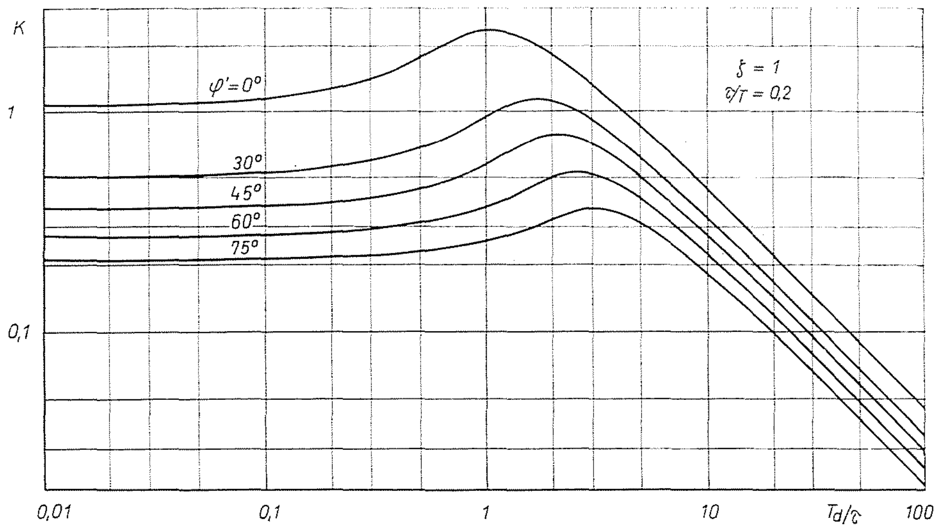

Fig. 11

From the diagrams the regularities described below follow:

a) For $T_d/\tau \to 0$ and simultaneously $\tau/T \to \infty$, the controller may be substituted by a proportional element and the plant by a pure dead time element. Accordingly, the limit position of the stability region approaches 1.

