PSEUDO-RANDOM NOISE IN THE TIME AND FREQUENCY DOMAINS

By

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Part 1

A problem in signal generation

Instruments for measuring the quantization noise in PCM telephone channels operate in the following manner. They produce a signal (u) which occupies the frequency band 450 to 550 Hz and feed this signal to the channel under test. At the receiving end of the channel a filter accepts only distortion products in the frequency range 850 to 3400 Hz and the r.m.s. value is measured [1, 2]. In this discussion we shall be concerned only with one aspect of the method, namely the production of a suitable waveform for the test signal u. It is clearly important that the frequency spectrum of u is confined within the specified limits, so it is inevitable that u appears at the output terminals of a filter. It is also important that the mean magnitude or the r.m.s. value of u is defined or measured and so can be maintained at some required value. Finally, it is important that the amplitude density distribution of u is defined, and preferable that this is the same distribution as that of human speech. Thus we arrive at the problem: What type of waveform should be fed to a bandpass filter if we want to produce a specified amplitude distribution at the output terminals? Similar problems have been discussed previously [3, 4] but in both these publications the input signal v to the filter is a random variable — though quite different in waveform in the two papers. The situation will now be considered where input signal v is a pseudo-random (p.r.) signal.

The advantage of p.r. signals is that in the form of m-sequences they can be produced easily and precisely by well-known circuit techniques using shift registers [5]. Only a brief description of m-sequences will be given here. Suppose a shift register with n stages is driven by a periodic clock pulse of frequency f_s. Suppose also that the condition of the first stage of the register is determined by feedback connections from later stages using logic circuits.

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Then with appropriate connections the sequence repeats after L clock pulses, where $L = 2^n - 1$. The feedback circuit is of the 'modulo 2' or 'exclusive OR' type such that the first stage will become a '1' if either but not both of the feedback stages are '1'. An output p.d. is available from anywhere in the register and will be a waveform of the type shown in Fig. 1.

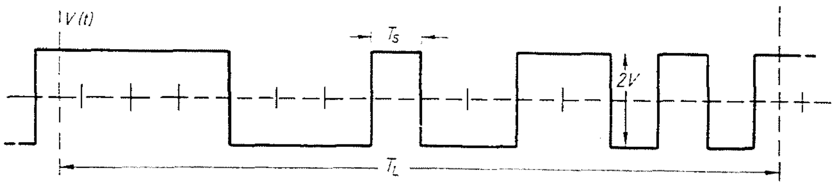

Fig. 1. Waveform derived from the m-sequence $n = 4$, $L = 15$, feedback stages 3 and 4

Figure 1 shows a rather short sequence and greater values of L are more common. A more typical situation would be: $n = 9$, $L = 511$. If we assume these values together with $f_s = 5110$ Hz, a suitable waveform for exciting a filter 100 Hz wide at 500 Hz will be generated. These values will be used to illustrate some general properties of the waveform which will be quoted without proof.

Fundamental frequency $= f_s/L$ 10 Hz

Spacing of line spectrum $= f_s/L$ 10 Hz

'Power' per line, for $L >> 1$, $\simeq \dfrac{2V^2}{L}\left[\dfrac{\sin(\pi f/f_s)}{\pi f/f_s}\right]^2$ volts^2

Amplitude of real sinusoidal line component, for $f << f_s = 2\sqrt{L}$ volts

The amplitude of these components is easily calculated from these simple expressions but their phase angles, though defined by the sequence, are not easy to find. It follows that when the signal v is fed to a filter of known properties only the mean square of the output u is easily calculated. The details of the waveform are difficult to estimate.

Consider, for example, an ideal filter of unity gain with bandwidth 100 Hz, containing ten lines of the spectrum of our example. The output signal u will be the sum of ten sinusoids with frequencies for example 450, 460, 470...530, 540 Hz.

The mean square value of u will be $10 \times 2V^2/L$, that is $20V^2/511$ volts^2. But in the application for PCM testing, and in many other circumstances, we are interested not merely in the m.s. value but also in the amplitude distribution of u, and particularly in the peak value. How do we estimate these? How do they vary as tolerances change in a practical circuit? How do we design a system to provide a specified amplitude distribution? These are the questions for which answers are being sought.

