INTERPOLATION WITH PR FUNCTIONS BASED ON F. FENYVES' METHOD

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Introduction

The problem of interpolation is often encountered in network theory. For instance, if the impedance function or the equivalent circuit of an aerial is to be determined from discrete measurement data the solution is obtained by interpolation. The same applies to broad band matching [1] and network modelling. Latter is illustrated by the problem of the determination of transistor models of various complexity from data measured at different frequencies [2].

The impedance-function interpolation can be formulated as follows: A positive real (PR) function, $Z(p)$ is sought such that for a set of complex frequencies $p_1, p_2, \ldots p_n$ and a set of complex values $Z_1, Z_2, \ldots Z_n$ the equation

$$Z(p_i) = Z_i \qquad i = 1, 2, \ldots n$$

holds.

Problems of this type were solved, among others, by SMILEN in 1965 [3], YOULA and SAITO in 1966 [4] as well as by BELEVITCH in 1967 [5]. In the present paper we should like to draw the attention to the Hungarian born F. Fenyves' doctor thesis, published in book form in 1938, dealing with impedance-function interpolation [6]. F. Fenyves was born in 1911 in Budapest, where he was educated up to university level. In 1929 he enroled in the Zurich Technical College (Eidgenössische Technische Hochschule, Zürich), where he graduated in electrical engineering. The above mentioned doctor thesis was supervised by Dr. M. Plancherel and Dr. F. Fischer. F. Fenyves disappeared during the second world war.

Interpolation within the unit circle

In his work FENYVES made use of PICK's and mainly NEVANLINNA's results associated with the investigations of bounded analytic functions ([7, 8] and [9] resp.). One of these results is the solution of the following interpolation problem [9]:

Let there be given two sets of n complex numbers, z_ν and w_ν, so that $|z_\nu| < 1$ and $|w_\nu| \leq 1$ $(\nu = 1, 2, \ldots n)$. Those bounded analytical functions $w(z)$ are to be determined for which

$$|w(z)| \leq 1 \quad \text{if} \quad |z| < 1 \tag{1}$$

and

$$w(z_\nu) = w_\nu \qquad \nu = 1, 2, \ldots n \tag{2}$$

For the sake of the better understanding of FENYVES' work it seems to be desirable to review some steps of the solution of this problem.

In the course of the solution the linear transformation

$$\xi(z) = \frac{z_1 - z}{1 - z_1^* z} \tag{3}$$

mapping the unit circle into itself, was used (*denotes conjugate). Using this linear transformation and choosing n arbitrary complex numbers c_1, c_2, ... c_n ($|c_i| < 1$), a set of series can be defined as follows:

$$\text{Series } w_\nu^{(1)} = w_\nu \qquad \nu = 1, 2, \ldots n \tag{4}$$

Series $w_\nu^{(2)}$ is defined in the form

$$\frac{c_1 - w_\nu^{(2)}}{1 - c_1^* w_\nu^{(2)}} = \frac{w_1^{(1)} - w_\nu^{(1)}}{1 - w_1^{(1)*} w_\nu^{(1)}} \cdot \frac{1 - z_1^* z_\nu}{z_1 - z_\nu} \qquad \nu = 2, 3, \ldots n. \tag{5}$$

In general, series $w_\nu^{(i+1)}$ is given in the form

$$\frac{c_i - w_\nu^{(i+1)}}{1 - c_i^* w_\nu^{(i+1)}} = \frac{w_i^{(i)} - w_\nu^{(i)}}{1 - w_i^{(i)*} w_\nu^{(i)}} \cdot \frac{1 - z_i^* z_\nu}{z_i - z_\nu} \qquad \nu = i+1, i+2, \ldots n \tag{6}$$

These series can be compiled in a table or triangular matrix having the form

$$\begin{matrix} w_1^{(1)} & w_2^{(1)} & \ldots & \ldots & w_{n-1}^{(1)} & w_n^{(1)} \\ 0 & w_2^{(2)} & \ldots & \ldots & w_{n-1}^{(2)} & w_n^{(2)} \\ 0 & 0 & w_3^{(3)} & \ldots & w_{n-1}^{(3)} & w_n^{(3)} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & w_n^{(n)} \end{matrix} \tag{7}$$

The magnitude of $w_\nu^{(\mu)}$-s cannot be greater than one. The solvability of and the number of solutions to the problem depend on the values of the table as follows.