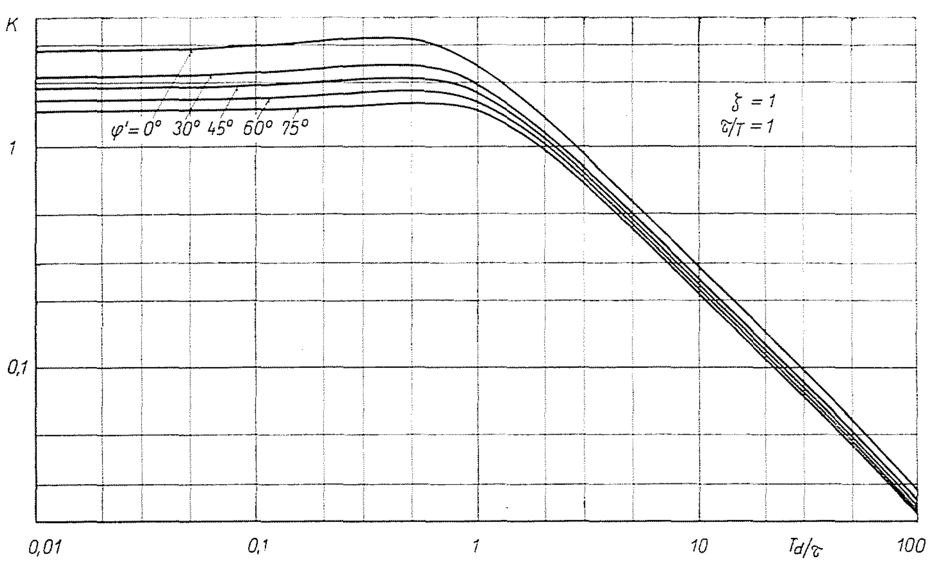

Fig. 12

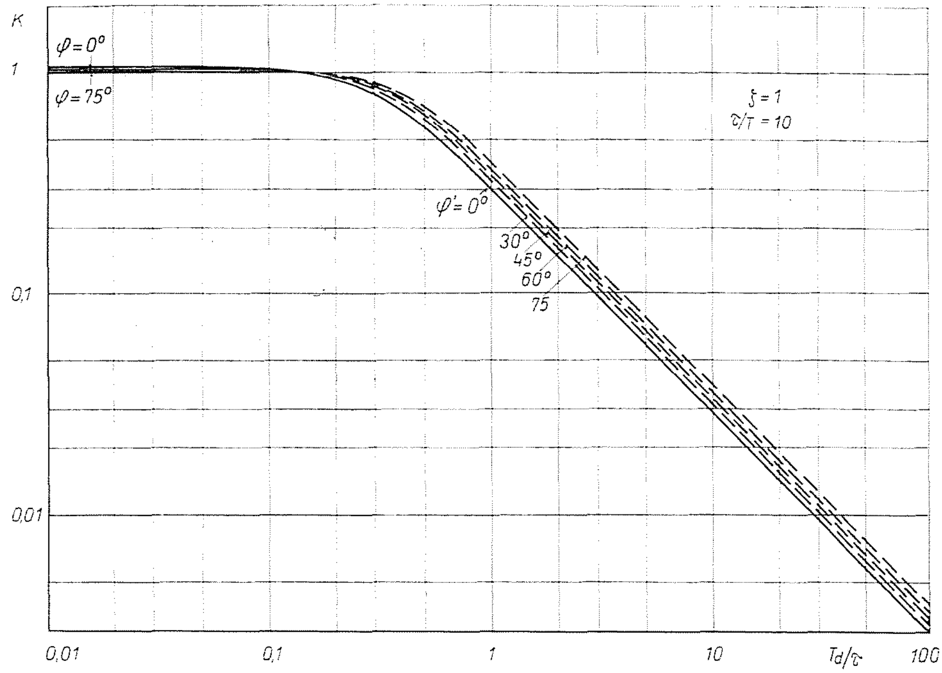

Fig. 13

b) When the dead time is lower than, or at the most equals the time constant of the second order lag, then the loop gain shows a maximum in the range of $0.1 \lessapprox T_d/\tau \lessapprox 10$.