Some light is thrown on the problem by considering the phasor diagram of the components of u. There are 10 phasors each of amplitude $2V/\sqrt{L}$ volts.

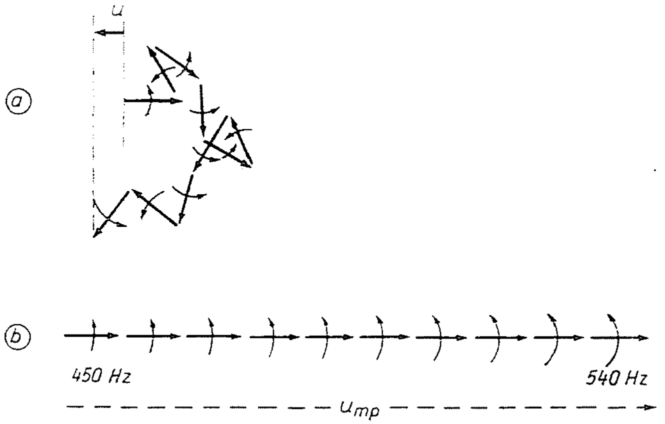

Fig. 2. Phasor diagram for ten sinousids of differing frequency, (a) in typical situation, (b) in phase to give maximum possible sum

At some instant in time they might be as shown in Fig. 2(a). The shape of the figure will depend on their original phases, the phase response of the filter, and the particular instant in time. If by chance all the components were in phase, as in Fig. 2(b), the maximum possible value of u would be obtained. For n_b components ($n_b = 10$ in Fig. 2) the corresponding peak would be $u_{mp} = n_b\, 2V/\sqrt{L}$, which is $\sqrt{2n_b}$ times the r.m.s. value of u. It is found in practice that waveforms in which u approaches u_{mp} are very rare. The phasor diagram seldom makes even an approximation to a straight line. Discussions of this approach have not yielded any answers to the particular problem, but the construction of Fig. 2 and of similar diagrams have given rise to the concepts discussed in Part 2 of this address.

Returning to the particular problem, there appears to be no simple and direct way of determining the amplitude density distribution. Both experiment and computation show that there is no obvious pattern of behaviour, except a general trend towards a normal distribution as sequence lengths increase. As an example of the nature of the observations, consider the following example. A computer program was written which defined a theoretical bandpass filter by its corner frequencies and the slope 'a' of its skirts assumed constant at $20a$ dB per decade. The phase response was computed from Bode's minimum phase shift theorem. The program then computed the response, a periodic waveform, to the fundamental periodic waveform arising from a '1' of an m-sequence. The program then followed the desired m-sequence automatically, summed the responses of all the '1's', plotted the envelope of the waveform and computed the r.m.s. and peak values and the amplitude distribution. A typical result was as follows. It concerns the particular example which has been used as an illustration throughout this paper, and is of particular interest because a real filter of similar properties was available and gave experimental results in accordance with the calculations.

Filter centre frequency 500 Hz, Bandwidth 100 Hz, $a = 15$.
Sequence $n = 9$, $L = 511$, $f_s = 5110$ Hz

$f_s = 5110$ Hz	$5110 + 10\%$	$5110 - 10\%$
$\frac{u_{\max}}{u_{\text{rms}}} = 2.41$	2.24	3.08

This result is typical of many, in that quite moderate changes in the system produce drastic changes in the amplitude distribution. A change of 10% in the clock frequency changed the waveform into one with little similarity to the original waveform.

It may be concluded, therefore, that when pseudo-noise is derived by feeding m-sequences to narrow band filters, the resulting waveform and its properties are quite sensitively dependent on the particular conditions. There appears to be no simple way of predicting a pattern of behaviour and each situation requires individual examination, either by computation or by experimental measurement.