Solutions can be found for two cases:

1. There is one solution if

$$|w_\nu^{(\mu)}| \begin{cases} < 1, \text{ if } \nu = 1, 2, \dots n; & \mu = 1, 2, \dots i \quad (i < n) \\ = 1, \text{ if } \nu = i+1, \dots n; & \mu = i + 1 \\ = 0, \text{ if } \nu = 1, 2, \dots n; & \mu = i + 2, \dots n \end{cases} \tag{8}$$

and $w_\nu^{(i+1)} = w_{i+1}^{(i+1)}$ for $\nu = i + 2,\ i + 3, \dots n$.

The solution is of i-th order in terms of z.

2. There is an infinite number of solutions if

$$|w_\nu^{(\mu)}| < 1, \text{ for } \nu = 1, 2, \dots n, \quad \mu = 1, 2, \dots n \tag{9}$$

Interpolation of PR function

The lossy impedance functions (representing networks containing lossy resistors series to the inductances and parallel to the capacitances) are bounded and analytic in the closed right half plane. FENYVES transformed the relations, concerning the interior of the unit circle introduced in the previous section to the right half plane, thus solving the interpolation problem. In order to be able to map the j axis he assumed the impedance to be lossy i.e. not to have singularities on the j axis.

This was taken into account by choosing a "right half plane" enlarged by the positive quantity δ to correspond with the interior of the unit circle.

The details of FENYVES' method are as follows. Let a positive number δ be given and two sets of complex numbers p_ν and Z_ν, $\nu = 1, 2, \dots n$, Re $p_\nu \geq 0$, Re $Z_\nu \geq 0$. We can define a number of series $Z_\chi^{(\nu)}$ ($\nu = 1, 2, \dots n$, $\chi = \nu$, $\nu + 1, \dots n$) in a way analogous to that used in (4), (5) and (6). For a start $Z_1^{(1)} = Z_1$. Using the notations

$$\gamma_1 = \text{Re } p_1 + \delta; \qquad \omega_1 = \text{Im } p_1$$

$$x_1 = \text{Re } Z_1^{(1)} \qquad y_1 = \text{Im } Z_1^{(1)}$$

the following polynomials are to be calculated

$$A_1(p) = -(p + \delta)(x_1\omega_1 + y_1\gamma_1)j - [x_1(\gamma_1^2 + \omega_1^2) + \gamma_1(x_1^2 + y_1^2)]$$

$$B_1(p) = (p + \delta)[x_1 + (x_1^2 + y_1^2)\gamma_1] - (x_1\omega_1 - y_1\gamma_1)j$$

$$C_1(p) = -(p+\delta)(x_1+\gamma_1) + (x_1\omega_1 + y_1\gamma_1)j$$

$$D_1(p) = (p+\delta)(x_1\omega_1 - y_1\gamma_1)j + [\gamma_1 + x_1(\gamma_1^2 + \omega_1^2)]$$

Series $Z_\chi^{(2)}$ can be determined by using the formula

$$Z_\chi^{(2)} = \left.\frac{A_1(p) - C_1(p)Z(p)}{B_1(p) - D_1(p)Z(p)}\right|_{p=p_\chi} \tag{10}$$

Here

$$Z(p_\chi) = Z_\chi, \text{ for } \chi = 2, 3, \dots n.$$

The series given in Eq. (10) for different χ's correspond to the second row of Table (7). The first row of the very same table is given by letting $Z_\chi^{(1)} = Z_\chi$, $\chi = 1,2, \dots n$. In general the ν-th row of the table is given in the following recurring way

$$Z^{(\nu)}(p_\chi) = Z_\chi^{(\nu)}$$

$$\gamma_\nu = \operatorname{Re} p_\nu + \delta, \qquad \omega_\nu = \operatorname{Im} p_\nu,$$

$$x_\nu = \operatorname{Re} Z_\nu^{(\nu)}, \qquad y_\nu = \operatorname{Im} Z_\nu^{(\nu)},$$

$$A_\nu(p) = -A_{\nu-1}(p)[(p+\delta)(x_\nu+\gamma_\nu) - j(x_\nu\omega_\nu + y_\nu\gamma_\nu)] + \\ + B_{\nu-1}(p)[(p+\delta)(x_\nu\omega_\nu + y_\nu\gamma_\nu)j + x_\nu(\gamma_\nu^2+\omega_\nu^2) + \gamma_\nu(x_\nu^2+y_\nu^2)]$$