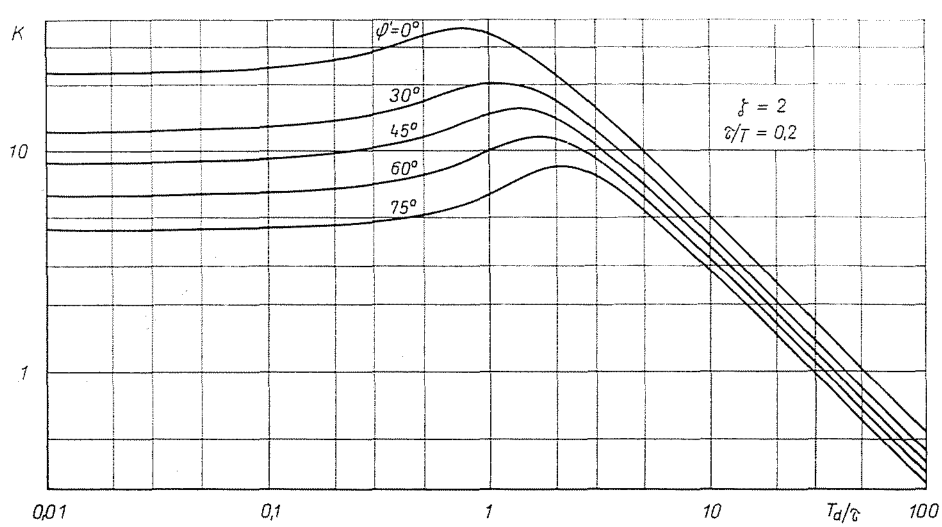

Fig. 14

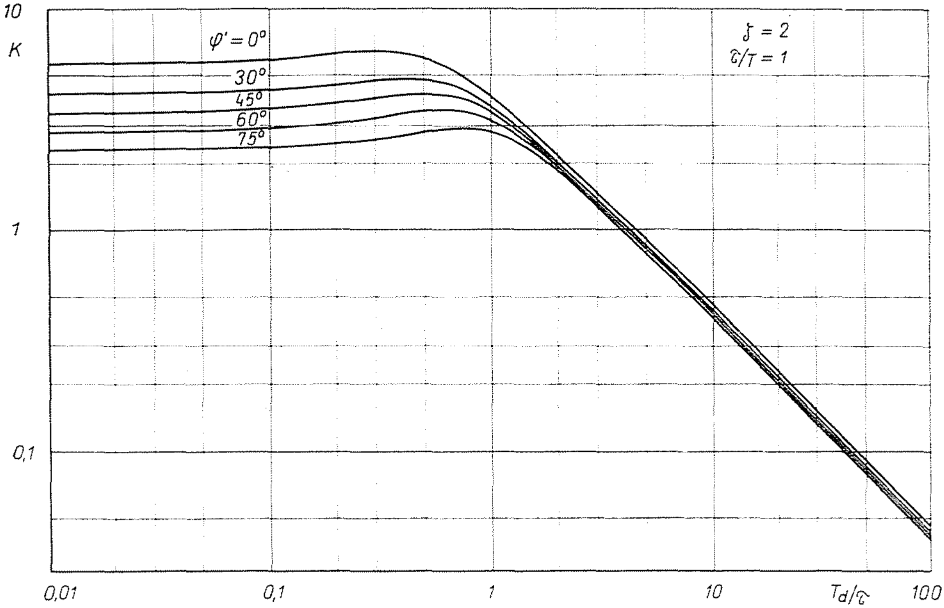
10
K
φ' = 0°
30°
45°
60°
75°
ζ = 2
τ/T = 1
1
0,1
0,01
0,1
1
10
Td/τ
100

Fig. 15

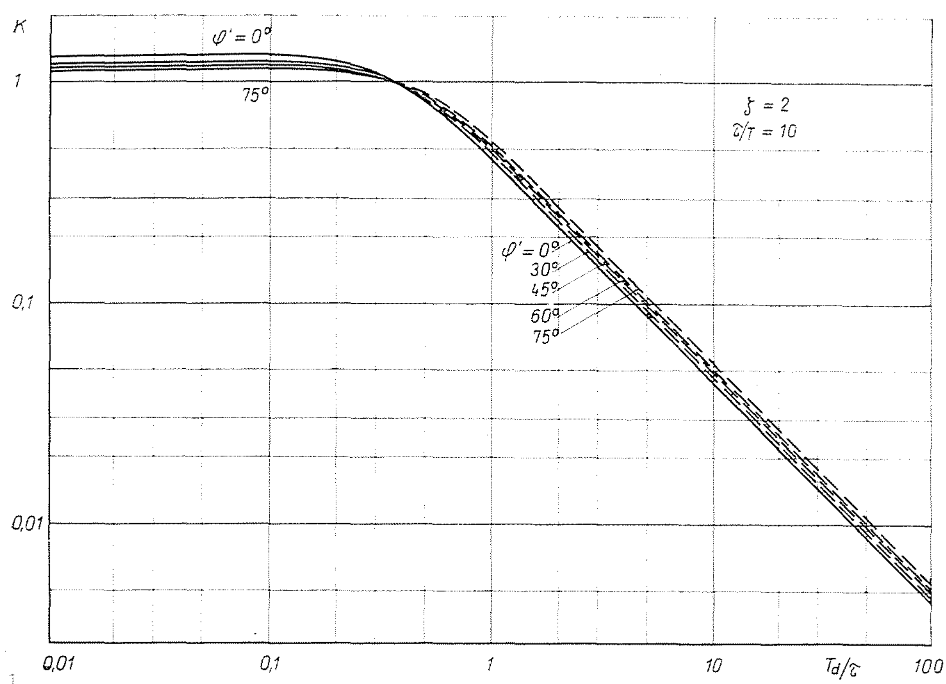
K
φ' = 0°
1
75°
ζ = 2
τ/T = 10
φ' = 0°
30°
45°
60°
75°
0,1
0,01
0,01
0,1
1
10
Td/τ
100