Part 2

Folding and curling phasor diagrams

The problem described in Part 1 gave rise to speculation about the relation of waveforms in the time and frequency domain, as shown for example by Fig. 2 which illustrates a folding phasor diagram. A phasor diagram representing a line spectrum may be imagined as folding with the passage of time, and a sketch of the folding diagram is useful as an aid to understanding. The brief discussion of folding phasor diagrams given here is quite general and does not refer specifically to the problem discussed in Part 1.

Consider the line spectrum of a periodic voltage waveform $u(t)$. We are accustomed to write:

$$u(t) = \sum_{n=-\infty}^{+\infty} C_n \exp(j\Phi_n) \exp(j2\pi n f_0 t)$$

where C_n at angle Φ_n is a component of the spectrum, $[C_n \exp(j\Phi_n)] \exp(j2\pi n f_0)$ is a rotating phasor in the complex plane, and the vector sum of all such phasors gives the waveform $u(t)$. The process can be sketched, in some cases quite simply, if the phasors are added in order of increasing

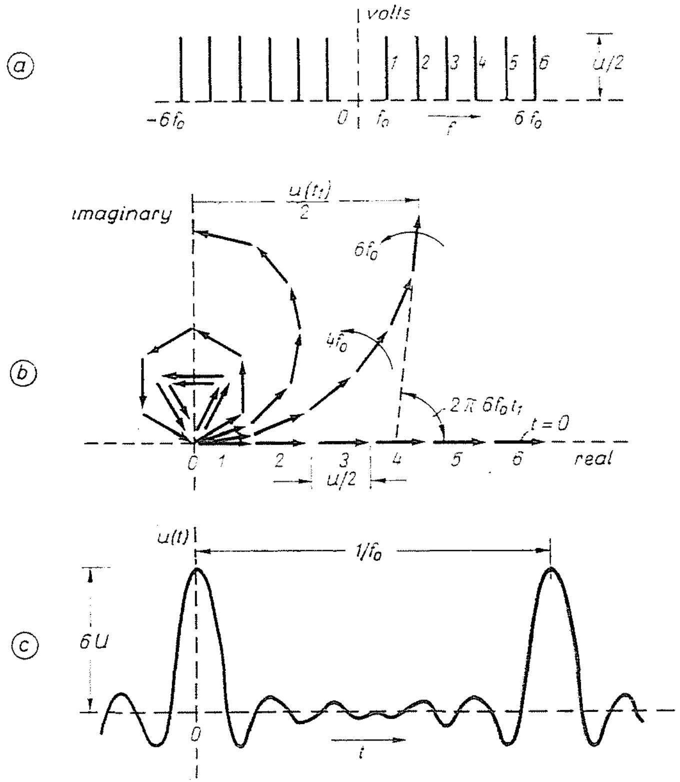

Fig. 3. A simple line spectrum (a) and its folding phasor diagramm (b) and the corresponding periodic waveform (c)

frequency. Consider, for example, the particular line spectrum shown in Fig. 3(a) where we assume $C_n = U/2$, $\Phi_n = 0$, up to the 6th Harmonic only. Fig. 3(b) shows the phasor diagram at various instants of time. Only positive frequency components need be shown since the negative frequencies add to produce the complex conjugate. One should try to imagine the change of shape with the passage of time, and to visualize the diagram folding in the manner of a model constructed of hinged links. The chain of hinged links folds round until at $t = 1/(2f_0)$ the links lie alongside each other with angles π, 2π, 3π, ... 6π after which the chain unfolds to provide the main pulse again at $t = 1/f_0$.

The waveform is envisaged by following the path of the tip of the link whose frequency is greatest.