$$B_\nu(p) = +A_{\nu-1}(p)[(p+\delta)(x_\nu\omega_\nu - y_\nu\gamma_\nu)j + \gamma_\nu + x_\nu(\gamma_\nu^2+\omega_\nu^2)] - \\ - B_{\nu-1}(p)\{(p+\delta)[x_\nu + \gamma_\nu(x_\nu^2+y_\nu^2)] - j(x_\nu\omega_\nu - y_\nu\gamma_\nu)\}$$

$$C_\nu(p) = A_\nu(p), \text{ if } \nu \neq 0$$

$$D_\nu(p) = B_\nu(p), \text{ if } \nu \neq 0$$

$$A_0 = D_0 = 0. \quad B_0 = -1. \quad C_0 = +1.$$

Making use of the above we obtain

$$Z_\chi^{(\nu+1)} = \left.\frac{A_\nu(p) - C_\nu(p)Z(p)}{B_\nu(p) - D_\nu(p)Z(p)}\right|_{p=p_\chi} \qquad \chi = \nu+1, \nu+2, \dots n. \tag{11}$$

Thus, Table (7) can be constructed from the calculated complex values $Z_\chi^{(\nu)}$. (Note that $Z_\chi^{(\nu)}$ here corresponds to $w_\nu^{(\mu)}$ in Table (7)). The following theorem can be concluded:

The necessary and sufficient condition that there exist(s) a positive function (positive functions) assuming the prescribed values $Z_1 \dots Z_n$ at complex frequencies $p_1 \dots p_n$ is

a) either that there be an integer k ($k < n$) separating the rows of Table (7) such as

$$\operatorname{Re} Z_\nu^{(\nu)} > 0 \qquad \text{for} \qquad \nu < k$$

$$\operatorname{Re} Z_k^{(k)} = 0 \qquad \text{and}$$

$$Z_\chi^{(k)} = Z_k^{(k)} \qquad \text{for} \qquad \chi = k + 1,\ k + 2, \ldots n,$$

b) or that

$$\operatorname{Re} Z_\nu^{(\nu)} > 0 \qquad \text{for} \qquad \nu = 1, 2, \ldots n.$$

In case *a)* the solution is a unique rational function of order $(k - 1)$ having the form

$$Z(p) = \frac{A_{k-1}(p) - B_{k-1}(p) Z_k^{(k)}}{C_{k-1}(p) - D_{k-1}(p) Z_k^{(k)}} \tag{12}$$

In case *b)* an infinite number of solutions exist in the form

$$Z(p) = \frac{A_n(p) - B_n(p) Z^{(n+1)}(p)}{C_n(p) - D_n(p) Z^{(n+1)}(p)} \tag{13}$$

where $Z^{(n+1)}(p)$ denotes an arbitrary positive function.

In network theory solutions are sought for in the form of positive real functions, a subclass of positive functions, requiring a mapping from real axis to real axis, too. The procedure previously described for the creation of positive functions can be applied to generating positive real functions, too. To achieve this every complex frequency p_i must have its complex conjugate p_i^* in the set of points where $Z(p)$ is prescribed and these prescribed values of $Z(p)$ must satisfy the condition $Z(p_i^*) = Z_i^*$. Further, complex conjugate pairs p_i and p_i^* must follow each other in the sequence of interpolation steps. Thus, for a set of n frequencies p_i, not containing complex conjugate pairs and containing m real numbers a rational function of order $(2\,n - m)$ can be determined.

Conclusion

The solutions to the interpolation problem given in FENYVES' dissertation differs from other solutions known to us in the literature. Considering his method, it is slightly similar to SMILEN's procedure, because the determination of a function satisfying n requirements is traced back to that of satisfying $(n - 1)$ requirements. This fact shows itself in the triangular shape of Table (7) as the subsequent rows contain values decreasing in number by one. SMILEN does the same when meeting one requirement, by subtracting one Brune-section, thus creating a remainder which is to meet requirements reduced in number by one.

Considering its conclusions FENYVES' solution resembles YOULA's and SAITO's results, as for the case of unique solution the last few rows of FENYVES' table are zero, which can be interpreted as the triangular matrix being singular. Likewise the A matrix by YOULA and SAITO (Nevanlinna-Pick matrix) is also singular if there exists a unique positive real Foster function. We note, however, that the definitions of the two matrices differ completely.

Summary

This paper gives a report on the Hungarian born F. Fenyves' contribution to the interpolation problem. He gave a rather simple solution to interpolation with positive or positive-real functions as early as 1938. In spite of the completeness of Fenyves' solution none of recent publications have made use of and referred to it.

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