Fig. 16

c) For $T_d/\tau \to \infty$ the controller operates as a differential element. The stability region variation may be characterized in this case by the relationship

$$K\frac{T_d}{\tau} = \frac{1}{s\tau}\,\frac{1 + 2\,\zeta(T/\tau)\,(s\tau) + (T/\tau)^2\,(s\tau)^2}{\exp(-s\tau)} = K\frac{T_d}{\tau}\left(\frac{T}{\tau};\zeta\right). \tag{4}$$

For high T_d/τ values the functionality $K = K(T_d/\tau)$ with φ', τ/T, ζ as parameters is linear, in accordance with (4). The significance of the above statement is merely theoretical, as no pure differential elements are used as controllers in practice.

Proportional-integral-differential control

The transfer function of the controller in the case of a PID-type compensation is as follows:

$$Y_1(s) = K\left(1 + \frac{1}{T_i\,s} + T_d\,s\right). \tag{5}$$

The limit of the stability region permitting to reach arbitrary phase margins may be determined by relationship (3) after the evaluation of (2) with any arbitrary accuracy.

For the proportional-integral-differential control the number of the parameters is more by one than either for the PI-, or the PD-controls. Therefore the number of the diagrams representing the stability region variation for various time constant values increases considerably as compared to the PI, or PD-type compensations. So in the case of the PID control the loop gain values belonging to the arbitrary phase margins will not be plotted with the time constants of the system as parameters.

However we specify the procedure written in the ALGOL programming language and supplied with a PID identifier suitable to determine the stability region of P, PI, PD and PID controls for any arbitrary time constant and phase margin. For selecting the series compensation element of the linear control with dead time shown in Fig. 1 the diagrams published in [5] may also prove satisfactory in many cases. These diagrams show the critical loop gain ($\varphi' = 0°$) versus the time constants of the system. If the stability region is to be determined in the case of $\varphi' > 0°$ as well, this can be easily done by activating the PROCEDURE PID given in the APPENDIX. By this means the variation of the loop gain permitting to reach arbitrary phase margins of the control system under investigation may be easily produced also in the case of a PID compensation for various T_i and T_d parameter values of the controller besides of the fixed T, ζ, τ parameters of the plant.

Description of the PROCEDURE PID

The PROCEDURE serves for the determination of the stability region permitting to reach arbitrary phase margins of the linear, one-loop control system with dead time shown in Fig. 1. In the general case the controller may be a PID element with the same transfer function as under (5). In the special case of the PID control the proportional, proportional-integral and proportion-

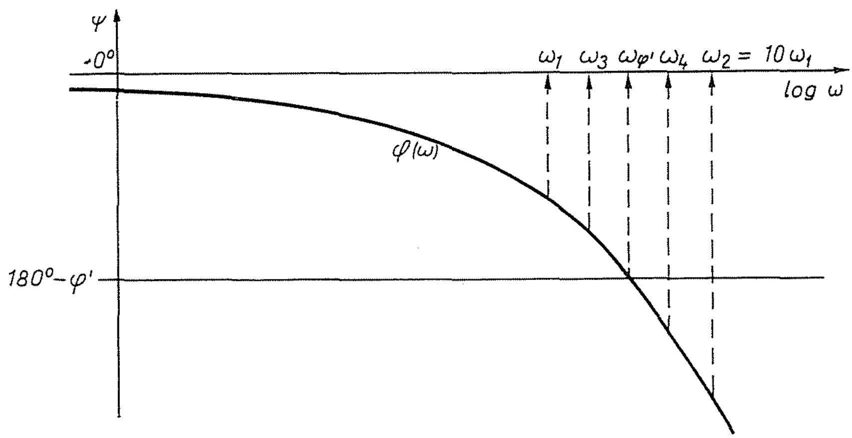

Fig. 17

al-differential compensated loop gain values may be calculated according to Table I.