Another interesting diagram is that of the square wave:

$$v(t) = \frac{4E}{\pi} (\cos \omega_0 t - 1/3 \cos 3\omega_0 t + 1/5 \cos 5\omega_0 t \ldots)$$

A further extension of the concept may be applied to waveforms which are not periodic, but transient. These are described in terms of the Fourier Transform by:

$$u(t) = \int_{-\infty}^{+\infty} S(f) \exp (j2\pi ft)\, df, \text{ where } S(f) = |S(f)| < \Theta_f$$

This equation states that $u(t)$ is composed of the vector sum of elementary components $S(f)\, df \exp (j2\pi ft)$. Each component is a rotating phasor of length $|S(f)|\, df$ and angle $(\Theta_f + 2\pi ft)$. The waveform can be derived by sketching the phasor diagram in a manner similar to the previous example. Only a simple example will be given here. Consider a uniform spectrum as shown in Fig. 4. It will be assumed that the phase angle is zero. Consider the positive frequencies only, starting at $f = 0$. Within each element df there

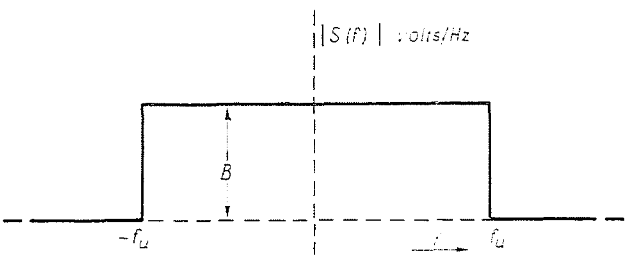

Fig. 4. A simple distributed spectrum

is an elementary component Bdf in length, and of angle $2\pi f_i$. At $t = 0$ these components lie in line along the real axis, the length of the line being Bf_u volts. As t increases, they form an arc which maintains its length Bf_u but which curls and eventually rolls into a circle of infinitely small diameter. This process and the corresponding waveform $a(t)$ is illustrated in Fig. 5. If we consider the passage of time from minus to plus infinity, we can envisage the phasor diagram first uncurling to produce the transient pulse and then recurling. The process is visualised more effectively by some manipulation with a strip of paper than by looking at a stationary drawing.

This example of a curling phasor diagram is the simplest possible case, but the concept can be extended to other waveforms and spectra. The prin-

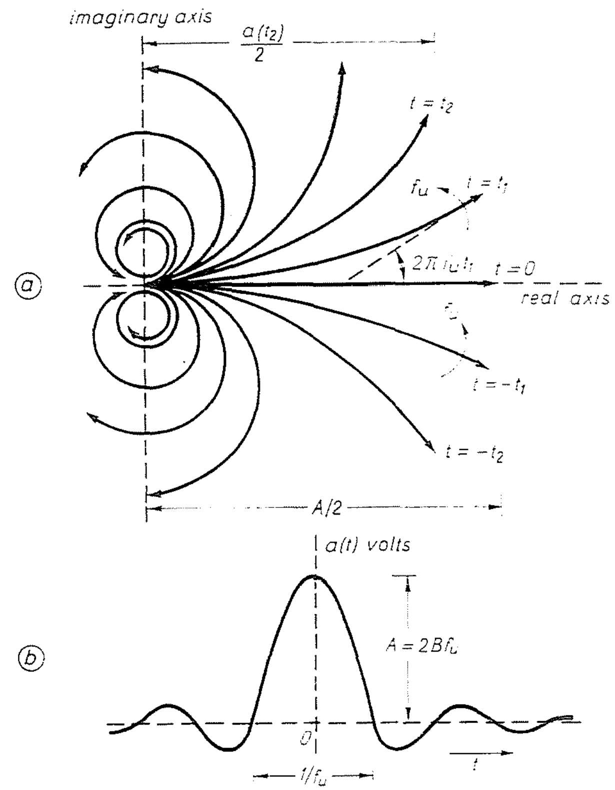

Fig. 5. Curling phasor diagram (a) and waveform (b) corresponding to the rectangular spectrum of figure 4

cipal advantage of the concept is that it helps the understanding of the relation between signals in the time and frequency domains.

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Summary

A particular method of testing pulse-code-modulated communication channels leads to the problem of determining the amplitude density distribution of the output signal of a filter when the input signal is a binary maximum length sequence. Experimental and computational methods can analyse the problem but a general design procedure has not been found. The problem leads to speculation about the representation of the Fourier Integral as a curling phasor diagram.

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