For the sake of tractability the PROCEDURE PID contains several PROCEDURE-s and REAL PROCEDURE-s with the additional advantage that when the plant contains more than two time lags, then the procedure can be easily adjusted, by a slight modification of the main bodies of the PROCE-

Table I

Controller	$1/T_i$	T_d
P	0	0
PI	variable	0
PD	0	variable
PID	variable	variable

DURE-s, FI, DFI and PID, to make it suitable for the determination of the stability of control systems with dead time containing more than two time lags. But as such systems are rarely met with in practice, or even if control systems of this type occur sometimes, a result of satisfactory accuracy can be obtained in most cases by substituting the plant by second order lag, the above adjustment is seldom needed. The calculation of the angular frequency giving the limit position of the stability region is based on the considerations as follows.

Let us trace the resultant phase-angular frequency diagram of the open loop of the control system shown in Fig. 1. The character of the curve $\varphi(\omega)$ is mostly as exhibited in Fig. 17. The angular frequency permitting to reach an arbitrary phase margin satisfying the equation $\varphi(\omega_{\varphi'}) = 180° - \varphi'$ is evaluated in the following steps:

Starting from an arbitrarily low angular frequency of $\omega_0 = V \cdot Q$ and assuming that $V = 10$, we find the frequency decade $[\omega_1, \omega_2]$ within which the required $\omega_{\varphi'}$ drops (PROCEDURE FIND). As to the value of Q, it is advisable to choose it between .0001 and .001.

In the second step we reduce the frequency decade by the method of bisection until the deviation of the phase margin from the required value of the function is under a not too low error limit specified in advance (PROCEDURE HALF).

When the conditions

$$|\varphi(\omega_3) - (180° - \varphi')| < \text{EPS}$$

and

$$|\varphi(\omega_4) - (180° - \varphi')| < \text{EPS}$$

respectively, are satisfied, the required angular frequency $\omega_{\varphi'}$ can be evaluated with arbitrary accuracy (DELT) by the NEWTON–RAPHSON iteration method (REAL PROCEDURE NEWTON). In the knowledge of this angular frequency the PROCEDURE PID supplies the value of the loop gain belonging to the given phase margin.

The REAL PROCEDURE FI and REAL PROCEDURE DFI produce the phase angle and its derivative, respectively.

In the interest of machine time economy the program has been developed in a way that first it should be run for the maximum phase margin, then the calculations are to be iterated with ever decreasing φ' values. If the critical loop gain is to be determined for one phase margin only, then the corresponding phase margin represents at the same time the maximum phase margin (FIMX).

The PROCEDURE PID is presented in the APPENDIX. For activating the procedure the programmer must know the meanings of the formal parameters appearing in it. These are:

ZETA	ζ
B	T_d/τ
A	τ/T
TAU	τ
FIPM	φ'
FIMX	φ' maximum

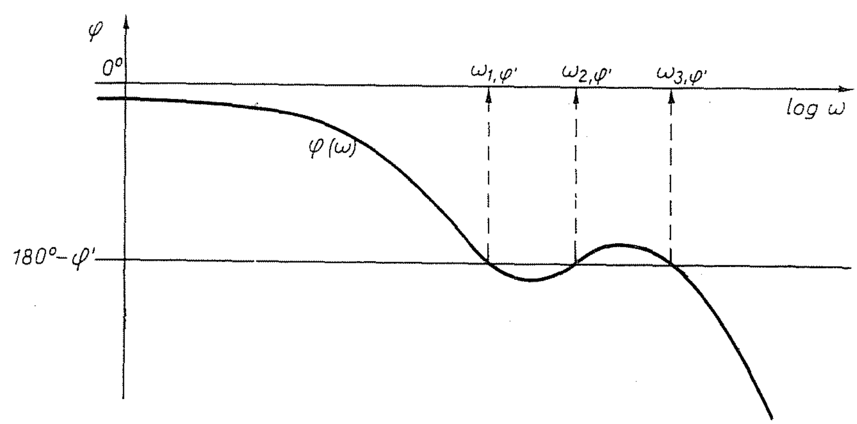

Fig. 18

The above parameters are input parameters. Their value must be specified before the PID procedure is activated. Also the values $C = 1/T_i$, Q, EPS and DELT must be specified before running the program. The role of Q has been mentioned already. By slightly varying EPS and DELT, the running time of the program may exhibit considerable deviations. If very strict EPS $>$ DELT error limits are imposed, then the solution will be convergent, but the running time may turn out to be very long, whereas if the EPS is chosen too high, a divergent solution might be obtained. The choice of the EPS value is decided by the time constants of the control system for which the course of the stability region is to be determined. The recommended values are: EPS $\approx 0.1-0.2$, DELT $\approx 10^{-4}$.

The course of the phase-angular frequency characteristic curve is not monotonously decreasing in every case. For instance, the introduction of the differentiation effect may result in the course of $\varphi(\omega)$ as traced in Fig. 18. In such cases it is advised to choose a lower EPS value.

The PROCEDURE PID does not contain any part-program for the determination of the loop gains belonging to each of the angular frequencies $\omega_{1,\varphi'}$, $\omega_{2,\varphi'}$ and $\omega_{3,\varphi'}$ respectively, in Fig. 18. The introduction of such a part-program would multiply the running time. But this deficiency has no specific significance, as the angular frequencies $\omega_{2,\varphi'}$ and $\omega_{3,\varphi'}$ refer to the conditionally stable region permitting to reach the phase margin $-\varphi'$ [1]. For dimensioning purposes it is advisable to use only the $K = K(\omega_{1,\varphi'})$ value in the calculation.

Neither does the procedure supplied with a PID identifier guarantee the production of the loop gain $K = K(\omega_{1,\varphi'})$ in all cases. But this is of no specific significance either, as this occurs only in a short τ/T_i interval for the increasing values of the differential time constant, as seen by the diagrams $K_{cr} = K_{cr}(\tau/T_i)$ traced for $\varphi' = 0°$ [5]. If the variation of the angular frequency in this short interval τ/T_i is not univocal, then it is advised to repeat the calculation for the involved time constants with reduced V and EPS values in PROCEDURE FIND and PROCEDURE HALF respectively, as needed.

Conclusions

The frequency region analysis of linear control systems with dead time imposed itself as little consideration of this problem is found in the literature so far, due to the difficulties in the calculation. Yet systems with dead time occur often in practice and they were dimensioned up to now by empiric thumb rules or by simplifying the control systems.

With the advent of the digital computers the possibility of more accurate numerical investigations permitting to speed up considerably the synthesis of the linear controls with dead time was opened up.

Investigation of the critical loop gain of the one-loop control system compensated in general by a PID controller versus the time constants of the system has been described in [8, 9].

[2, . . .,7] and the present paper deal with the determination of the stability region variation of a one-loop control system for the plant with dead time and second order lag, for various phase margin values when P, I, PI, PD and PID controllers are chosen respectively. Considering that systems with plant containing more than two time lags are rarely met with in control technics, the results in [2, . . ., 9] and the present paper are suitable for investigating the stability of any linear control system with dead time, where the transfer function of the open loop can be written in the form $Y(s) = Y_1(s)\exp(-s\tau)$, with $Y_1(s)$ not containing the factor $\exp(-s\tau)$ anymore. If the plant contains more than two time lags, then the substitution of the plant by two time lags proves to be a satisfactory approximation in the majority of the practical cases.

Summary

Diagrams produced by a digital computer, and representing the stability regions permitting to reach arbitrary phase margins of a linear, one-loop control system with dead time and second order lag and compensated in series by a proportional-differential element are presented. The PROCEDURE PID of the APPENDIX written in the ALGOL language permits to determine the stability region variation versus the phase margin and the time constants of the system for the above control compensated by proportional, proportional-integral, proportional-differential and proportional-integral-differential elements, respectively. The diagrams in [6, 7] and in the present paper, along with the PROCEDURE PID permit the easy choice of the P, I, PI, PD and PID compensations, the most advantageous in the case of the actually involved linear, one-loop control system with dead time and second order lag.

APPENDIX

```
PROCEDURE PID (ZETA,B,A,TAU,FIPM,FIMX,K);
VALUE ZETA,B,A,TAU,FIPM,FIMX;
REAL ZETA,B,A,TAU,FIPM,FIMX,K;
REAL PROCEDURE FI(OMEG);
VALUE OMEG; REAL OMEG;
  BEGIN REAL M8,M9;
    M2:=OMEG ↑ 2; M3:=M2×TD;
    M4:=T ↑ 2×M2; M5:=C−M3;
    M6:=1−M4; M7:=M1×OMEG;
    IF ABS(M5)<₁₀−30
      THEN BEGIN M8:=1.570796;
                    GO TO L1
           END
      ELSE M8:=ARCTAN(OMEG/M5);
    IF M8<0 THEN M8:=3.141593+M8;
    L1:IF ABS(M6)<₁₀−30
      THEN BEGIN M9:=1.570796;
                    GO TO L2
           END
      ELSE M9:=ARCTAN(M7/M6);
    IF M9<0 THEN M9:=3.141593+M9;
    L2:FI:=1.570796+OMEG×TAU+M9−M8;
  END FI;
REAL PROCEDURE DFI;
  BEGIN
    M5:=M5×M5; M6:=M6×M6;
    M7:=M7×M7;
    DFI:=TAU+M1×(1+M4)/(M6+M7)−(C+M3)/(M5+M2)
  END DFI;
```

```
PROCEDURE FIND(V,Q,OM1,OM2);
  VALUE V,Q; REAL V,Q,OM1,OM2;
  BEGIN OMEG:=Q; OM1:=0;
      FOR OMEG:=V×OMEG WHILE FI(OMEG)< F DO OM1:=
                              OMEG;
      OM2:=V×OM1
    END FIND;
PROCEDURE HALF(OM1,OM2,EPS,OMEG);
  VALUE OM1,OM2,EPS; REAL OM1,OM2,EPS,OMEG;
    BEGIN FOR OMEG:=(OM1+OM2)×0.5 WHILE ABS(FI(OMEG)
                              —F)>EPS DO
                    BEGIN IF FI(OMEG)<F
                       THEN OM1:=OMEG
                       ELSE OM2:=OMEG
              END;
            OMEG:=0.5×(OM1+OM2)
  END HALF;
REAL PROCEDURE NEWT(OMEG,DELT);
  VALUE DELT; REAL OMEG,DELT;
  BEGIN REAL Z;
    Z:=OMEG;
    FOR Z:=Z—(FI(OMEG)—F)/DFI WHILE ABS(Z—OMEG)>DELT
                              DO
      OMEG:=Z;
    NEWT:=OMEG
  END NEWT;
BEGIN
  T:=TAU/A; TD:=B×TAU; M1:=2×ZETA×T;
  F:=3.141593×(1—FIPM/180);
  IF FIMX—0.1>FIPM
    THEN FIND(2,Q,OM1,OM2)
    ELSE FIND(10,Q,OM1,OM2);
  HALF(OM1,OM2,EPS,OMEG);
  K:=NEWT(OMEG,DELT)×SQRT( (M6+M7)/(M5+M2) )
END PID;
```

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